

OSCILLATORY BEHAVIOR OF SECOND ORDER UNSTABLE TYPE NEUTRAL DIFFERENCE EQUATIONS

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Abstract. This paper deals with the oscillatory behavior of all bounded/ unbounded solutions of second order neutral type difference equation of the form

$$\Delta(a_n(\Delta_c y_n + p y_{n-k}))^\alpha - g_n f(y_{\sigma(n)}) = 0,$$

where p is real, α is a ratio of odd positive integers, k is a positive integer and $\{\sigma(n)\}$ is a sequence of integers. Examples are provided to illustrate the results.

1. Introduction

Consider the difference equation of the form

$$\Delta a_n(\Delta_c y_n + p y_{n-k})^\alpha - g_n f(y_{\sigma(n)}) = 0 \quad (1)$$

where $n \in \mathbb{N}(n_0) = \{n_0, n_0 + 1, \dots\}$, n_0 a nonnegative integer, Δ is the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$, subject to the conditions:

- (c₁) p is a real number, k is a positive integer and α is a ratio of odd positive integers;
- (c₂) $\{a_n\}$ is a positive sequence such that $\sum_{n=n_0}^{\infty} \frac{1}{a_n^\alpha} = \infty$ and $\{g_n\}$ is a non negative sequence of real numbers and g_n is not identically zero for infinitely many values of n ;
- (c₃) $\{\sigma(n)\}$ is a sequence of integers such that $\sigma(n) \rightarrow \infty$ as $n \rightarrow \infty$;
- (c₄) $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function with $uf(u) > 0$ and $\frac{f(u)}{u^\alpha} \geq M > 0$ for $u/0$.

Let $\theta = \max\{k, \inf \sigma(n)\}$. By a solution of equation (1), we mean a real sequence $\{y_n\}$ defined for $n \geq n_0 - \theta$ and satisfying equation (1) for $n \in \mathbb{N}(n_0)$ and $\sup\{|y_s| : s \geq n\} > 0$ for $n \in \mathbb{N}(n_0)$. Such a solution $\{y_n\}$ is called oscillatory if for any $n_1 \in \mathbb{N}(n_0)$, there are integers $n_2, n_3 \geq n_1$ such that $y_{n_2} y_{n_3} \leq 0$ and is called nonoscillatory otherwise.

Equations of this type arise in a number of important applications such as problems in population dynamics when maturation and gestation are included, in cobweb models in economics where demand depends on current price but supply depends on the price

Received September 8, 2003; revised October 28, 2003.

2000 *Mathematics Subject Classification.* 39A10.

Key words and phrases. Neutral difference equation, bounded solution, unbounded solution, oscillation.

at an earlier time, and in electric networks containing lossless transmission lines. Hence it is important and useful to study the oscillatory properties of solution of equation (1).

In most of the papers [1, 2, 6, 7, 10, 11], the authors established conditions for the oscillation and nonoscillation of solutions of equation of type (1) with $\alpha = 1$ and treating the deviations are constant. In [1, 3, 5, 8, 9, 12], the authors consider the particular cases of equation (1) in the form

$$\Delta(a_n(\Delta y_n)^\alpha) - g_n f(y_{\alpha(n)}) = 0 \quad (2)$$

or

$$\Delta(a_n \Delta y_n) - g_n f(y_{n+1}) = 0 \quad (3)$$

and discussed the oscillatory and asymptotic behavior of solutions of equations (2) and (3).

Motivated by this observation in this paper our aim is to study the oscillatory behavior of solutions of equation (1) under the assumption that the deviation $\sigma(n) - n$ is not necessarily constant and may be unbounded. In Section 2, we obtain conditions for the oscillation of all bounded solutions of equation (1) under the assumption that $\{\sigma(n)\}$ is increasing such that $\sigma(n) \leq n$ and in Section 3, we establish conditions for all unbounded solution of equation (1) to be oscillatory under the assumption that $\{\sigma(n)\}$ in increasing such that $\sigma(n) > n$. Examples are inserted to illustrate our results.

2. Bounded Oscillation of Equation (1)

From Theorem 3.1 of [6] it is clear that it is not possible to find criteria for all the solutions of equation (1) to be oscillatory when $\{\sigma(n)\}$ is increasing with $\sigma(n) \leq n$. However in this section we establish conditions for the oscillation of all bounded solutions of equation (1).

Theorem 1. *With respect to the difference equation (1) assume that $-1 < p \leq 0$. If*

$$\limsup_{n \rightarrow \infty} \sum_{s=\sigma(n)}^n \left(\frac{1}{a_s} \sum_{t=s}^n g_t \right)^{\frac{1}{\alpha}} > \frac{1}{M^{\frac{1}{\alpha}}} \quad (4)$$

then every bounded solution of equation (1) is oscillatory.

Proof. Without loss of generality, we may assume that $\{y_n\}$ is an eventually positive solution of equation (1). Define

$$z_n = y_n + p y_{n-k}. \quad (5)$$

From equation (1), we have $\Delta(a_n(\Delta z_n)^\alpha) \leq 0$ for all large $n \in \mathbb{N}(n_0)$. If $a_n(\Delta z_n)^\alpha > 0$ eventually, then $\lim_{n \rightarrow \infty} z_n = \infty$, which contradicts the boundedness of $\{y_n\}$. Therefore $a_n \Delta z_n < 0$ since α is a ratio of odd positive integers and $\{a_n\}$ is eventually positive we may take $\Delta z_n < 0$ for $n \in \mathbb{N}(n_0)$. Now, we have two possibilities for $\{z_n\}$:

- (a) $z_n > 0$ for $n \geq n_1 \in \mathbb{N}(n_0)$;
- (b) $z_n < 0$ for $n \geq n_1$.

In case (a), equation (1) can be written as

$$\Delta(a_n(\Delta z_n)^\alpha) \geq M g_n y_{\sigma(n)}^\alpha.$$

Using (5), we obtain

$$\Delta(a_n(\Delta z_n)^\alpha) \geq M g_n z_{\sigma(n)}^\alpha. \tag{6}$$

Summing (6) from s to n yields

$$a_{n+1}(\Delta z_{n+1})^\alpha - a_s(\Delta z_s)^\alpha \geq M \sum_{t=s}^n q_t z_{\sigma(t)}^\alpha.$$

or

$$-\Delta z_s \geq z_{\sigma(n)} \left(\frac{M}{a_s} \sum_{t=s}^n q_t \right)^{\frac{1}{\alpha}}.$$

Summing the last inequality in s from $\sigma(n)$ to n , we see that

$$-z_{n+1} + z_{\sigma(n)} \geq M^{\frac{1}{\alpha}} z_{\sigma(n)} \sum_{s=\sigma(n)}^n \left(\frac{1}{a_s} \sum_{t=s}^n q_t \right)^{\frac{1}{\alpha}}.$$

Hence for $n \geq n_1$, we have,

$$z_{n+1} + z_{\sigma(n)} \left(M^{\frac{1}{\alpha}} \sum_{s=\sigma(n)}^n \left(\frac{1}{a_s} \sum_{t=s}^n q_t \right)^{\frac{1}{\alpha}} - 1 \right) \leq 0 \tag{7}$$

which contradiction (4). For the case (b), we obtain

$$y_n < -p y_{n-k} < (-p)^2 y_{n-2k} < \dots < (-p)^j y_{n-jk}$$

for $n \geq n_1 + jk$ and we are led to that $\lim_{n \rightarrow \infty} y_n = 0$. Hence $\lim_{n \rightarrow \infty} z_n = 0$. which is again a contradiction. This completes the proof of the theorem.

Remark 1. If $\alpha = 1$, $a_n \equiv 1$ and $\sigma(n) = n - l$, then Theorem 1 reduces to Theorem 4.1 of Lalli and Zhang [6].

To prove our next result, we need the following lemma.

Lemma 1. *Let $\{y_n\}$ be an eventually positive sequence and $z_n = y_n + p y_{n-k}$ be such that $\Delta z_n < 0$ for all $n \in \mathbb{N}(n_0)$. Then there is an integer $N \in \mathbb{N}(n_0)$ such that*

$$y_n \geq \frac{z_n}{1+p}, \quad n \geq N$$

if $-1 < p \leq 0$ and

$$y_n \geq \frac{z_{n+k}}{1+p}, \quad n \geq N$$

if $p > 0$.

Proof. Since $\{z_n\}$ is decreasing, from $z_n = y_n + py_{n-k}$, we may assume without loss of generality that, $\{y_n\}$ is also decreasing for $n \geq N \in \mathbb{N}(n_0)$, (see [4]). Hence for $-1 < p \leq 0$, we have

$$z_n = y_n + py_{n-k} \leq y_n + py_n, \quad n \geq N,$$

and therefore $y_n \geq z_{n/1+p}$, $n \geq N$. If $p > 0$, then we have

$$z_n \leq y_{n-k} + py_{n-k}, \quad n \geq N,$$

and we have

$$y_n \geq \frac{z_{n+k}}{1+p}, \quad n \geq N.$$

This completes the proof.

Theorem 2. *With respect to the difference equation (1) assume that $-1 < p \leq 0$. If*

$$\limsup_{n \rightarrow \infty} \sum_{s=\sigma(n)}^n \left(\frac{1}{a_s} \sum_{t=s}^n q_t \right)^{\frac{1}{\alpha}} > \frac{1+p}{M^{\frac{1}{\alpha}}} \quad (8)$$

then every bounded solution of equation (1) is oscillatory.

Proof. Assume that $\{y_n\}$ is an eventually positive bounded solution of equation (1). We can proceed exactly as in the proof of Theorem 1, to see that there are two possibilities for $\{z_n\}$:

- (a) $z_n > 0$, $\Delta z_n < 0$, $\Delta(a_n(\Delta z_n)^\alpha) \geq 0$ for $n \geq n_1 \in \mathbb{N}(n_0)$;
- (b) $z_n < 0$, $\Delta z_n < 0$, $\Delta(a_n(\Delta z_n)^\alpha) \geq 0$ for $n \geq n_1$.

In case (a), from equation (1), we have

$$\Delta(a_n(\Delta z_n)^\alpha) \geq M g_n y_{\sigma(n)}^\alpha, \quad n \geq n_1.$$

Now, using Lemma 1, we obtain

$$\Delta(a_n(\Delta z_n)^\alpha) \geq \frac{M g_n}{(1+p)^\alpha} z_{\sigma(n)}^\alpha, \quad n \geq n_1.$$

Proceeding now exactly as in the proof of Theorem 1, we obtain

$$z_{n+1} + z_{\sigma(n)} \left(\frac{M^{\frac{1}{\alpha}}}{1+p} \sum_{s=\sigma(n)}^n \left(\frac{1}{a_s} \sum_{t=s}^n q_t \right)^{\frac{1}{\alpha}} - 1 \right) \leq 0$$

which contradicts condition (8). The case (b) can be handled similarly as in the proof of Theorem 1. The proof is now complete.

Remark 2. It is easy to see that the results obtained in Theorem 2 is better than that of Theorem 1 since the constant p includes in condition (8). Further Theorem 2 improves and generalize Theorem 4.1 of Lalli and Zhang [6].

Remark 3. Theorem 2 is true for $p = 0$. This is due to Wong and Agarwal [12]. Further, if $a_n = 1$, $\sigma(n) = n - 1$ and $a_n \equiv 1$, then Theorem 2 reduces to Theorem 1 of Thandapani, Arul and Raja [11].

Example 1. Consider the difference equation

$$\Delta((\Delta(y_n - \frac{1}{2}y_{n-k}))^\alpha) = cy_{n-l}, \quad n \in \mathbb{N}(n_0). \tag{9}$$

Here $p = -\frac{1}{2}$, $a_n = 1$, $q_n = c$, $\sigma(n) = n - l$ and the condition (8) becomes

$$\sum_{s=1}^{l+1} c^{\frac{1}{\alpha}} s^{\frac{1}{\alpha}} > \frac{1}{2}. \tag{10}$$

Thus, by Theorem 2 if (10) is satisfied then all bounded solutions of equation (9) are oscillatory. This is particularly so when $\alpha = 3$, $l = 2$ and $c > 0.034$.

Theorem 3. *With respect to the difference equation (1) assume that $p = -1$. If*

$$\limsup_{n \rightarrow \infty} \sum_{s=\sigma(n)}^n \left(\frac{1}{a_s} \sum_{t=s}^n q_t \right)^{\frac{1}{\alpha}} > \frac{1}{M^{\frac{1}{\alpha}}} \tag{11}$$

then every bounded solution of equation (1) is oscillatory.

Proof. Assume that $\{y_n\}$ is an eventually positive solution of equation (1). Proceeding as in the proof of Theorem 1, we see that there are two possibilities for $\{z_n\}$:

- (a) $z_n > 0$, $\Delta z_n < 0$, $\Delta(z_n(\Delta z_n)^\alpha) \geq 0$, $n \geq n_1 \in \mathbb{N}(n_0)$;
- (b) $z_n < 0$, $\Delta z_n < 0$, $\Delta(z_n(\Delta z_n)^\alpha) \geq 0$, $n \geq n_1$

In case (a), we are led to (7) which contradicts condition (11). In the case (b), we have $\lim_{n \rightarrow \infty} z_n = -\beta$ where $\beta > 0$ is a finite number. So there is an integer $n_2 \in \mathbb{N}(n_0)$ such that $-\beta < z_n < -\frac{\beta}{2}$ for $n \geq n_2 \geq n_1$. Hence

$$-\beta < y_n - y_{n-k} < -\frac{\beta}{2}, \quad n \geq n_2.$$

Then,

$$y_n - \frac{\beta}{2} + y_{n+k} < -2\frac{\beta}{2} + y_{n-2k} < \dots < -j\frac{\beta}{2} + y_{n-jk}$$

for $n \geq n_2 + jk$. Choose a sequence $\{n_j\}$ such that $n_i = n_2 + jk$. Then

$$y_{n_2+jk} < j\frac{\beta}{2} + y_{n_2}$$

and therefore $\lim_{j \rightarrow \infty} y_{n_j} = -\infty$. This is a contradiction to the boundedness of $\{y_n\}$. This completes the proof.

In the following, we present another oscillatory criterion for equation (1) where $p \leq -1$.

Theorem 4. *With respect to the difference equation (1) assume that $p \leq -1$. If*

$$\sum_{n=n_0}^{\infty} \left(\frac{1}{a_n} \sum_{s=n}^{\infty} q_s \right)^{\frac{1}{\alpha}} = \infty \quad (12)$$

then every bounded solution of equation (1) is oscillatory.

Proof. Assume without loss of generality that $\{y_n\}$ is a bounded positive solution of equation (1) and z_n is defined by (5). Thus there are two possibilities for $\{z_n\}$:

- (a) $z_n > 0$, $\Delta z_n < 0$ and $\Delta(a_n(\Delta z_n)^\alpha) \geq 0$, $n \geq n_1 \in \mathbb{N}(n_0)$;
- (b) $z_n < 0$, $\Delta z_n < 0$ and $\Delta(a_n(\Delta z_n)^\alpha) \geq 0$, $n \geq n_1$.

In case (a), we have $y_n > -py_{n-k}$ for $n \geq n_1$ and there exists a constant $\delta > 0$ such that $y_n \geq \delta$ for $n \geq n_1$. Hence from equation (1), we have

$$\Delta(a_n(\Delta z_n)^\alpha) \geq M\delta^\alpha q_n, \quad n \geq n_1.$$

In case (b), there exists a finite number $\beta > 0$ such that $\lim_{n \rightarrow \infty} z_n = -\beta$. Then there exists an integer $n_2 \geq n_1$ such that $-\beta < z_n < -\frac{\beta}{2}$ for $n \geq n_2$, that is,

$$-\beta < y_n + py_{n-k} < -\frac{\beta}{2}$$

for $n \geq n_2$. Hence $y_{n-k} > \frac{\beta}{2p}$, $n \geq n_2$. Then there exists an integer $n_3 \geq n_2$ such that $y_{\sigma(n)} > \frac{\beta}{2}$, $n \geq n_3$. From equation (1), we have

$$\Delta(a_n(\Delta(z_n)^\alpha)) \geq M\left(\frac{P}{2}\right)^\alpha q_n, \quad n \geq n_3.$$

Thus in both the cases we are led to the inequality

$$\Delta(a_n(\Delta(z_n)^\alpha)) \geq Bq_n, \quad n \geq n_3 \quad (13)$$

where B is a constant. Summing (13) from n to N for $N > n \geq n_3$, we have

$$a_{N+1}(\Delta z_{N+1})^\alpha - a_n(\Delta z_n)^\alpha \geq B \sum_{s=n}^N q_s, \quad n_3 \leq n < N.$$

Hence

$$-a_n(\Delta z_n)^\alpha \geq B \sum_{s=n}^N q_s, \quad n_3 \leq n < N$$

which implies that $\sum_{s=n_0}^\infty q_s < \infty$ and so

$$-\Delta z_n \geq \left(\frac{B}{a_n} \sum_{s=n}^\infty q_s \right)^{\frac{1}{\alpha}}.$$

Summing the last inequality from n to $N - 1$ for $N - 1 > n$, we have

$$z_n \geq z_N + B^{\frac{1}{\alpha}} \sum_{s=n}^{N-1} \left(\frac{1}{a_s} \sum_{t=s}^\infty q_t \right)^{\frac{1}{\alpha}}$$

or

$$z_{n_0} \geq B^{\frac{1}{\alpha}} \sum_{s=n_0}^\infty \left(\frac{1}{a_s} \sum_{t=s}^\infty q_t \right)^{\frac{1}{\alpha}}$$

which contradicts condition (12). This completes the proof.

Example 2. Consider the difference equation.

$$\Delta((\Delta(y_n - 2y_{n-k}))^3) = \frac{128}{(n-4)^3} y_{n-5}^3, \quad n \geq 5. \tag{14}$$

It is easy to see that all assumptions of Theorem 4 are satisfied. Therefore, every bounded solution of equation (14) is oscillatory.

Remark 4. If $\alpha = 1$, $a_n \equiv 1$ and $\sigma(n) = n - 1$, then Theorem 4 reduces to Theorem 4.2 of Lalli and Zhang [6].

Our final result in this section deals with the case $p > 0$.

Theorem 5. *With respect to the difference equation (1), assume that $p > 0$ and $\sigma(n) = n - l$ where l is a positive integer such that $l \geq k + 1$. If*

$$\limsup_{n \rightarrow \infty} \sum_{s=n+k-l}^n \left(\frac{1}{a_s} \sum_{t=s}^n q_t \right)^{\frac{1}{\alpha}} > \frac{1+p}{M^{\frac{1}{\alpha}}}$$

then all bounded solutions of equation (1) are oscillatory.

Proof. The proof is similar to that of Theorem 2 and hence the details are omitted.

Example 3. Consider the difference equation

$$\Delta((\Delta(y_n + py_{n+2}))^3) + 8(1+p)^3 y_{n-4}^3 (1 + |y_{n-4}|) = 0. \tag{15}$$

All conditions of Theorem 5 are satisfied and hence all bounded solutions of equation (15) are oscillatory. In fact; $\{y_n\} = \{(-1)^n\}$ is such a solution of equation (15).

3. Unbounded Oscillation of Equation (1)

In this section we present sufficient conditions for the oscillation of all unbounded solution of equation (1) when $\sigma(n) = n + l$, where l is a positive integer.

Theorem 6. *With respect to the difference equation (1) assume that*

$$0 \leq p < 1 \quad (16)$$

and

$$\limsup_{n \rightarrow \infty} \sum_{s=n}^{n+l-1} \left(\frac{1}{a_s} \sum_{t=n}^{s-1} q_t \right)^{\frac{1}{\alpha}} > M^{\frac{1}{\alpha}}(1-p) \quad (17)$$

hold. Then every unbounded solution of equation (1) oscillates.

Proof. Assume for the sake of contradiction that equation (1) has an eventually positive unbounded solution $\{y_n\}$. Then $y_n > 0$ for $n \geq n_1 \in \mathbb{N}(n_0)$. Let z_n be defined by (5). Then $z_n > 0$ for $n \geq n_1$ and $\{z_n\}$ is unbounded and

$$\Delta(a_n(\Delta z_n)^\alpha) \geq 0.$$

Thus $\{\Delta z_n\}$ is of constant sign and $\Delta z_n > 0$ for all $n \geq n_2 \in \mathbb{N}(n_1)$ since $\{z_n\}$ is unbounded. From (5) and inview of $\{z_n\}$ is increasing we have

$$y_{n+l} \geq (1-p)z_{n+l} \quad (18)$$

for $n \geq n_2$. From equation (1) and (18), we obtain

$$\Delta(a_n(\Delta z_n)^\alpha) \geq q_n M(1-p)^\alpha z_{n+l}^\alpha, \quad n \geq n_2. \quad (19)$$

Now summing (19) from n to $s-1$, we have

$$a_s(\Delta z_s)^\alpha - a_n(\Delta z_n)^\alpha \geq M(1-p)^\alpha \sum_{t=n}^{s-1} q_t z_{s+l}^\alpha \geq M(1-p)^\alpha z_{n+l}^\alpha \sum_{t=n}^{s-1} q_t$$

or

$$\Delta z_s \geq M^{\frac{1}{\alpha}}(1-p)z_{n+l} \left(\frac{1}{a_s} \sum_{t=n}^{s-1} q_t \right)^{\frac{1}{\alpha}}.$$

Again summing the last inequality in s from n to $n+l-1$, we obtain

$$z_{n+l} - z_n \geq M^{\frac{1}{\alpha}}(1-p)z_{n+l} \sum_{s=n}^{n+l-1} \left(\frac{1}{a_s} \sum_{t=n}^{s-1} q_t \right)^{\frac{1}{\alpha}}. \quad (20)$$

Hence

$$z_{n+l}(M^{\frac{1}{\alpha}}(1-p) \sum_{s=n}^{n+l-1} \left(\frac{1}{a_s} \sum_{t=n}^{s-1} q_t \right)^{\frac{1}{\alpha}} - 1) \leq 0$$

which contradicts condition (17). This completes the proof of the theorem.

Remark 5. For difference equation without neutral term ($p = 0$) Theorem 6 provides the result of Wong and Agarwal [12].

Example 4. Consider the following difference equation

$$\Delta \left(\frac{1}{(2n+3)^3} \left(\Delta \left(y_n + \frac{1}{2}y_{n+1} \right) \right)^3 \right) = \frac{1}{4(n-3)^3} y_{n+3}^3, \quad n \geq 4. \quad (21)$$

All conditions of Theorem 6 are satisfied and hence every unbounded solution of equation (21) oscillates. In fact $\{y_n\} = \{n(-1)^n\}$ is such a solution of equation (21).

Next we consider the case $\alpha = 1$ and $a_n \equiv 1$ in equation (1) and discuss the oscillation of all unbounded solutions of equation (1) by relaxing the condition (16).

Theorem 7. *With respect to the difference equation (1) assume that $p > 0$. If*

$$\limsup_{n \rightarrow \infty} \sum_{s=n}^{n+l-1} (n+l-s)q_s^* > \frac{1+p}{M} \quad (22)$$

where $q_n^* = \min\{q_n, q_{n-k}\}$, then every unbounded solution of equation (1) is oscillatory.

Proof. Assume without loss of generality that equation (1) has an eventually positive unbounded solution $\{y_n\}$. Then $z_n > 0$ for $n \geq n_1 \in \mathbb{N}(n_0 + k)$, $\{z_n\}$ is unbounded and from equation (1), we have

$$\Delta^2 z_n \geq Mq_n y_{n+l}.$$

Thus $\Delta^2 z_n \geq 0$ and this implies that Δz_n is of constant sign. But if we take $\Delta z_n < 0$, then $\{z_n\}$ would be bounded. Therefore $\Delta z_n > 0$ for $n \geq n_2 \in \mathbb{N}(n_1)$. Let

$$x_n = z_n + pz_{n-k}$$

then $x_n > 0$ and $\Delta x_n > 0$ for all large n and further

$$\begin{aligned} \Delta^2 x_n &\geq \Delta^2 z_n + p\Delta^2 z_{n-k} \\ &\geq Mq_n y_{n+l} + pMq_{n-k} y_{n+l-k} \\ &\geq Mq_n^* (y_{n+l} + py_{n+l-k}) \\ &\geq Mq_n^* z_{n+l} \\ &\geq \frac{Mq_n^*}{(1+p)} x_{n+l}. \end{aligned}$$

Hence $\{x_n\}$ is a positive solution of the inequality

$$\Delta^2 x_n - \frac{Mq_n^*}{1+p} x_{n+l} \geq 0 \quad (23)$$

for $n \geq n_2$. Summing (23) from n to $j-1$, we have

$$\Delta x_j - \Delta x_n \geq \frac{M}{1+p} \sum_{s=n}^{j-1} q_s^* x_{s+l}.$$

Now summing in j from n to $n+l-1$, we are led to

$$x_{n+l} - x_n - \Delta x_n(l) \geq \frac{M}{1+p} \sum_{s=n}^{n+l-1} (n+l-s) q_s^* x_{s+l}.$$

Consequently, using the monotonicity of $\{x_n\}$, we obtain

$$x_{n+l} \left[\frac{M}{1+p} \sum_{s=n}^{n+l-1} q_s^* (n+l-s) - 1 \right] \leq 0$$

which contradicts condition (22). The proof is now complete.

In our next theorem we consider the case when the condition (22) is not satisfied.

Theorem 8. *With respect to the difference equation (1) assume that $p > 0$. Further assume that there exists a sequence $\{b_n\}$ and a positive integer such that*

$$b_n > 0, \quad \Delta b_n \leq 0, \quad n \in \mathbb{N}(n_0) \quad (24)$$

$$l \geq 2m \quad (25)$$

and

$$q_n^* \geq \frac{1+p}{M} b_{n+1} b_{n+m} \geq 0 \quad (26)$$

where q_n^* is the same as defined in Theorem 8. If the difference inequality

$$\Delta u_n - b_{n+m} u_{n+m} \geq 0 \quad (27)$$

has no eventually positive solution then all unbounded solutions of equation (1) are oscillatory.

Proof. Assume that $y_n > 0$ is an unbounded solution of equation (1). Let z_n and x_n be the same as defined in Theorem 8. Then proceeding exactly as in the proof of Theorem 8, we obtain (23). We put

$$d_n = \Delta x_n + b_n x_{n+m}$$

then $\{d_n\}$ is positive and

$$\Delta d_n - b_{n+1}d_{n+m} = \Delta^2 x_n + \Delta b_n x_{n+m} - b_{n+1}b_{n+m}x_{n+2m}.$$

Hence, in view of (23)-(26) one gets for all large n .

$$\begin{aligned} \Delta d_n - \frac{\Delta b_n}{b_n}d_n - b_{n+1}d_{n+m} &\geq \Delta^2 x_n - \frac{\Delta b_n}{b_n}\Delta z_n - b_{n+1}b_{n+m}x_{n+2m} \\ &\geq \Delta^2 x_n - b_{n+1}b_{n+m}x_{n+2m} \\ &\geq \Delta^2 x_n - \frac{Mq_n^*}{1+p}x_{n+l} \geq 0. \end{aligned}$$

Setting

$$d_n = b_n v_n$$

we conclude that $\{v_n\}$ is a positive solutions of (27). This contradiction completes the proof.

Corollary 1. *With respect to the difference equation (1) assume that $p > 0$. Moreover assume that conditions (23)-(26) hold.*

If

$$\limsup_{n \rightarrow \infty} \left[\frac{1}{m} \sum_{s=n}^{n+m} b_{s+m} \right] > \frac{m^m}{(m+1)^{m+1}} \tag{28}$$

then equation (1) does not allow unbounded non-oscillatory solution.

Proof. It is known that [5] condition (28) is sufficient for (27) to have no eventually positive solutions. Hence the assertion follows from Theorem 8.

We conclude this paper with the following remarks.

Remark 6. Theorem 8 permits to apply any sufficient condition for absence of eventually positive solution of (27) to obtain unbounded oscillation criteria for equation (1) provided that conditions (23)-(26) hold.

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