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ON A SHARPER FORM OF HALF-DISCRETE HILBERT INEQUALITY

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Abstract. In this paper we introduce a general inequality by using the half-discrete Hilbert inequality. As an application we give a sharper form of the half-discrete Hilbert inequality and some carlson's type inequalities.

1. Introduction

If $f(x), g(x) \geq 0$, $0 < \int_0^\infty f^2(x)dx < \infty$, and $0 < \int_0^\infty g^2(x)dx < \infty$, then (see [3])

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \pi \left\{ \int_0^\infty f^2(x)dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(x)dx \right\}^{\frac{1}{2}}. \tag{1.1}$$

Inequality (1.1) is called Hilbert's integral inequality which has been extended by Hardy [3] as: if $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x), g(x) > 0$, $0 < \int_0^\infty f^p(x)dx < \infty$, and $0 < \int_0^\infty g^q(x)dx < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin\frac{\pi}{p}} \left\{ \int_0^\infty f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(x)dx \right\}^{\frac{1}{q}}. \tag{1.2}$$

The corresponding inequalities in the discrete case are (a_m, $b_n > 0$):

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m+n} < \pi \left\{ \sum_{n=1}^\infty a_n^2 \right\}^{\frac{1}{2}} \left\{ \sum_{n=1}^\infty b_n^2 \right\}^{\frac{1}{2}} \tag{1.3}$$

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{\sin\frac{\pi}{p}} \left\{ \sum_{n=1}^\infty a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^\infty b_n^q \right\}^{\frac{1}{q}}, \tag{1.4}$$

provided that the series on the right-hand side of (1.3) and (1.4) are convergent. The constant factor π is the best possible in both (1.1) and (1.3), and the constant $\frac{\pi}{\sin\frac{\pi}{p}}$ is the best possible in (1.2) and (1.4). Recently in [4] the following sharper forms of (1.2) and (1.4) were given

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \sqrt{L} \left(\int_0^\infty x^{pqA_1-1} f^p(x)dx \right)^{\frac{1}{2p}} \left(\int_0^\infty y^{pqA_2-1} g^q(y)dy \right)^{\frac{1}{2q}}$$

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$$\times\left\{\int_0^\infty\int_0^\infty \frac{xf(x)g(y)}{(x+y)^2}dxdy\right\}^{\frac{pA_2}{2}}\left\{\int_0^\infty\int_0^\infty \frac{yf(x)g(y)}{(x+y)^2}dxdy\right\}^{\frac{qA_1}{2}} \quad (1{,}5)$$

and

$$\sum_{n=1}^\infty\sum_{m=1}^\infty \frac{a_m b_n}{m+n} < \sqrt{L}\left\{\sum_{m=1}^\infty m^{pqA_1-1}a_m^p\right\}^{\frac{1}{2p}}\left\{\sum_{n=1}^\infty n^{pqA_2-1}b_n^q\right\}^{\frac{1}{2q}}$$
$$\times\left\{\sum_{n=1}^\infty\sum_{m=1}^\infty \frac{na_m b_n}{(m+n)^2}\right\}^{\frac{qA_1}{2}}\left\{\sum_{n=1}^\infty\sum_{m=1}^\infty \frac{ma_m b_n}{(m+n)^2}\right\}^{\frac{pA_2}{2}}, \quad (1.6)$$

here $A_1 \in \left(0,\frac{1}{q}\right)$, $A_2 \in \left(0,\frac{1}{p}\right)$, $pA_2+qA_1=1$ and the constant $L=\frac{B(pA_2,1-pA_2)}{(pA_2)^{pA_2}(qA_1)^{qA_1}}$ ($B(x,y)$ is the Beta function). Note that both (1.5) and (1.6) reduces to (1.2) and (1.4) if we put $A_1=A_2=\frac{1}{pq}$ and apply Young's Inequality, see [4].

In [1] Yang introduced the following half-discrete Hilbert's inequality

$$\int_0^\infty f(x)\sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda}dx < B(\lambda_1,\lambda_2)\left(\int_0^\infty x^{p(1-\lambda_1)-1}f^p(x)dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{q(1-\lambda_2)-1}a_n^q\right)^{\frac{1}{q}}, \quad (1.7)$$

here, $\lambda_1,\lambda_2>0, \lambda_1+\lambda_2=\lambda, 0<\lambda_1<1$, and the constant $B(\lambda_1,\lambda_2)$ is the best possible. In [2] B. Yang and Q. Chen obtained a general half-discrete Hilbert's inequality for a homogeneous kernel. In fact they proved the following theorem:

Theorem A. *Suppose that* $\lambda_1, \lambda_2 \in \mathbb{R}, \lambda_1+\lambda_2=\lambda, k_\lambda(u(x),v(y))$ *is a non-negative finite homogeneous function of degree* $-\lambda$ *in* $\mathbb{R}^2_+$, $u(x)(x\in(b,c), -\infty\le b<c\le\infty)$ *and* $v(y)(y\in[n_0,\infty), n_0\in\mathbb{N})$ *are strictly increasing differential functions with* $u(b^+)=0$, $v(n_0)>0, u(c^-)=v(\infty)=\infty$, $k(\lambda_1)\in\mathbb{R}_+$. *If* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $f(x), a_n\ge 0$, $\int_b^c \frac{[u(x)]^{p(1-\lambda_1)-1}}{[u'(x)]^{p-1}}f^p(x)dx<\infty, 0<\sum_{n=n_0}^\infty \frac{[v(n)]^{q(1-\lambda_2)-1}}{[v'(n)]^{q-1}}a_n^q <\infty$ *then we have the following inequality*:

$$\sum_{n=n_0}^\infty a_n\int_b^c k_\lambda(u(x),v(n))f(x)dx = \int_b^c f(x)\sum_{n=n_0}^\infty a_n k_\lambda(u(x),v(n))dx$$
$$< k(\lambda_1)\left(\int_b^c \frac{[u(x)]^{p(1-\lambda_1)-1}}{[u'(x)]^{p-1}}f^p(x)dx\right)^{\frac{1}{p}}\left(\sum_{n=n_0}^\infty \frac{[v(n)]^{q(1-\lambda_2)-1}}{[v'(n)]^{q-1}}a_n^q\right)^{\frac{1}{q}}. \quad (1.8)$$

Moreover, if $\frac{v'(y)}{v(y)}(y\ge n_0)$ *is decreasing and there exist constants* $\blacksquare<\lambda_1$ *and* $M>0$, *such that* $k_\lambda(t,1)\le\frac{M}{t^\delta}$ $t\in\left(0,\frac{1}{v(n_0)}\right]$, *then the constant factor* $k(\lambda_1)$ *is the best possible.*

In particular, if we put $k_1(u(x),v(n))=\frac{1}{\alpha u(x)+\beta v(n)}, \lambda_1+\lambda_2=1(\alpha,\beta>0)$, we get from (1.8)

$$\sum_{n=n_0}^\infty a_n\int_b^c \frac{1}{\alpha u(x)+\beta v(n)}f(x)dx < \frac{B(\lambda_1,\lambda_2)}{\alpha^{\lambda_1}\beta^{\lambda_2}}\left(\int_b^c \frac{[u(x)]^{p(1-\lambda_1)-1}}{[u'(x)]^{p-1}}f^p(x)dx\right)^{\frac{1}{p}}\left(\sum_{n=n_0}^\infty \frac{[v(n)]^{q(1-\lambda_2)-1}}{[v'(n)]^{q-1}}a_n^q\right)^{\frac{1}{q}}. \quad (1.9)$$

We need the following formula for the beta function

$$B(s,t+1)=\frac{t}{s+t}B(s,t). \tag{1.10}$$

For the sequence of real numbers (a_n), Carlson's inequality is given as

$$\sum_{n=1}^{\infty}a_n<\sqrt{\pi}\left(\sum_{n=1}^{\infty}a_n^2\right)^{\frac{1}{4}}\left(\sum_{n=1}^{\infty}n^2a_n^2\right)^{\frac{1}{4}},$$

the constant $\sqrt{\pi}$ is the best possible. The continuous version of (1.10) is

$$\int_0^{\infty}f(x)dx\le\sqrt{\pi}\left(\int_0^{\infty}f^2(x)dx\right)^{\frac{1}{4}}\left(\int_0^{\infty}x^2f^2(x)dx\right)^{\frac{1}{4}},$$

the constant $\sqrt{\pi}$ is sharp. For more details about these inequalities and their extensions we refer the reader to the book [5].

In this paper, we introduce a new inequality with a best constant factor which gives an upper estimate for the quantity $\sum\limits_{n=n_0}^{\infty}\int_b^c F(x,n)\,dx$, where $F(x,n)$ is a positive function defined on $(b,c)\times(n_0,\infty)$. As an application, we obtain a sharper form of half-discrete Hilbert inequality (1.9). Some examples of Carlson's type inequalities are also considered.

2. Main results

Theorem 2.1. *Let* $p>1,\frac{1}{p}+\frac{1}{q}=1$, *the functions* u,v,f *and* a_n *are as in Theorem A,* $0<\int_b^c\frac{[u(x)]^{p(1-\lambda_1)-1}}{[u'(x)]^{p-1}}f^p(x)dx<\infty, 0<\sum\limits_{n=n_0}^{\infty}\frac{[v(n)]^{q(1-\lambda_2)-1}}{[v'(n)]^{q-1}}a_n^q<\infty$. *Let* $F(x,n)$ *be a positive function defined on* $(b,c)\times(n_0,\infty)$ *such that* $\sum\limits_{n=n_0}^{\infty}\int_b^c\frac{u(x)F^2(x,n)}{a_nf(x)}dx<\infty$ *and* $\sum\limits_{n=n_0}^{\infty}\int_b^c\frac{v(n)F^2(x,n)}{a_nf(x)}dx<\infty$ *then the following inequality holds*

$$\left[\sum_{n=n_0}^{\infty}\int_b^c F(x,n)\,dx\right]^2<K\left(\int_b^c\frac{[u(x)]^{p(1-\lambda_1)-1}}{[u'(x)]^{p-1}}f^p(x)dx\right)^{\frac{1}{p}}\left(\sum_{n=n_0}^{\infty}\frac{[v(n)]^{q(1-\lambda_2)-1}}{[v'(n)]^{q-1}}a_n^q\right)^{\frac{1}{q}}$$
$$\times\left(\sum_{n=n_0}^{\infty}\int_b^c\frac{u(x)F^2(x,n)}{a_nf(x)}dx\right)^{\lambda_1}\left(\sum_{n=n_0}^{\infty}\int_b^c\frac{v(n)F^2(x,n)}{a_nf(x)}dx\right)^{\lambda_2}, \tag{2.1}$$

here $\lambda_1,\lambda_2>0$, $\lambda_1+\lambda_2=1$ *and the constant* $K=\frac{B(\lambda_1,\lambda_2)}{\lambda_1^{\lambda_1}\lambda_2^{\lambda_2}}$ *is the best possible.*

Proof. Let $\alpha,\beta>0$, applying Cauchy's inequality for integrals, Cauchy's inequality for series and inequality (1.9), respectively, we get

$$\left[\sum_{n=n_0}^{\infty}\int_b^c F(x,n)\,dx\right]^2$$

$$= \left[\sum_{n=n_0}^{\infty}\int_b^c \left\{\frac{\sqrt{a_n f(x)}}{\sqrt{\alpha u(x)+\beta v(n)}}\right\}\left\{\frac{\sqrt{\alpha u(x)+\beta v(n)}}{\sqrt{a_n f(x)}}F(x,n)\right\}dx\right]^2$$
$$\leq \left[\sum_{n=n_0}^{\infty}\left(\int_b^c \frac{a_n f(x)dx}{\alpha u(x)+\beta v(n)}\right)^{\frac{1}{2}}\left(\int_b^c \frac{\alpha u(x)+\beta v(n)}{a_n f(x)}F^2(x,n)\,dx\right)^{\frac{1}{2}}\right]^2$$
$$\leq \sum_{n=n_0}^{\infty}\int_b^c \frac{a_n f(x)dx}{\alpha u(x)+\beta v(n)}\sum_{n=n_0}^{\infty}\int_b^c \frac{\alpha u(x)+\beta v(n)}{a_n f(x)}F^2(x,n)\,dx$$
$$< \frac{B(\lambda_1,\lambda_2)}{\alpha^{\lambda_1}\beta^{\lambda_2}}\left(\int_b^c \frac{[u(x)]^{p(1-\lambda_1)-1}}{[u'(x)]^{p-1}}f^p(x)dx\right)^{\frac{1}{p}}\left(\sum_{n=n_0}^{\infty}\frac{[v(n)]^{q(1-\lambda_2)-1}}{[v'(n)]^{q-1}}a_n^q\right)^{\frac{1}{q}}$$
$$\times\left[\alpha\sum_{n=n_0}^{\infty}\int_b^c \frac{u(x)}{a_n f(x)}F^2(x,n)\,dx+\beta\sum_{n=n_0}^{\infty}\int_b^c \frac{v(n)}{a_n f(x)}F^2(x,n)\,dx\right]$$
$$= B(\lambda_1,\lambda_2)\left(\int_b^c \frac{[u(x)]^{p(1-\lambda_1)-1}}{[u'(x)]^{p-1}}f^p(x)dx\right)^{\frac{1}{p}}\left(\sum_{n=n_0}^{\infty}\frac{[v(n)]^{q(1-\lambda_2)-1}}{[v'(n)]^{q-1}}a_n^q\right)^{\frac{1}{q}}$$
$$\times\left[\left(\frac{\alpha}{\beta}\right)^{\lambda_2}\sum_{n=n_0}^{\infty}\int_b^c \frac{u(x)}{a_n f(x)}F^2(x,n)\,dx+\left(\frac{\beta}{\alpha}\right)^{\lambda_1}\sum_{n=n_0}^{\infty}\int_b^c \frac{v(n)}{a_n f(x)}F^2(x,n)\,dx\right].$$

Set $S=\sum_{n=n_0}^{\infty}\int_b^c \frac{u(x)}{a_n f(x)}F^2(x,n)\,dx$, $T=\sum_{n=n_0}^{\infty}\int_b^c \frac{v(n)}{a_n f(x)}F^2(x,n)\,dx$, $y=\frac{\beta}{\alpha}$ and consider the function $g(y)=y^{-\lambda_2}S+y^{\lambda_1}T$. Since $g'(y)=y^{\lambda_1-2}(\lambda_1 Ty-\lambda_2 S)$, we conclude that the minimum of this function attains for $y=\frac{\lambda_2 S}{\lambda_1 T}$. Therefore, if we let $\alpha=\lambda_1 T$ and $\beta=\lambda_2 S$, we get (2.1).

It remains to show that the constant K in (2.1) is the best possible. There exists $d\in(b,c)$, satisfying $u(d)=1$. For $0<\varepsilon<p\lambda_1$, setting $\widetilde{f}(x)=0, x\in(b,d); \widetilde{f}(x)=[u(x)]^{\lambda_1-\frac{\varepsilon}{p}-1}u'(x), x\in[d,c)$, $\widetilde{a}_n=[v(n)]^{\lambda_2-\frac{\varepsilon}{q}-1}v'(n), n\geq n_0$ and $\widetilde{F}(x,n)=\frac{\widetilde{a}_n\widetilde{f}(x)}{u(x)+v(n)}$. Suppose that there exists a positive constant $C<K$ such that (2.1) is still valid if we replace K by C, then we find

$$\widetilde{I}=:\left[\sum_{n=n_0}^{\infty}\int_b^c \widetilde{F}(x,n)\,dx\right]^2=\left[\sum_{n=n_0}^{\infty}\int_b^c \frac{\widetilde{a}_n\widetilde{f}(x)}{u(x)+v(n)}dx\right]^2$$
$$< C\left(\int_b^c \frac{[u(x)]^{p(1-\lambda_1)-1}}{[u'(x)]^{p-1}}\widetilde{f}^p(x)dx\right)^{\frac{1}{p}}\left(\sum_{n=n_0}^{\infty}\frac{[v(n)]^{q(1-\lambda_2)-1}}{[v'(n)]^{q-1}}\widetilde{a}_n^q\right)^{\frac{1}{q}}$$
$$\times\left(\sum_{n=n_0}^{\infty}\int_b^c \frac{u(x)\widetilde{a}_n\widetilde{f}(x)}{(u(x)+v(n))^2}dx\right)^{\lambda_1}\left(\sum_{n=n_0}^{\infty}\int_b^c \frac{v(n)\widetilde{a}_n\widetilde{f}(x)}{(u(x)+v(n))^2}dx\right)^{\lambda_2}$$
$$:= C(I_1^{\frac{1}{p}}+I_2^{\frac{1}{q}}+I_3^{\lambda_1}+I_4^{\lambda_2}). \tag{2.2}$$

Computing these quantities we obtain

$$I_1=\int_d^c [u(x)]^{-\varepsilon-1}u'(x)dx=\int_1^{\infty}t^{-1-\varepsilon}dt=\frac{1}{\varepsilon},$$

$$I_2 = \sum_{n=n_0}^{\infty} [v(n)]^{-\varepsilon-1} v'(n) < \frac{v'(n_0)}{[v(n_0)]^{1+\varepsilon}} + \int_{n_0}^{\infty} [v(t)]^{-\varepsilon-1} v'(t)\,dt = \frac{v'(n_0)}{[v(n_0)]^{1+\varepsilon}} + \frac{1}{\varepsilon\,[v(n_0)]^{\varepsilon}},$$

$$\begin{aligned}
I_3 &= \sum_{n=n_0}^{\infty} [v(n)]^{\lambda_2-\frac{\varepsilon}{q}-1} v'(n) \int_d^c \frac{[u(x)]^{\lambda_1-\frac{\varepsilon}{p}} u'(x)}{(u(x)+v(n))^2}dx \\
&= \left(t = \frac{u(x)}{v(n)}\right) = \sum_{n=n_0}^{\infty} [v(n)]^{-\varepsilon-1} v'(n) \int_{\frac{1}{v(n)}}^{\infty} \frac{t^{\lambda_1-\frac{\varepsilon}{p}}}{(t+1)^2}dx \\
&< \sum_{n=n_0}^{\infty} [v(n)]^{-\varepsilon-1} v'(n) \int_0^{\infty} \frac{t^{\lambda_1-\frac{\varepsilon}{p}}}{(t+1)^2}dx = [B(\lambda_1+1,\lambda_2)+o(1)] \sum_{n=n_0}^{\infty} \frac{v'(n)}{[v(n)]^{\varepsilon+1}} \\
&< [B(\lambda_1+1,\lambda_2)+o(1)] \left(\frac{v'(n_0)}{[v(n_0)]^{\varepsilon+1}} + \frac{1}{\varepsilon\,[v(n_0)]^{\varepsilon}}\right).
\end{aligned}$$

Similarly we obtain

$$I_4 < [B(\lambda_1,\lambda_2+1)+o(1)] \left(\frac{v'(n_0)}{[v(n_0)]^{\varepsilon+1}} + \frac{1}{\varepsilon\,[v(n_0)]^{\varepsilon}}\right).$$

Now

$$\begin{aligned}
\sqrt{\widetilde{I}} &= \sum_{n=n_0}^{\infty} \int_b^c \widetilde{F}(x,n)\,dx = \sum_{n=n_0}^{\infty} \int_b^c \frac{\widetilde{a}_n \widetilde{f}(x)}{u(x)+v(n)}dx \\
&= \sum_{n=n_0}^{\infty} \int_d^c \frac{[u(x)]^{\lambda_1-\frac{\varepsilon}{p}-1} u'(x)\,[v(n)]^{\lambda_2-\frac{\varepsilon}{q}-1} v'(n)}{u(x)+v(n)}dx = \left(t = \frac{u(x)}{v(n)}\right) \\
&= \sum_{n=n_0}^{\infty} [v(n)]^{-\varepsilon-1} v'(n) \int_{\frac{1}{v(n)}}^{\infty} \frac{t^{\lambda_1-\frac{\varepsilon}{p}-1}}{t+1}dt \\
&= \sum_{n=n_0}^{\infty} [v(n)]^{-\varepsilon-1} v'(n) \left(\int_0^{\infty} \frac{t^{\lambda_1-\frac{\varepsilon}{p}-1}}{t+1}dt - \int_0^{\frac{1}{v(n)}} \frac{t^{\lambda_1-\frac{\varepsilon}{p}-1}}{t+1}dt\right) \\
&= [B(\lambda_1,\lambda_2)+o(1)] \sum_{n=n_0}^{\infty} [v(n)]^{-\varepsilon-1} v'(n) - \sum_{n=n_0}^{\infty} [v(n)]^{-\varepsilon-1} v'(n) \int_0^{\frac{1}{v(n)}} \frac{t^{\lambda_1-\frac{\varepsilon}{p}-1}}{t+1}dt \\
&> [B(\lambda_1,\lambda_2)+o(1)] \left(\frac{v'(n_0)}{[v(n_0)]^{\varepsilon+1}} + \frac{1}{\varepsilon\,[v(n_0)]^{\varepsilon}}\right) - \sum_{n=n_0}^{\infty} [v(n)]^{-\varepsilon-1} v'(n) \int_0^{\frac{1}{v(n)}} t^{\lambda_1-\frac{\varepsilon}{p}-1}dt \\
&= [B(\lambda_1,\lambda_2)+o(1)] \left(\frac{v'(n_0)}{[v(n_0)]^{\varepsilon+1}} + \frac{1}{\varepsilon\,[v(n_0)]^{\varepsilon}}\right) - \sum_{n=n_0}^{\infty} \frac{[v(n)]^{-\frac{\varepsilon}{q}-\lambda_1-1} v'(n)}{\lambda_1-\frac{\varepsilon}{p}} \\
&> [B(\lambda_1,\lambda_2)+o(1)] \left(\frac{v'(n_0)}{[v(n_0)]^{\varepsilon+1}} + \frac{1}{\varepsilon\,[v(n_0)]^{\varepsilon}}\right) - \frac{[v(n_0)]^{-\frac{\varepsilon}{q}-\lambda_1-1} v'(n_0)}{\lambda_1-\frac{\varepsilon}{p}} \\
&\quad - \int_{n_0}^{\infty} \frac{[v(n)]^{-\frac{\varepsilon}{q}-\lambda_1-1} v'(n)}{\lambda_1-\frac{\varepsilon}{p}} \\
&= [B(\lambda_1,\lambda_2)+o(1)] \left(\frac{v'(n_0)}{[v(n_0)]^{\varepsilon+1}} + \frac{1}{\varepsilon\,[v(n_0)]^{\varepsilon}}\right) - O(1).
\end{aligned}$$

Substituting the above inequalities in (2.1) we get

$$\left\{[B(\lambda_1,\lambda_2)+o(1)]\left(\frac{v'(n_0)}{[v(n_0)]^{\varepsilon+1}}+\frac{1}{\varepsilon\,[v(n_0)]^{\varepsilon}}\right)-O(1)\right\}^2$$
$$< C\left(\frac{1}{\varepsilon}\right)^{\frac{1}{p}}\left(\frac{v'(n_0)}{[v(n_0)]^{1+\varepsilon}}+\frac{1}{\varepsilon\,[v(n_0)]^{\varepsilon}}\right)^{\frac{1}{q}}(B(\lambda_1+1,\lambda_2)+o(1))^{\lambda_1}\left(\frac{v'(n_0)}{[v(n_0)]^{\varepsilon+1}}+\frac{1}{\varepsilon\,[v(n_0)]^{\varepsilon}}\right)^{\lambda_1}$$
$$\times (B(\lambda_1,\lambda_2+1)+o(1))^{\lambda_2}\left(\frac{v'(n_0)}{[v(n_0)]^{\varepsilon+1}}+\frac{1}{\varepsilon\,[v(n_0)]^{\varepsilon}}\right)^{\lambda_2}. \tag{2.3}$$

Multiplying inequality (2.3) by $\varepsilon^2\left(\varepsilon=\varepsilon^{\lambda_1}\varepsilon^{\lambda_2}\right)$ and then let $\varepsilon\to 0^+$, we have

$$B^2(\lambda_1,\lambda_2)\le CB^{\lambda_1}(\lambda_1+1,\lambda_2)B^{\lambda_2}(\lambda_1,\lambda_2+1). \tag{2.4}$$

Using (1.10) we find

$$B^{\lambda_1}(\lambda_1+1,\lambda_2)=\lambda_1^{\lambda_1}B^{\lambda_1}(\lambda_1,\lambda_2), \tag{2.5}$$

and

$$B^{\lambda_2}(\lambda_1,\lambda_2+1)=\lambda_2^{\lambda_2}B^{\lambda_2}(\lambda_1,\lambda_2). \tag{2.6}$$

Substituting (2.5) and (2.6) in (2.4) we obtain the contradiction $C\ge K=\frac{B(\lambda_1,\lambda_2)}{\lambda_1^{\lambda_1}\lambda_2^{\lambda_2}}$. The Theorem is proved.

Some Applications

1. If we put $F(x,n)=\frac{a_nf(x)}{u(x)+v(n)}$ in (2.1), then we have the following form of half-discrete Hilbert's inequality

$$\left(\sum_{n=n_0}^{\infty}\int_b^c\frac{a_nf(x)}{u(x)+v(n)}dx\right)^2<\frac{B(\lambda_1,\lambda_2)}{\lambda_1^{\lambda_1}\lambda_2^{\lambda_2}}\left(\int_b^c\frac{[u(x)]^{p(1-\lambda_1)-1}}{[u'(x)]^{p-1}}f^p(x)dx\right)^{\frac{1}{p}}\left(\sum_{n=n_0}^{\infty}\frac{[v(n)]^{q(1-\lambda_2)-1}}{[v'(n)]^{q-1}}a_n^q\right)^{\frac{1}{q}}$$
$$\times\left(\sum_{n=n_0}^{\infty}\int_b^c\frac{u(x)f(x)a_n}{(u(x)+v(n))^2}dx\right)^{\lambda_1}\left(\sum_{n=n_0}^{\infty}\int_b^c\frac{v(n)f(x)a_n}{(u(x)+v(n))^2}dx\right)^{\lambda_2}, \tag{2.7}$$

Inequality (2.7) is a sharper form of (1.9). To see that, let us rewrite (2.7) in the following form

$$\left(\sum_{n=n_0}^{\infty}\int_b^c\frac{a_nf(x)}{u(x)+v(n)}dx\right)^2<\frac{B(\lambda_1,\lambda_2)}{\lambda_1^{\lambda_1}\lambda_2^{\lambda_2}}\left(\int_b^c\frac{[u(x)]^{p(1-\lambda_1)-1}}{[u'(x)]^{p-1}}f^p(x)dx\right)^{\frac{1}{p}}\left(\sum_{n=n_0}^{\infty}\frac{[v(n)]^{q(1-\lambda_2)-1}}{[v'(n)]^{q-1}}a_n^q\right)^{\frac{1}{q}}S^{\lambda_1}T^{\lambda_2},$$

where $S=\sum_{n=n_0}^{\infty}\int_b^c\frac{u(x)f(x)a_n}{(u(x)+v(n))^2}dx$ and $T=\sum_{n=n_0}^{\infty}\int_b^c\frac{v(n)f(x)a_n}{(u(x)+v(n))^2}dx$. Since we may write $\sum_{n=n_0}^{\infty}\int_b^c\frac{a_nf(x)}{u(x)+v(n)}dx=S+T$, dividing both sides of the last inequality by $\sum_{n=n_0}^{\infty}\int_b^c\frac{a_nf(x)}{u(x)+v(n)}dx$, we get

$$\sum_{n=n_0}^{\infty}\int_b^c \frac{a_n f(x)}{u(x)+v(n)}dx < B(\lambda_1,\lambda_2)\left(\int_b^c \frac{[u(x)]^{p(1-\lambda_1)-1}}{[u'(x)]^{p-1}} f^p(x)dx\right)^{\frac{1}{p}}\left(\sum_{n=n_0}^{\infty}\frac{[v(n)]^{q(1-\lambda_2)-1}}{[v'(n)]^{q-1}}a_n^q\right)^{\frac{1}{q}} \times \frac{\left(\frac{S}{\lambda_1}\right)^{\lambda_1}\left(\frac{T}{\lambda_2}\right)^{\lambda_2}}{S+T}. \tag{2.8}$$

Applying Young's inequality ($x^{\theta}y^{\mu} \le \theta x+\mu y,\ \theta+\mu=1$) to the product $\left(\frac{S}{\lambda_1}\right)^{\lambda_1}\left(\frac{T}{\lambda_2}\right)^{\lambda_2}$ with $x=\frac{S}{\lambda_1}$ and $y=\frac{T}{\lambda_2}$, we obtain

$$\frac{\left(\frac{S}{\lambda_1}\right)^{\lambda_1}\left(\frac{T}{\lambda_2}\right)^{\lambda_2}}{T+S} \le 1.$$

Therefore, inequality (2.7) is a sharper form of (1.9). In particular if we set $u(x)=x, x\in(0,\infty)$ and $v(n)=n, n\ge 1$ we obtain

$$\left(\sum_{n=1}^{\infty}\int_0^{\infty}\frac{a_n f(x)}{x+n}dx\right)^2 < \frac{B(\lambda_1,\lambda_2)}{\lambda_1^{\lambda_1}\lambda_2^{\lambda_2}}\left(\int_0^{\infty}x^{p(1-\lambda_1)-1}f^p(x)dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{q(1-\lambda_2)-1}a_n^q\right)^{\frac{1}{q}} \times\left(\sum_{n=1}^{\infty}\int_0^{\infty}\frac{xf(x)a_n}{(x+n)^2}dx\right)^{\lambda_1}\left(\sum_{n=1}^{\infty}\int_0^{\infty}\frac{nf(x)a_n}{(x+n)^2}dx\right)^{\lambda_2}. \tag{2.9}$$

If we put $p=q=2,\ \lambda_1=\frac{1}{2}=\lambda_2$ in (2.9), we obtain the following sharper form of the half-discrete Hilbert inequality (1.8)(for $\lambda=1$)

$$\left(\sum_{n=1}^{\infty}\int_0^{\infty}\frac{a_n f(x)}{x+n}dx\right)^2 < 2\pi\left(\int_0^{\infty}f^2(x)dx\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}a_n^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\int_0^{\infty}\frac{xf(x)a_n}{(x+n)^2}dx\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\int_0^{\infty}\frac{nf(x)a_n}{(x+n)^2}dx\right)^{\frac{1}{2}}.$$

2. If we put $F(x,n)=\frac{a_n f(x)}{u(x)+v(n)+\mu}$ ($\mu>0$) in (2.1), then we have

$$\left(\sum_{n=n_0}^{\infty}\int_b^c \frac{a_n f(x)}{u(x)+v(n)+\mu}dx\right)^2 < \frac{B(\lambda_1,\lambda_2)}{\lambda_1^{\lambda_1}\lambda_2^{\lambda_2}}\left(\int_b^c \frac{[u(x)]^{p(1-\lambda_1)-1}}{[u'(x)]^{p-1}}f^p(x)dx\right)^{\frac{1}{p}} \times\left(\sum_{n=n_0}^{\infty}\frac{[v(n)]^{q(1-\lambda_2)-1}}{[v'(n)]^{q-1}}a_n^q\right)^{\frac{1}{q}} \times\left(\sum_{n=n_0}^{\infty}\int_b^c\frac{u(x)f(x)a_n}{\left(u(x)+v(n)+\mu\right)^2}dx\right)^{\lambda_1} \times\left(\sum_{n=n_0}^{\infty}\int_b^c\frac{v(n)f(x)a_n}{\left(u(x)+v(n)+\mu\right)^2}dx\right)^{\lambda_2},$$

since

$$\sum_{n=n_0}^{\infty}\int_b^c\frac{a_n f(x)}{u(x)+v(n)+\mu}dx = \sum_{n=n_0}^{\infty}\int_b^c\frac{u(x)f(x)a_n}{\left(u(x)+v(n)+\mu\right)^2}dx+\sum_{n=n_0}^{\infty}\int_b^c\frac{v(n)f(x)a_n}{\left(u(x)+v(n)+\mu\right)^2}dx$$

$$
\begin{aligned}
&+\sum_{n=n_0}^{\infty}\int_b^c \frac{\mu f(x)a_n}{\left(u(x)+v(n)+\mu\right)^2}dx\\
&:= S_1+S_2+S_3,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\sum_{n=n_0}^{\infty}\int_b^c \frac{a_n f(x)}{u(x)+v(n)+\mu}dx &< B(\lambda_1,\lambda_2)\left(\int_b^c \frac{[u(x)]^{p(1-\lambda_1)-1}}{[u'(x)]^{p-1}} f^p(x)dx\right)^{\frac{1}{p}}\\
&\times\left(\sum_{n=n_0}^{\infty}\frac{[v(n)]^{q(1-\lambda_2)-1}}{[v'(n)]^{q-1}}a_n^q\right)^{\frac{1}{q}}\frac{\left(\frac{S_1}{\lambda_1}\right)^{\lambda_1}\left(\frac{S_2}{\lambda_2}\right)^{\lambda_2}}{S_1+S_2+S_3},
\end{aligned}
$$

applying Young's inequality we get $\frac{\left(\frac{S_1}{\lambda_1}\right)^{\lambda_1}\left(\frac{S_2}{\lambda_2}\right)^{\lambda_2}}{S_1+S_2+S_3} \le \frac{S_1+S_2}{S_1+S_2+S_3} < 1$, thus we arrive at

$$
\begin{aligned}
\sum_{n=n_0}^{\infty}\int_b^c \frac{a_n f(x)}{u(x)+v(n)+\mu}dx &< B(\lambda_1,\lambda_2)\left(\int_b^c \frac{[u(x)]^{p(1-\lambda_1)-1}}{[u'(x)]^{p-1}} f^p(x)dx\right)^{\frac{1}{p}}\\
&\times\left(\sum_{n=n_0}^{\infty}\frac{[v(n)]^{q(1-\lambda_2)-1}}{[v'(n)]^{q-1}}a_n^q\right)^{\frac{1}{q}}.
\end{aligned}
$$

3. Let $p=q=2$, $\lambda_1=\frac{1}{2}=\lambda_2$, $u(x)=x, x\in(0,\infty)$, $v(n)=n, n\ge 1, f(x)=\frac{1}{x+1}, a_n=\frac{1}{n}$, then by (2.1) we obtain

$$
\begin{aligned}
&\left[\sum_{n=1}^{\infty}\int_0^{\infty}F(x,n)\,dx\right]^2\\
&< 2\pi\,(\zeta(2))^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\int_0^{\infty}nx(x+1)F^2(x,n)dx\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\int_0^{\infty}n^2(x+1)F^2(x,n)dx\right)^{\frac{1}{2}}, \qquad (2.10)
\end{aligned}
$$

where $\zeta(\alpha)$ is the Riemann zeta function. In particular if we set $F(x,n)=\frac{c_n}{(x+1)^2}, F(x,n)=\frac{g(x)}{n^2}$, respectively in (3.1) then we get the following Carlson's type inequalities

$$
\sum_{n=1}^{\infty}c_n < \frac{\pi}{6^{\frac{1}{4}}}\left\{\sum_{n=1}^{\infty}nc_n^2\right\}^{\frac{1}{4}}\left\{\sum_{n=1}^{\infty}n^2c_n^2\right\}^{\frac{1}{4}},
$$

and

$$
\int_0^{\infty}g(x)dx < \left(\frac{2\pi\sqrt{\zeta(3)}}{\zeta(2)}\right)^{\frac{1}{2}}\left\{\int_0^{\infty}x(x+1)g^2(x)dx\right\}^{\frac{1}{4}}\left\{\int_0^{\infty}(x+1)g^2(x)dx\right\}^{\frac{1}{4}}.
$$

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