EXTENSION OF AN INEQUALITY OF H. ALZER FOR NEGATIVE POWERS

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Abstract. In this paper, we show that let $n$ be a natural number, then for all real numbers $r$,
\[ \frac{n}{n+1} < \left( \frac{1}{n} \sum_{i=1}^{n} i^r \right)^{1/r} < 1. \]
Both bounds are best possible. This extends a result of H. Alzer, who established this inequality for $r > 0$.

In [1, 2, 4, 5], it was shown that
\[ \frac{n}{n+1} < \left( \frac{1}{n} \sum_{i=1}^{n} i^r \right)^{1/r} \]
for $r > 0$, $n \in \mathbb{N} =: \{1, 2, \ldots\}$. The lower bound is best possible. (1) is called Alzer’s inequality [1].

The main purpose of this note is to provide an extension of the result given by Alzer. In order to show that the inequality (1) holds not only for $r \in (0, +\infty)$ but even for $r \in (-\infty, +\infty)$, we need the following lemma.

Lemma. Let $r \neq -1, 0$ be a real number, define the function $f$ by
\[ f(x) = \frac{(x + 1)^r}{(x + 1)^{r+1} - x^{r+1}} \quad (x > 0). \]

Then
(a) For $r \in (-\infty, -1) \cup (0, +\infty)$, the function $f$ is strictly decreasing on $(0, +\infty)$;

Received September 26, 2003.
2000 Mathematics Subject Classification. 26D15.
Key words and phrases. Inequality, power mean, Lagrange’s mean value theorem, mathematical induction.

The authors were supported in part by NNSF (#10001016) of China, SF for the Prominent Youth of Henan Province (#0112000200), SF of Henan Innovation Talents at Universities, Doctor Fund of Jiaozuo Institute of Technology, China.
(b) For $r \in (-1, 0)$, the function $f$ is strictly increasing on $(0, +\infty)$.

**Proof.** Easy computation yields

$$f'(x) = -\frac{(x + 1)^{r-1}[(x + 1)^{r+1} - x^{r+1} - (r + 1)x^r]}{[(x + 1)^{r+1} - x^{r+1}]^2}.$$ 

By Lagrange’s mean value theorem, there exists at least one point $\xi \in (x, x + 1)$ such that

$$(x + 1)^{r+1} - x^{r+1} = (r + 1)\xi^r, \quad x < \xi < x + 1.$$ 

Further, we have

$$f'(x) = -\frac{(r + 1)(x + 1)^{r-1}(\xi^r - x^r)}{[(x + 1)^{r+1} - x^{r+1}]^2}.$$ 

It is easy to see that for $r \in (-\infty, -1) \cup (0, +\infty)$, $f'(x) < 0(x > 0)$, and for $r \in (-1, 0)$, $f'(x) > 0(x > 0)$. The proof is complete.

We are now in a position to establish our result.

**Theorem.** Let $n$ be a natural number. Then for all real numbers $r$,

$$\frac{n}{n + 1} < \left(\frac{\frac{1}{n} \sum_{i=1}^{n} i^r}{\frac{1}{n + 1} \sum_{i=1}^{n+1} i^r}\right)^{1/r} < 1. \tag{2}$$

Both bounds are best possible.

**Proof.** It was shown in [3] that for all natural numbers $n$,

$$\frac{n}{n + 1} < \frac{\sqrt[n]{n!}}{n^{1/(n + 1)!}} < 1. \tag{3}$$

For $r = 0$, (2) can be interpreted as (3) because of

$$\lim_{r \to 0} \left(\frac{\frac{1}{n} \sum_{i=1}^{n} i^r}{\frac{1}{n + 1} \sum_{i=1}^{n+1} i^r}\right)^{1/r} = \frac{\sqrt[n]{n!}}{n^{1/(n + 1)!}}.$$ 

For $r = -1$, the inequality (2) holds clearly.

For $r \in (-\infty, -1) \cup (0, +\infty)$, the left-hand inequality of (2) is equivalent to

$$\sum_{i=1}^{n} i^r > \frac{n^{r+1}(n + 1)^r}{(n + 1)^{r+1} - n^{r+1}}. \tag{4}$$

Clearly, the inequality (4) holds for $n = 1$. Suppose (4) holds for some $n \geq 1$. Adding $(n + 1)^r$ to the both sides of (4) leads to

$$\sum_{i=1}^{n+1} i^r > \frac{(n + 1)^{2r+1}}{(n + 1)^{r+1} - n^{r+1}}. \tag{5}$$
By mathematical induction, it remains to show that
\[ \sum_{i=1}^{n+1} i^r > \frac{(n + 1)^{r+1}(n + 2)^r}{(n + 2)^{r+1} - (n + 1)^{r+1}}. \] (6)

From (5) and (6) it suffices to show that
\[ \frac{(n + 1)^r}{(n + 1)^{r+1} - n^{r+1}} > \frac{(n + 2)^r}{(n + 2)^{r+1} - (n + 1)^{r+1}} \]
which was shown in Lemma (a).

For \( r \in (-1, 0) \), the left-hand inequality of (2) is equivalent to
\[ \sum_{i=1}^{n} i^r < n(n + 1)^r \] (7)
Clearly, the inequality (7) holds for \( n = 1 \). Now accepting (7) for \( n \geq 1 \), we try to obtain it for \( n + 1 \). It is easy to see that the induction step can be written as
\[ \frac{(n + 1)^r}{(n + 1)^{r+1} - n^{r+1}} < \frac{(n + 2)^r}{(n + 2)^{r+1} - (n + 1)^{r+1}} \]
which was shown in Lemma (b). Hence, the left-hand inequality of (2) holds for all real numbers \( r \).

For \( r > 0 \), the right-hand inequality of (2) is equivalent to
\[ \sum_{i=1}^{n} i^r < n(n + 1)^r \] (8)
which follows obviously.

For \( r < 0 \), the right-hand inequality of (2) is equivalent to
\[ \frac{1}{n} \sum_{i=1}^{n} i^r > \frac{1}{n + 1} \sum_{i=1}^{n+1} i^r. \] (9)
Setting \( s = -r \), then (9) can be written as
\[ \sum_{i=1}^{n} \frac{1}{i^s} > \frac{n}{(n + 1)^s} \]
which follows obviously. Hence, the right-hand inequality of (2) holds for all real numbers \( r \).

It is easy to see that
\[ \lim_{r \to +\infty} \left( \frac{\frac{1}{n} \sum_{i=1}^{n} i^r}{\frac{1}{n + 1} \sum_{i=1}^{n+1} i^r} \right)^{1/r} = \frac{n}{n + 1}, \] (10)
\[ \lim_{r \to -\infty} \left( \frac{\frac{1}{n} \sum_{i=1}^{n} i^r}{\frac{1}{n + 1} \sum_{i=1}^{n+1} i^r} \right)^{1/r} = 1. \] (11)
Thus, the both bounds given in (2) are best possible. The proof is complete.

In view of (2), (10) and (11), it is natural to pose the following conjecture.

**Conjecture.** For any given natural number $n$, define the function $f$ by

$$f(r) = \begin{cases} 
\left( \frac{\frac{1}{n} \sum_{i=1}^{n} i^r}{\frac{1}{n+1} \sum_{i=1}^{n+1} i^r} \right)^{1/r}, & r \neq 0, \\
\frac{\sqrt[n+1]{n!}}{(n+1)!}, & r = 0.
\end{cases}$$

Then, the function $f(r)$ is strictly decreasing on $(-\infty, +\infty)$.

**References**


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