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NEW INEQUALITIES FOR CO-ORDINATED CONVEX FUNCTIONS VIA RIEMANN-LIOUVILLE FRACTIONAL CALCULUS

MARCELA V. MIHAI

Abstract. We provide some new Hermite-Hadamard type inequalities for co-ordinated convex functions, via Riemann-Liouville fractional integration.

1. Introduction

The Hermite-Hadamard inequality states that if a function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is convex, then one has

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $a, b \in I$ with $a < b$. Both inequalities hold in reversed direction if f is concave.

In recent years many researchers have returned to Hermite-Hadamard inequality and found many variations and generalizations of it for various types of convexity. Some of this research are related to functions convex on the co-ordinates (see, for instance, [1], [3], [4], [5], [6], [7], and the references therein).

Definition 1 ([3]). Let us consider the bidimensional interval $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b, c < d$. A function $f : \Delta \to \mathbb{R}$ will be called convex on the co-ordinates if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are convex where defined for all $y \in [c,d]$ and $x \in [a,b]$. Recall mapping $f : \Delta \to \mathbb{R}$ is convex on the co-ordinates on Δ if the following inequality holds,

$$\begin{aligned} f\left(tx+(1-t)\,y, su+(1-s)\,w\right) \leq\ & tsf(x,u)+t\,(1-s)\,f(x,w) \\ & +s\,(1-t)\,f(y,u)+(1-t)\,(1-s)\,f(y,w) \end{aligned}$$

for all $(x,u), (y,w) \in \Delta$ and $t, s \in [0,1]$.

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In [6], Dragomir established the following inequalities of Hadamard's type for co-ordinated convex functions on a rectangle from the plane $\mathbb{R}^2$.

Theorem 1. *Suppose that $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ is a convex on the co-ordinates on Δ. Then one has the inequalities:*

$$\begin{aligned} f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)\mathrm{d}x+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)\mathrm{d}y\right] \\ &\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,\mathrm{d}x\mathrm{d}y \\ &\le \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,\mathrm{d}x+\frac{1}{b-a}\int_a^b f(x,d)\,\mathrm{d}x\right. \\ &\quad \left.+\frac{1}{d-c}\int_c^d f(a,y)\,\mathrm{d}y+\frac{1}{d-c}\int_c^d f(b,y)\,\mathrm{d}y\right] \\ &\le \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned}$$

The above inequalities are sharp.

The purpose of our paper is to establish, via the Riemann-Liouville fractional calculus, some Hermite-Hadamard type inequalities for co-ordinated convex functions, via Riemann-Liouville fractional integration.

Let $f\in L^1[a,b]$, where $a\ge 0$. The Riemann-Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$, of order $\alpha>0$, are defined by

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\mathrm{d}t,\ \text{for } x>a,$$

and

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)\mathrm{d}t,\ \text{for } x<b,$$

respectively. Here, $\Gamma(\alpha)=\int_0^{\infty}e^{-t}t^{\alpha-1}\mathrm{d}t$ is the Gamma function. We also make the convention

$$J_{a+}^{0}f(x)=J_{b-}^{0}f(x)=f(x).$$

For details about the Riemann-Liouville fractional integrals see [2].

2. A lemma

We assume throughout the present paper that $\Delta=[a,b]\times[c,d]$ in $[0,\infty)^2$ and $f:\Delta\to\mathbb{R}$ is a differentiable mapping on Δ and $\frac{\partial^2 f}{\partial s\partial t}\in L^1(\Delta)$, where $\alpha,\beta>0$. Before stating the results we establish the notation.

We define the *cumulative to the left* (α,β)*–gap* by

$$\begin{aligned}\mathcal{L}_{\Delta}\left(\alpha,\beta\right) &= f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{2^{\alpha-1}\Gamma\left(\alpha+1\right)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-}f\left(a,\frac{c+d}{2}\right)+J^{\alpha}_{b-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]\\ &-\frac{2^{\beta-1}\Gamma\left(\beta+1\right)}{(d-c)^{\beta}}\left[J^{\beta}_{\frac{c+d}{2}-}f\left(\frac{a+b}{2},c\right)+J^{\beta}_{d-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]\\ &+\frac{2^{\alpha+\beta-2}\Gamma\left(\alpha+1\right)\Gamma\left(\beta+1\right)}{(b-a)^{\alpha}(d-c)^{\beta}}\left[J^{\alpha,\beta}_{\frac{a+b}{2}-,\frac{c+d}{2}-}f\left(a,c\right)+J^{\alpha,\beta}_{b-,d-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.\\ &\left.+J^{\alpha,\beta}_{\frac{a+b}{2}-,d-}f\left(a,\frac{c+d}{2}\right)+J^{\alpha,\beta}_{b-,\frac{c+d}{2}-}f\left(\frac{a+b}{2},c\right)\right],\end{aligned}$$

where

$$J^{\alpha,\beta}_{b-,d-}f\left(x,y\right)=\frac{1}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)}\int_x^b\int_y^d (u-x)^{\alpha-1}\left(v-y\right)^{\beta-1}f\left(u,v\right)\mathrm{d}v\mathrm{d}u,\, x<b,\, y<d,$$

is Riemann-Liouville integral and Γ is the Euler Gamma function.

Remark 1. The particular case $\alpha=1$ and $\beta=1$ gives

$$\begin{aligned}\mathcal{L}_{\Delta}\left(1,1\right) &= f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{b-a}\int_a^b f\left(u,\frac{c+d}{2}\right)\mathrm{d}u-\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},v\right)\mathrm{d}v\\ &+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f\left(u,v\right)\mathrm{d}v\mathrm{d}u.\end{aligned}$$

The right hand side term has its origins in the inequalities of the Theorem 1.

In order to prove our main results we need the following lemma.

Lemma 1. *It holds*

$$\begin{aligned}\mathcal{L}_{\Delta}\left(\alpha,\beta\right) &= \frac{(b-a)(d-c)}{16}\\ &\times\left[\int_0^1\int_0^1 t^{\alpha}s^{\beta}\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)\,a,s\frac{c+d}{2}+(1-s)\,c\right)\mathrm{d}s\mathrm{d}t\right.\\ &+\int_0^1\int_0^1\left(t^{\alpha}-1\right)\left(s^{\beta}-1\right)\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\,\frac{a+b}{2},sd+(1-s)\,\frac{c+d}{2}\right)\mathrm{d}s\mathrm{d}t\\ &+\int_0^1\int_0^1 t^{\alpha}\left(s^{\beta}-1\right)\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)\,a,sd+(1-s)\,\frac{c+d}{2}\right)\mathrm{d}s\mathrm{d}t\\ &\left.+\int_0^1\int_0^1\left(t^{\alpha}-1\right)s^{\beta}\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\,\frac{a+b}{2},s\frac{c+d}{2}+(1-s)\,c\right)\mathrm{d}s\mathrm{d}t\right],\end{aligned}$$

for all $t,s\in[0,1]$.

Proof. Calculate the four integrals by parts and change of variables $u=t\frac{a+b}{2}+(1-t)\,a$, $v=s\frac{c+d}{2}+(1-s)\,c$ and similar such

$$I_1=\int_0^1\int_0^1 t^{\alpha}s^{\beta}\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)\,a,s\frac{c+d}{2}+(1-s)\,c\right)\mathrm{d}s\mathrm{d}t$$

$$
\begin{aligned}
&= \int_0^1 t^{\alpha} \left[\int_0^1 s^{\beta} \frac{\partial^2 f}{\partial t \partial s} \left(t\frac{a+b}{2} + (1-t)\, a, s\frac{c+d}{2} + (1-s)\, c \right) \mathrm{d}s \right] \mathrm{d}t \\
&= \frac{2}{d-c} \int_0^1 t^{\alpha} \frac{\partial f}{\partial t} \left(t\frac{a+b}{2} + (1-t)\, a, \frac{c+d}{2} \right) \mathrm{d}t \\
&\quad - \frac{2\beta}{d-c} \int_0^1 s^{\beta-1} \left[\int_0^1 t^{\alpha} \frac{\partial f}{\partial t} \left(t\frac{a+b}{2} + (1-t)\, a, s\frac{c+d}{2} + (1-s)\, c \right) \mathrm{d}t \right] \mathrm{d}s \\
&= \frac{4}{(b-a)(d-c)} f\left(\frac{a+b}{2}, \frac{c+d}{2} \right) \\
&\quad - \frac{4\alpha}{(b-a)(d-c)} \int_0^1 t^{\alpha-1} f\left(t\frac{a+b}{2} + (1-t)\, a, \frac{c+d}{2} \right) \mathrm{d}t \\
&\quad - \frac{4\beta}{(b-a)(d-c)} \int_0^1 s^{\beta-1} f\left(\frac{a+b}{2}, s\frac{c+d}{2} + (1-s)\, c \right) \mathrm{d}s \\
&\quad - \frac{4\alpha\beta}{(b-a)(d-c)} \int_0^1 t^{\alpha-1} s^{\beta-1} f\left(t\frac{a+b}{2} + (1-t)\, a, s\frac{c+d}{2} + (1-s)\, c \right) \mathrm{d}s\mathrm{d}t \\
&= \frac{4}{(b-a)(d-c)} f\left(\frac{a+b}{2}, \frac{c+d}{2} \right) \\
&\quad - \frac{2^{\alpha+2}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}(d-c)} \cdot \frac{1}{\Gamma(\alpha)} \int_a^{\frac{a+b}{2}} (u-a)^{\alpha-1} f\left(u, \frac{c+d}{2} \right) \mathrm{d}u \\
&\quad - \frac{2^{\beta+2}\Gamma(\beta+1)}{(b-a)(d-c)^{\beta+1}} \cdot \frac{1}{\Gamma(\beta)} \int_c^{\frac{c+d}{2}} (v-c)^{\beta-1} f\left(\frac{a+b}{2}, v \right) \mathrm{d}v \\
&\quad + \frac{2^{\alpha+\beta+2}\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{\alpha+1}(d-c)^{\beta+1}} \cdot \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} (u-a)^{\alpha-1} (v-c)^{\beta-1} f(u,v) \mathrm{d}u\mathrm{d}v \\
&= \frac{4}{(b-a)(d-c)} f\left(\frac{a+b}{2}, \frac{c+d}{2} \right) \\
&\quad - \frac{2^{\alpha+2}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}(d-c)} J^{\alpha}_{\frac{a+b}{2}-} f\left(a, \frac{c+d}{2} \right) - \frac{2^{\beta+2}\Gamma(\beta+1)}{(b-a)(d-c)^{\beta+1}} J^{\beta}_{\frac{c+d}{2}-} f\left(\frac{a+b}{2}, c \right) \\
&\quad + \frac{2^{\alpha+\beta+2}\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{\alpha+1}(d-c)^{\beta+1}} J^{\alpha,\beta}_{\frac{a+b}{2}-, \frac{c+d}{2}-} f\,(a,c).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_2 &= \int_0^1 \int_0^1 \left(t^{\alpha} - 1 \right) \left(s^{\beta} - 1 \right) \frac{\partial^2 f}{\partial t \partial s} \left(tb + (1-t)\, \frac{a+b}{2}, sd + (1-s)\, \frac{c+d}{2} \right) \mathrm{d}s\mathrm{d}t \\
&= \frac{4}{(b-a)(d-c)} f\left(\frac{a+b}{2}, \frac{c+d}{2} \right) \\
&\quad - \frac{2^{\alpha+2}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}(d-c)} J^{\alpha}_{b-} f\left(\frac{a+b}{2}, \frac{c+d}{2} \right) - \frac{2^{\beta+2}\Gamma(\beta+1)}{(b-a)(d-c)^{\beta+1}} J^{\beta}_{d-} f\left(\frac{a+b}{2}, \frac{c+d}{2} \right) \\
&\quad + \frac{2^{\alpha+\beta+2}\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{\alpha+1}(d-c)^{\beta+1}} J^{\alpha,\beta}_{b-,d-} f\left(\frac{a+b}{2}, \frac{c+d}{2} \right), \\
I_3 &= \int_0^1 \int_0^1 t^{\alpha} \left(s^{\beta} - 1 \right) \frac{\partial^2 f}{\partial t \partial s} \left(t\frac{a+b}{2} + (1-t)\, a, sd + (1-s)\, \frac{c+d}{2} \right) \mathrm{d}s\mathrm{d}t
\end{aligned}
$$

$$= \frac{4}{(b-a)(d-c)} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{2^{\alpha+2}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}(d-c)} J^{\alpha}_{\frac{a+b}{2}-} f\left(a, \frac{c+d}{2}\right)$$
$$- \frac{2^{\beta+2}\Gamma(\beta+1)}{(b-a)(d-c)^{\beta+1}} J^{\beta}_{d-} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + \frac{2^{\alpha+\beta+2}\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{\alpha+1}(d-c)^{\beta+1}} J^{\alpha,\beta}_{\frac{a+b}{2}-,d-} f\left(a, \frac{c+d}{2}\right)$$

and

$$I_4 = \int_0^1 \int_0^1 \left(t^{\alpha}-1\right) s^{\beta} \frac{\partial^2 f}{\partial t \partial s}\left(tb+(1-t)\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)c\right) \mathrm{d}s\mathrm{d}t$$
$$= \frac{4}{(b-a)(d-c)} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{2^{\alpha+2}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}(d-c)} J^{\alpha}_{b-} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)$$
$$- \frac{2^{\beta+2}\Gamma(\beta+1)}{(b-a)(d-c)^{\beta+1}} J^{\beta}_{\frac{c+d}{2}-} f\left(\frac{a+b}{2}, c\right) + \frac{2^{\alpha+\beta+2}\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{\alpha+1}(d-c)^{\beta+1}} J^{\alpha,\beta}_{b-,\frac{c+d}{2}-} f\left(\frac{a+b}{2}, c\right).$$

Multiplying the sum of I_1, I_2, I_3, I_4 with $\frac{(b-a)(d-c)}{16}$ we get $\mathcal{L}_{\Delta}(\alpha,\beta)$ and the proof is complete. □

3. Inequalities of Hermite-Hadamard type

We are now in a position to state and prove the following:

Theorem 2. *Assume* $\left|\frac{\partial^2 f}{\partial t \partial s}\right|$ *is co-ordinated convex function on* Δ. *Then the following inequality holds:*

$$\left|\mathcal{L}_{\Delta}(\alpha,\beta)\right| \le \frac{(b-a)(d-c)}{16} \left[A\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right| + B\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, c\right)\right| \right.$$
$$+ C\left|\frac{\partial^2 f}{\partial t \partial s}\left(a, \frac{c+d}{2}\right)\right| + D\left|\frac{\partial^2 f}{\partial t \partial s}\left(b, \frac{c+d}{2}\right)\right| + E\left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{a+b}{2}, d\right)\right|$$
$$\left. + F\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right| + G\left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right| + H\left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right| + I\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right| \right]$$

where $A = \frac{\alpha^2\beta^2 + 5\alpha^2\beta + 5\alpha\beta^2 + 2\alpha^2 + 2\beta^2 + 25\alpha\beta + 10\alpha + 10\beta + 4}{4(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}$,
$B = \frac{\alpha^2 + 5\alpha + 2}{2(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}$, $C = \frac{\beta^2 + 5\beta + 2}{2(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}$,
$D = \frac{\alpha(\beta^2 + 5\beta + 2)}{4(\alpha+2)(\beta+1)(\beta+2)}$, $E = \frac{\beta(\alpha^2 + 5\alpha + 2)}{4(\alpha+1)(\alpha+2)(\beta+2)}$,
$F = \frac{1}{(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}$, $G = \frac{\alpha\beta}{4(\alpha+2)(\beta+2)}$,
$H = \frac{\beta}{2(\alpha+1)(\alpha+2)(\beta+2)}$ *and* $I = \frac{\alpha}{2(\alpha+2)(\beta+1)(\beta+2)}$.

Proof. From Lemma 1 and by using the property of modulus, we can write

$$\left|\mathcal{L}_{\Delta}(\alpha,\beta)\right| \le \frac{(b-a)(d-c)}{16}$$

$$\begin{aligned}
&\times\left[\int_0^1\int_0^1 t^\alpha s^\beta\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)\,a, s\frac{c+d}{2}+(1-s)\,c\right)\right|\mathrm{d}s\mathrm{d}t\right.\\
&+\int_0^1\int_0^1\left(1-t^\alpha\right)\left(1-s^\beta\right)\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\,\frac{a+b}{2}, sd+(1-s)\,\frac{c+d}{2}\right)\right|\mathrm{d}s\mathrm{d}t\\
&+\int_0^1\int_0^1 t^\alpha\left(1-s^\beta\right)\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)\,a, sd+(1-s)\,\frac{c+d}{2}\right)\right|\mathrm{d}s\mathrm{d}t\\
&\left.+\int_0^1\int_0^1\left(1-t^\alpha\right)s^\beta\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\,\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)\,c\right)\right|\mathrm{d}s\mathrm{d}t\right]\\
&=\frac{(b-a)(d-c)}{16}\left(J_1+J_2+J_3+J_4\right). \qquad (3.1)
\end{aligned}$$

By co-ordinated convexity of $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$, we have

$$\begin{aligned}
J_1 &= \int_0^1\int_0^1 t^\alpha s^\beta\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)\,a, s\frac{c+d}{2}+(1-s)\,c\right)\right|\mathrm{d}s\mathrm{d}t\\
&\le \int_0^1\int_0^1 t^\alpha s^\beta\left[ts\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|+t(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|\right.\\
&\quad\left.+s(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|+(1-t)(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|\right]\mathrm{d}s\mathrm{d}t\\
&= \frac{1}{(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|+\frac{1}{(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|\\
&\quad+\frac{1}{(\alpha+1)(\alpha+2)(\beta+1)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|+\frac{1}{(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|.
\end{aligned}$$

Similarly

$$\begin{aligned}
J_2 &= \int_0^1\int_0^1\left(1-t^\alpha\right)\left(1-s^\beta\right)\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\,\frac{a+b}{2}, sd+(1-s)\,\frac{c+d}{2}\right)\right|\mathrm{d}s\mathrm{d}t\\
&\le \frac{\alpha\beta}{4(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|+\frac{\alpha\beta(\beta+3)}{4(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|\\
&\quad+\frac{\alpha\beta(\alpha+3)}{4(\alpha+1)(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|\\
&\quad+\frac{\alpha\beta(\alpha+3)(\beta+3)}{4(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|,
\end{aligned}$$

$$\begin{aligned}
J_3 &= \int_0^1\int_0^1 t^\alpha\left(1-s^\beta\right)\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)\,a, sd+(1-s)\,\frac{c+d}{2}\right)\right|\mathrm{d}s\mathrm{d}t\\
&\le \frac{\beta}{2(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|+\frac{\beta(\beta+3)}{2(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\\
&\quad+\frac{\beta}{2(\alpha+1)(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|+\frac{\beta(\beta+3)}{2(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|
\end{aligned}$$

and

$$J_4 = \int_0^1\int_0^1\left(1-t^\alpha\right)s^\beta\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\,\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)\,c\right)\right|\mathrm{d}s\mathrm{d}t$$

$$\leq \frac{\alpha}{2(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|+\frac{\alpha}{2(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|$$
$$+\frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|$$
$$+\frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|.$$

Considering the results J_1, J_2, J_3, J_4 in (3.1) and making appropriate calculations, we get the conclusion of the Theorem 2. □

We recall that the Beta function (the Euler integral of the first kind), is defined by

$$\mathrm{B}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t$$

for $x, y > 0$.

Our next result reads as:

Theorem 3. *Assume* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q, q>1$ *is co-ordinated convex function on* Δ*. Then the following inequality holds:*

$$\left|\mathscr{L}_\Delta(\alpha,\beta)\right|\leq\frac{(b-a)(d-c)}{16\cdot 4^{1/q}}\left\{\left[\frac{1}{(\alpha p+1)(\beta p+1)}\right]^{1/p}\right.$$
$$\times\left[\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|^q\right]^{1/q}$$
$$+\left[\frac{1}{\alpha\beta}\mathrm{B}\left(p+1,\frac{1}{\alpha}\right)\mathrm{B}\left(p+1,\frac{1}{\beta}\right)\right]^{1/p}$$
$$\times\left[\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|^q\right]^{1/q}$$
$$+\left[\frac{1}{\alpha(\beta p+1)}\mathrm{B}\left(p+1,\frac{1}{\alpha}\right)\right]^{1/p}$$
$$\times\left[\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|^q\right]^{1/q}$$
$$+\left[\frac{1}{(\alpha p+1)\beta}\mathrm{B}\left(p+1,\frac{1}{\beta}\right)\right]^{1/p}$$
$$\times\left[\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|^q\right]^{1/q}$$

for $\frac{1}{p}+\frac{1}{q}=1$.

Proof. According to Lemma 1 and Hölder's inequality, we have

$$\left|\mathscr{L}_\Delta(\alpha,\beta)\right|\leq\frac{(b-a)(d-c)}{16}$$

$$\times\left\{\left[\int_0^1\int_0^1\left(t^\alpha s^\beta\right)^p\mathrm{d}s\mathrm{d}t\right]^{1/p}K_1^{1/q}+\left[\int_0^1\int_0^1\left(\left(1-t^\alpha\right)\left(1-s^\beta\right)\right)^p\mathrm{d}s\mathrm{d}t\right]^{1/p}K_2^{1/q}\right.$$
$$\left.+\left[\int_0^1\int_0^1\left(t^\alpha\left(1-s^\beta\right)\right)^p\mathrm{d}s\mathrm{d}t\right]^{1/p}K_3^{1/q}+\left[\int_0^1\int_0^1\left(\left(1-t^\alpha\right)s^\beta\right)^p\mathrm{d}s\mathrm{d}t\right]^{1/p}K_4^{1/q}\right\}$$

Since $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ is co-ordinated convex function, we have:

$$\begin{aligned}
K_1 &= \int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a, s\frac{c+d}{2}+(1-s)c\right)\right|^q\mathrm{d}s\mathrm{d}t\\
&\le \left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\int_0^1\int_0^1 ts\mathrm{d}s\mathrm{d}t+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|^q\int_0^1\int_0^1 t(1-s)\mathrm{d}s\mathrm{d}t\\
&\quad+\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|^q\int_0^1\int_0^1(1-t)s\mathrm{d}s\mathrm{d}t+\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q\int_0^1\int_0^1(1-t)(1-s)\mathrm{d}s\mathrm{d}t\\
&= \frac{1}{4}\left[\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q\right]
\end{aligned}$$

and similarly

$$\begin{aligned}
K_2 &= \int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)\right|^q\mathrm{d}s\mathrm{d}t\\
&\le \frac{1}{4}\left[\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right],\\
K_3 &= \int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a, sd+(1-s)\frac{c+d}{2}\right)\right|^q\mathrm{d}s\mathrm{d}t\\
&\le \frac{1}{4}\left[\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|^q\right],\\
K_4 &= \int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)c\right)\right|^q\mathrm{d}s\mathrm{d}t\\
&\le \frac{1}{4}\left[\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|^q\right].
\end{aligned}$$

A simple computation shows that

$$\int_0^1\int_0^1(t^\alpha s^\beta)^p\mathrm{d}s\mathrm{d}t=\frac{1}{(\alpha p+1)(\beta p+1)},$$
$$\int_0^1\int_0^1\left(1-t^\alpha\right)^p\left(1-s^\beta\right)^p\mathrm{d}s\mathrm{d}t=\frac{1}{\alpha\beta}\mathrm{B}\left(p+1,\frac{1}{\alpha}\right)\mathrm{B}\left(p+1,\frac{1}{\beta}\right),$$
$$\int_0^1\int_0^1(t^\alpha)^p(1-s^\beta)^p\mathrm{d}s\mathrm{d}t=\frac{1}{(\alpha p+1)\beta}\mathrm{B}\left(p+1,\frac{1}{\beta}\right),$$
$$\int_0^1\int_0^1(1-t^\alpha)^p(s^\beta)^p\mathrm{d}s\mathrm{d}t=\frac{1}{\alpha(\beta p+1)}\mathrm{B}\left(p+1,\frac{1}{\alpha}\right)$$

and the proof is complete. □

Theorem 4. *If* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ *, $q>1$ is co-ordinated convex function on Δ, then the following inequality holds:*

$$
\begin{aligned}
\left|\mathscr{L}_\Delta(\alpha,\beta)\right| \le & \frac{(b-a)(d-c)}{16\left[(\alpha+1)(\beta+1)\right]^{1/p}\left[(\alpha+2)(\beta+2)\right]^{1/q}} \\
& \times\left\{\left[\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\frac{1}{\beta+1}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|^q\right.\right. \\
& \left.+\frac{1}{\alpha+1}\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|^q+\frac{1}{(\alpha+1)(\beta+1)}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q\right]^{1/q} \\
& +\frac{\alpha\beta}{4^{1/q}}\left[\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q+\frac{\beta+3}{\beta+1}\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|^q\right. \\
& \left.+\frac{\alpha+3}{\alpha+1}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|^q+\frac{(\alpha+3)(\beta+3)}{(\alpha+1)(\beta+1)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right]^{1/q} \\
& +\frac{\beta}{2^{1/q}}\left[\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|^q+\frac{\beta+3}{\beta+1}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right. \\
& \left.+\frac{1}{\alpha+1}\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\frac{\beta+3}{(\alpha+1)(\beta+1)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|^q\right]^{1/q} \\
& +\frac{\alpha}{2^{1/q}}\left[\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|^q+\frac{1}{\beta+1}\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q\right. \\
& \left.\left.+\frac{\alpha+3}{\alpha+1}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q+\frac{\alpha+3}{(\alpha+1)(\beta+1)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|^q\right]^{1/q}\right\}
\end{aligned}
$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 1 and the power mean inequality, we have

$$
\begin{aligned}
\left|\mathscr{L}_\Delta(\alpha,\beta)\right| \le & \frac{(b-a)(d-c)}{16} \\
& \times\left\{\left[\int_0^1\int_0^1 t^\alpha s^\beta \mathrm{d}s\mathrm{d}t\right]^{1/p} M_1^{1/q}+\left[\int_0^1\int_0^1 (1-t^\alpha)\left(1-s^\beta\right)\mathrm{d}s\mathrm{d}t\right]^{1/p} M_2^{1/q}\right. \\
& \left.+\left[\int_0^1\int_0^1 t^\alpha\left(1-s^\beta\right)\mathrm{d}s\mathrm{d}t\right]^{1/p} M_3^{1/q}+\left[\int_0^1\int_0^1 (1-t^\alpha)s^\beta\mathrm{d}s\mathrm{d}t\right]^{1/p} M_4^{1/q}\right\}
\end{aligned}
$$

Since $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ is co-ordinated convex function, we have:

$$
\begin{aligned}
M_1 &= \int_0^1\int_0^1 t^\alpha s^\beta\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a, s\frac{c+d}{2}+(1-s)c\right)\right|^q \mathrm{d}s\mathrm{d}t \\
&\le \left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\int_0^1\int_0^1 t^{\alpha+1}s^{\beta+1}\mathrm{d}s\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&+\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|^q \int_0^1\int_0^1 t^{\alpha+1}(s^\beta - s^{\beta+1})\mathrm{d}s\mathrm{d}t\\
&+\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|^q \int_0^1\int_0^1 (t^\alpha - t^{\alpha+1})s\mathrm{d}s\mathrm{d}t\\
&+\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q \int_0^1\int_0^1 (t^\alpha - t^{\alpha+1})(s^\beta - s^{\beta+1})\mathrm{d}s\mathrm{d}t\\
=&\ \frac{1}{(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q + \frac{1}{(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|^q\\
&+\frac{1}{(\alpha+1)(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|^q + \frac{1}{(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q
\end{aligned}
$$

and similarly

$$
\begin{aligned}
M_2 =&\int_0^1\int_0^1 \left(1-t^\alpha\right)\left(1-s^\beta\right)\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2}, sd+(1-s)\frac{c+d}{2}\right)\right|^q \mathrm{d}s\mathrm{d}t\\
\le&\ \frac{\alpha\beta}{4(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\\
&+\frac{\alpha\beta(\beta+3)}{4(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|^q + \frac{\alpha\beta(\alpha+3)}{4(\alpha+1)(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|^q\\
&+\frac{\alpha\beta(\alpha+3)(\beta+3)}{4(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q,\\
M_3 =&\int_0^1\int_0^1 t^\alpha\left(1-s^\beta\right)\left|\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)a, sd+(1-s)\frac{c+d}{2}\right)\right|^q \mathrm{d}s\mathrm{d}t\\
\le&\ \frac{\beta}{2(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},d\right)\right|^q + \frac{\beta(\beta+3)}{2(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\\
&+\frac{\beta}{2(\alpha+1)(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q + \frac{\beta(\beta+3)}{2(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,\frac{c+d}{2}\right)\right|^q,\\
M_4 =&\int_0^1\int_0^1 \left(1-t^\alpha\right)s^\beta\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)\frac{a+b}{2}, s\frac{c+d}{2}+(1-s)c\right)\right|^q \mathrm{d}s\mathrm{d}t\\
\le&\ \frac{\alpha}{2(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,\frac{c+d}{2}\right)\right|^q + \frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\\
&+\frac{\alpha}{2(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q + \frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{a+b}{2},c\right)\right|^q.
\end{aligned}
$$

Hence the proof of the theorem is complete. □

4. Remarks

Remark 2. For $\alpha = 1$ and $\beta = 1$ in the Theorems 2, 3, respectively 4, we recover the results stated in ([4, Theorems 8-10]). Also for $\alpha = 1$ and $\beta = 1$ in Lemma 1, we get ([4, Lemma 1]).

We end our paper by considering the *cumulative to the right* $(\alpha,\beta)-$*gap* defined as

$$\begin{aligned}\mathscr{R}_{\Delta}\left(\alpha,\beta\right) = {} & \frac{f(a,c)+f(b,d)+f(a,d)+f(b,c)}{4} - \frac{2^{\alpha-2}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \\ & \times \left[J^{\alpha}_{a+} f\left(\frac{a+b}{2},c\right) + J^{\alpha}_{\frac{a+b}{2}+} f(b,d) + J^{\alpha}_{a+} f\left(\frac{a+b}{2},d\right) + J^{\alpha}_{\frac{a+b}{2}+} f(b,c) \right] \\ & - \frac{2^{\beta-2}\Gamma(\beta+1)}{(d-c)^{\beta}} \left[J^{\beta}_{c+} f\left(a,\frac{c+d}{2}\right) + J^{\beta}_{\frac{c+d}{2}+} f(b,d) + J^{\beta}_{\frac{c+d}{2}+} f(a,d) + J^{\beta}_{c+} f\left(b,\frac{c+d}{2}\right) \right] \\ & + \frac{2^{\beta-2}\Gamma(\beta+1)}{(d-c)^{\beta}} \left[J^{\alpha,\beta}_{a+,c+} f\left(\frac{a+b}{2},\frac{c+d}{2}\right) + J^{\alpha,\beta}_{\frac{a+b}{2}+,\frac{c+d}{2}+} f(b,d) \right. \\ & \left. + J^{\alpha,\beta}_{a+,\frac{c+d}{2}+} f\left(\frac{a+b}{2},d\right) + J^{\alpha,\beta}_{\frac{a+b}{2}+,c+} f\left(b,\frac{c+d}{2}\right) \right],\end{aligned}$$

where $f:\Delta\to\mathbb{R}$ be a differentiable function on Δ and

$$J^{\alpha,\beta}_{a+,c+} f\left(x,y\right) = \frac{1}{\Gamma(\alpha)\Gamma\left(\beta\right)} \int_a^x \int_c^y (x-u)^{\alpha-1}\left(y-v\right)^{\beta-1} f(u,v)\,\mathrm{d}v\mathrm{d}u,\ x>a,\ y>c,$$

is Riemann-Liouville integral and Γ is the Euler Gamma function.

Remark 3. The particular case $\alpha=1$ and $\beta=1$ gives

$$\begin{aligned}\mathscr{R}_{\Delta}(1,1) = {} & \frac{f(a,c)+f(b,d)+f(a,d)+f(b,c)}{4} - \frac{1}{2(b-a)}\left(\int_a^b f(x,c)\mathrm{d}x + \int_a^b f(x,d)\mathrm{d}x\right) \\ & - \frac{1}{2(d-c)}\left(\int_c^d f(a,y)\mathrm{d}y + \int_c^d f(b,y)\mathrm{d}y\right) + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\mathrm{d}y\mathrm{d}x\end{aligned}$$

The right hand side term can be recognized from the Hermite-Hadamard's inequality concerning the co-cordinated convex functions. See [5].

Using the above technique, the reader can find companions of the results we proved for the cumulative to the right $(\alpha,\beta)-$gap. We give below one of the results, but omit the proof.

Lemma 2. *It holds*

$$\begin{aligned}\mathscr{R}_{\Delta}\left(\alpha,\beta\right) = {} & \frac{(b-a)(d-c)}{16} \\ & \times \left[\int_0^1\int_0^1 t^{\alpha}s^{\beta}\frac{\partial^2 f}{\partial t\partial s}\left(ta+(1-t)\frac{a+b}{2}, sc+(1-s)\frac{c+d}{2}\right)\mathrm{d}s\mathrm{d}t \right. \\ & + \int_0^1\int_0^1 \left(t^{\alpha}-1\right)\left(s^{\beta}-1\right)\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)b, \frac{c+d}{2}+(1-s)d\right)\mathrm{d}s\mathrm{d}t \\ & + \int_0^1\int_0^1 t^{\alpha}\left(s^{\beta}-1\right)\frac{\partial^2 f}{\partial t\partial s}\left(ta+(1-t)\frac{a+b}{2}, \frac{c+d}{2}+(1-s)d\right)\mathrm{d}s\mathrm{d}t \\ & \left. + \int_0^1\int_0^1 \left(t^{\alpha}-1\right)s^{\beta}\frac{\partial^2 f}{\partial t\partial s}\left(t\frac{a+b}{2}+(1-t)b, sc+(1-s)\frac{c+d}{2}\right)\mathrm{d}s\mathrm{d}t \right],\end{aligned}$$

for all $t,s\in[0,1]$.

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Department of Mathematics, University of Craiova, Street A. I. Cuza 13, Craiova, RO-200585, Romania.

E-mail: mmihai58@yahoo.com