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NORMAL FAMILIES OF MEROMORPHIC FUNCTIONS WITH MULTIPLE POLES

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Abstract. Let $\mathscr{F}$ be a family of meromorphic functions defined in a domain $\mathscr{D}$, and a, b be two constants such that $a \neq 0$, ∞ and $b \neq \infty$. If for each $f \in \mathscr{F}$, all poles of $f(z)$ are of multiplicity at least 3 in $\mathscr{D}$, and $f'(z) + af^2(z) - b$ has at most 1 zero in $\mathscr{D}$, ignoring multiplicity, then $\mathscr{F}$ is normal in $\mathscr{D}$.

1. Introduction and main results

Let $\mathscr{D}$ be a domain in $\mathbb{C}$, and $\mathscr{F}$ be a family of meromorphic functions defined in the domain $\mathscr{D}$. $\mathscr{F}$ is said to be normal in $\mathscr{D}$, in the sense of Montel, if for every sequence $\{f_n\} \subseteq \mathscr{F}$ contains a subsequence $\{f_{n_j}\}$ such that f_{n_j} converges spherically uniformly on compact subsets of $\mathscr{D}$ (See [1, Definition 3.1.1]).

$\mathscr{F}$ is said to be normal at a point $z_0 \in \mathscr{D}$ if there exists a neighborhood of z_0 in which $\mathscr{F}$ is normal. It is well known that $\mathscr{F}$ is normal in a domain $\mathscr{D}$ if and only if it is normal at each of its points (see [1, Theorem 3.3.2]).

Let $f(z)$ and $g(z)$ be two meromorphic functions in a domain $\mathscr{D} \subseteq \mathbb{C}$, and let a be a finite complex number. If $f(z) - a$ and $g(z) - a$ assume the same zeros, then we say that f and g share the value a in $\mathscr{D}$ IM (ignoring multiplicity)(see [2, pp.115-116]).

In 1959, W. K. Hayman [3] proved the following well-known result.

Theorem A. *Let f be a non-constant meromorphic function in the complex plane $\mathbb{C}$, n be a positive integer and a, b be two constants such that $n \geqslant 5$, $a \neq 0, \infty$ and $b \neq \infty$. If $f' - af^n \neq b$, then f is a constant.*

Corresponding to Theorem A there are the following theorems which confirmed a Hayman's well-known conjecture about normal families in [4, Problem 5.14].

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Theorem B. *Let $\mathscr{F}$ be a family of meromorphic functions in a complex domain $\mathscr{D}$, n be a positive integer and a, b be two constants such that $n \geqslant 3$, $a \neq 0, \infty$ and $b \neq \infty$. If $f' - af^n \neq b$, then $\mathscr{F}$ is normal in $\mathscr{D}$.*

Theorem C. *Let $\mathscr{F}$ be a family of holomorphic functions in a complex domain $\mathscr{D}$, n be a positive integer and a, b be two constants such that $n \geqslant 2$, $a \neq 0, \infty$ and $b \neq \infty$. If $f' - af^n \neq b$, then $\mathscr{F}$ is normal in $\mathscr{D}$.*

Theorem B is due to S. Li [5, $n \geqslant 5$], X. Li [6, $n \geqslant 5$], J. Langley [7, $n \geqslant 5$], X. Pang [8, $n = 4$], H. Chen and M. Fang [9, $n = 3$] and L. Zalcman [10, $n = 3$] independently. Theorem C is due to D. Drasin [11, $n \geqslant 3$] and Y. Ye [12, $n = 2$].

Generally speaking, for $n = 2$, the result of Theorem B is not valid. For examples we refer the reader to [13]. However, in [13], M. Fang and W. Yuan had

Theorem D. *Let $\mathscr{F}$ be a family of meromorphic functions in a complex domain $\mathscr{D}$, and $a \neq 0, \infty$, $b \neq \infty$. If, for every $f \in \mathscr{F}$, $f' - af^2 \neq b$ and the poles of f are of multiplicity at least 3, then $\mathscr{F}$ is normal in $\mathscr{D}$.*

It is natural to ask whether the condition in Theorem D that $f' - af^2 \neq b$ can be relaxed. In this paper we investigate this problem and prove the following result.

Theorem 1. *Let $\mathscr{F}$ be a family of meromorphic functions defined in a domain $\mathscr{D}$, and a, b be two constants such that $a \neq 0$, ∞ and $b \neq \infty$. If for each $f \in \mathscr{F}$, all poles of $f(z)$ are of multiplicity at least 3 in $\mathscr{D}$, and $f'(z) + af^2(z) - b$ has at most 1 zero in $\mathscr{D}$, ignoring multiplicity, then $\mathscr{F}$ is normal in $\mathscr{D}$.*

By the idea of shared values, Q. Zhang [14, Theorem 2] proved.

Theorem E. *Let $\mathscr{F}$ be a family of holomorphic functions defined in a domain $\mathscr{D}$ and a, b be two constants such that $a \neq 0$, ∞ and $b \neq \infty$. If for every pair of functions f, $g \in \mathscr{F}$, $f' + af^2$ and $g' + ag^2$ share the value b in $\mathscr{D}$, then $\mathscr{F}$ is normal in $\mathscr{D}$.*

It is natural to ask whether Theorems D can be improved by the idea of shared values. In this paper, we study the problem and obtain the following theorem.

Theorem 2. *Let $\mathscr{F}$ be a family of meromorphic functions defined in a domain $\mathscr{D}$. Let $a \neq 0$, b be two finite complex number. If for each $f \in \mathscr{F}$, all poles of $f(z)$ are of multiplicity at least 3 in $\mathscr{D}$, and if $f' + af^2$ and $g' + ag^2$ share the value b in $\mathscr{D}$ for every pair of functions f, $g \in \mathscr{F}$, then $\mathscr{F}$ is normal in $\mathscr{D}$.*

Example 1. Let $\mathscr{D} = \{z : |z| < 1\}$. Let $\mathscr{F} = \{f_m\}$ where $f_m := \frac{1}{mz^k}$, then $f_m' + f^2 = \frac{1 - kmz^{k-1}}{m^2 z^{2k}}$, so $f_m' + f^2$ has exactly one zero for $k = 2$ and $f_m' + f^2$ has two distinct zeros for $k = 3$. However, it is easily obtained that $\mathscr{F}$ is not normal at the point $z = 0$.

This shows that the condition that all poles of $f(z)$ are of multiplicity at least 3 and $f'(z)+af^2(z)$ has at most 1 zero in Theorems 1 is sharp.

2. Some lemmas

To prove our results, we need some preliminary results.

Lemma 1 ([15], [16] Lemma 1 (Zalcman's Lemma)). *Let $\mathscr{F}$ be a family of functions meromorphic on a domain $\mathscr{D}$, all of whose poles have multiplicity at least j; Then if $\mathscr{F}$ is not normal at $z_0 \in \mathscr{D}$, there exist, for each $j < \alpha < 1$,*

(a) *points z_n, $z_n \to z_0$;*

(b) *functions $f_n \in \mathscr{F}$; and*

(c) *positive numbers $\rho_n \to 0^+$*

such that $\rho_n^\alpha f_n(z_n + \rho_n \xi) = g_n(\xi) \to g(\xi)$ locally uniformly with respect to the spherical metric, where $g(\xi)$ is a nonconstant meromorphic function on $\mathbb{C}$. Moreover, the order of $g(\xi)$ is less than 2 and the poles of $g(\xi)$ are of multiplicity at least j.

Here, as usual, $g^\#(\xi) = \frac{|g'(\xi)|}{1+|g(\xi)|^2}$ is the spherical derivative.

Lemma 2 ([17], Theorem 2). *Let $f(z)$ be a transcendental meromorphic function in $\mathbb{C}$. If all zeros of $f(z)$ have multiplicity at least 3, for any positive integer k, then $f^{(k)}(z)$ assumes every non-zero finite value infinitely often.*

Lemma 3 ([17]). *Let $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 + q(z)/p(z)$, where $a_0, a_1, \cdots, a_n$ are constants with $a_n \neq 0$, and q and p are two co-prime polynomials, neither of which vanishes identically, with $\deg q < \deg p$; and let k be a positive integer and b a nonzero complex number. If $f^{(k)} \neq b$, and the zeros of f all have multiplicity at least $k+1$, then*

$$f(z) = \frac{b(z-d)^{k+1}}{k!(z-c)},$$

where c and d are distinct complex numbers.

Lemma 4 ([19], Lemma 4). *Let f be a nonconstant rational function and k, m be positive integers. If f has no zeros in $\mathbb{C}$ while all poles of f have multiplicity at least m, then $f^{(k)} - 1$ has at least $k+m$ distinct zeros in $\mathbb{C}$.*

Lemma 5. *Let f be a non-constant meromorphic function, and $a \neq 0$ be a finite complex number. Let all poles of $f(z)$ have multiplicity at least 3, then $f' + af^2$ has at least two distinct zeros.*

Proof. Set $f = \frac{1}{a\varphi}$, then all zeros of φ have multiplicity at least 3 and $f' + af^2 = -\frac{\varphi'-1}{a\varphi^2}$.

Case 1. If $\frac{\varphi'-1}{a\varphi^2}$ has only one zero z_0, then z_0 is a multiple pole of φ, or else a zero of $\varphi' - 1$. If z_0 is a multiple pole of φ, since $\frac{\varphi'-1}{a\varphi^2}$ has only one zero, then $\varphi' - 1 \neq 0$. By Lemma 2 and Lemma 3, this is a contradiction.

So φ has no multiple pole and $\varphi' - 1$ has just one unique zero z_0. By the Lemma 2, φ is not any transcendental function.

Case 1.1. φ is a non-constant polynomial.

Since $\varphi' - 1$ has only unique one zero z_0, set

$$\varphi' - 1 = A(z - z_0)^l$$

where A is non-zero constant, l is a positive integer. Then

$$\varphi'' = Al(z - z_0)^{l-1}$$

Note that φ is a a non-constant polynomial and all zeros of φ have multiplicity at least 3, so $l \geqslant 3 - 1 = 2$. But φ'' has only one zero z_0, so φ' has only the same zero z_0 too. Hence $\varphi'(z_0) = 0$, which contradicts $\varphi'(z_0) = 1 \neq 0$. Therefore φ is a rational function which is not a polynomial.

Case 1.2. If φ is rational but not a polynomial and has at least one zero.

Under the conditions of Lemma 5 on the rational function φ,

$$\varphi(z) = A\frac{(z+\alpha_1)^{m_1}(z+\alpha_2)^{m_2}\cdots(z+\alpha_s)^{m_s}}{(z+\beta_1)(z+\beta_2)\cdots(z+\beta_t)} \tag{2.1}$$

where A is a non-zero constant, $m_i \geqslant 3$ $(i = 1, 2, \cdots, s)$, $\alpha_i (i = 1, 2, \cdots, s)$, and $\beta_j (j = 1, 2, \cdots, t)$ are distinct complex numbers.

For simplicity, we denote

$$m_1 + m_2 + \cdots + m_s = M \geqslant 3s. \tag{2.2}$$

Let

$$g(z) = z - \varphi(z)$$

We use $\deg(g)$ to denote the degree of a polynomial. Next we use some results from complex dynamics, cf. [18, Chapter 3, 19 Lemma 5]. Since $\varphi' - 1$ has only unique one zero, so that g has only unique one critical point. Since $m_i \geqslant 3$, we also see that every zero point $\alpha_i = -z_i$ of $-\varphi$ is a fixed point of g of multiplicity m_i. Moreover, since near z_i

$$g(z) = z + c_i(z - z_i)^{m_i}[1 + o(1)]$$

there are $m_i - 1$ parabolic basins associated with the fixed point z_i.

Case 1.2.1. If ∞ is not a fixed point of g, each of these parabolic basins, with at most one exception, contains a critical point of g which is not a pole of g, so the function $g(z) = z - \varphi(z)$ has $1 \geqslant M - s - 1$ parabolic basins associated with the zero points of φ, we deduce that $s = 1$ and $m_1 = 3$. We note that ∞ is not a fixed point of g, so

$$\varphi(z) = \frac{(z+\alpha_1)^3}{(z+\beta_1)(z+\beta_2)} \tag{2.3}$$

In order to simplify the calculation, set $Z = z + \beta_1$. From (2.3), we have

$$\varphi(Z) = \frac{(Z+\alpha)^3}{Z(Z+\beta)}$$

where $\alpha = \alpha_1 - \beta_1 \neq 0$ and $\beta = \beta_2 - \beta_1 \neq 0$.

We have

$$\varphi' = \frac{(Z+\alpha)^2(Z^2 + 2(\beta-\alpha)Z - \alpha\beta)}{Z^2(Z+\beta)^2} \tag{2.4}$$

Since $\varphi'(z) - 1$ has exactly one zero z_0, from (2.3) we obtain

$$\varphi'(Z) = 1 + \frac{B(Z-Z_0)^l}{Z^2(Z+\beta)^2} = \frac{Z^2(Z+\beta)^2 + B(Z-Z_0)^l}{Z^2(Z+\beta)^2} \tag{2.5}$$

where B is a non-zero constant and $Z_0 = z_0 - \beta_1$ and l is a positive integer. From (2.4) and (2.5), we have $l \leq 3$.

From (2.4) we have

$$\varphi'' = 2\frac{(Z+\alpha)[(\beta^2 + 3\alpha^2 - 3\alpha\beta)Z^2 + (3\alpha^2\beta - \alpha\beta^2)Z + \alpha^2\beta^2]}{Z^3(Z+\beta)^3} \tag{2.6}$$

From (2.5), we have

$$\varphi'' = B\frac{(Z-Z_0)^{l-1}[(l-4)Z^2 + (l\beta + 4Z_0 - 2\beta)Z + 2\beta Z_0]}{Z^3(Z+\beta)^3} \tag{2.7}$$

If $l = 1$, from (2.4) and (2.5), we have

$$(Z+\alpha)^2(Z^2 + 2(\beta-\alpha)Z - \alpha\beta) = Z^2(Z+\beta)^2 + B(Z-Z_0)$$

By comparing coefficients of both sides, we have

$$\begin{cases} 3\alpha(\beta-\alpha) = \beta^2 \\ -2\alpha^3 = B \\ -\alpha^3\beta = BZ_0 \end{cases}$$

i.e.

$$\begin{cases} 3\alpha\beta - 3\alpha^2 - \beta^2 = 0 \\ B = -2\alpha^3 \\ Z_0 = \beta/2 \end{cases}$$

By (2.6), we have $\varphi''(-\alpha) = 0$. From (2.7), we get

$$(1-4)(-\alpha)^2 + (\beta + 4\frac{\beta}{2} - 2\beta)(-\alpha) + 2\beta\frac{\beta}{2} = 0$$

i.e.

$$\alpha\beta + 3\alpha^2 - \beta^2 = 0$$

Combining this with $3\alpha\beta - 3\alpha^2 - \beta^2 = 0$ yields

$$\alpha = \beta = 0$$

which is a contradiction.

If $l = 2$, from (2.4) and (2.5), we have

$$(Z+\alpha)^2(Z^2 + 2(\beta - \alpha)Z - \alpha\beta) = Z^2(Z+\beta)^2 + B(Z - Z_0)^2$$

By comparing coefficients of both sides, we have

$$\begin{cases} 3\alpha(\beta - \alpha) = \beta^2 + B \\ -2\alpha^3 = -2BZ_0 \\ -\alpha^3\beta = BZ_0^2 \end{cases}$$

i.e.

$$\begin{cases} 3\alpha\beta^2 - 3\alpha^2\beta - \beta^3 + \alpha^3 = 0 \\ B = -\frac{\alpha^3}{\beta} \\ Z_0 = -\beta \end{cases}$$

By (2.6), we have $\varphi''(-\alpha) = 0$. From (2.7), we get

$$\begin{aligned} \varphi''(-\alpha) &= -\frac{\alpha^3}{\beta}\frac{(\beta-\alpha)[(2-4)(-\alpha)^2 + (2\beta - 4\beta - 2\beta)(-\alpha) - 2\beta^2]}{(-\alpha)^3(\beta-\alpha)^3} \\ &= -\frac{2}{\beta} \neq 0 \end{aligned}$$

which is a contradiction. If $l = 3$, then $\deg((Z+\alpha)[(\beta^2 + 3\alpha^2 - 3\alpha\beta)Z^2 + (3\alpha^2\beta - \alpha\beta^2)Z + \alpha^2\beta^2]) < \deg((Z - Z_0)^{l-1}[(l-4)Z^2 + (l\beta + 4Z_0 - 2\beta)Z + 2\beta Z_0])$, which is a contradiction.

Case 1.2.2. If ∞ is a fixed point of g, each parabolic basin contains a critical point of g which is not a pole of g. Thus each parabolic basins contains a zero point of g', and hence $2 \le M - s \le 1$, which is a contradiction.

Case 1.3. If ϕ is rational but not a polynomial and has no zero. By Lemma 4, we have $\varphi' - 1$ has at least two distinct zeros, which contradicts the fact that $\varphi' - 1$ has just one unique zero z_0.

Case 2. If $\frac{\varphi'-1}{a\varphi^2} \neq 0$.

Case 2.1. Since all zeros of $\varphi(z)$ have the multiple at least 3 and φ is a non-constant function, it easily obtained that φ is not a polynomial.

Case 2.2. If φ is rational but not a polynomial. If there exist a point z_0 such that $\varphi'(z_0) = 1$, we note that $\frac{\varphi'-1}{a\varphi^2} \neq 0$, so $\varphi(z_0) = 0$. Since all zeros of $\varphi(z)$ have the multiple at least 3, we have $\varphi'(z_0) = 0$, we get a contradiction.

If $\varphi' \neq 1$, by Lemma 3, we have

$$\varphi = \frac{(z-d)^2}{(z-c)},$$

where c and d are distinct complex numbers. This contradicts the fact that all zeros of $\varphi(z)$ have the multiple at least 3.

The proof is complete. □

3. Proofs of Theorems

Proof of Theorem 1. Suppose that $\mathscr{F}$ is not normal in $\mathscr{D}$. Then there exists at least one point z_0 such that $\mathscr{F}$ is not normal at the point $z_0 \in \mathscr{D}$. Without loss of generality we assume that $z_0 = 0$. By Zalcman's lemma, there exist:

(a) points z_n, $z_n \to z_0$;
(b) functions $f_n \in \mathscr{F}$; and
(c) positive numbers $\rho_n \to 0^+$

such that

$$g_j(\xi) = \rho_j f_j(z_j + \rho_j \xi) \to g(\xi) \tag{3.1}$$

spherically uniformly on compact subsets of $\mathbb{C}$, where $g(\xi)$ is a non-constant meromorphic function in $\mathbb{C}$ and the poles of $g(\xi)$ are of multiplicity at least 3.

From (3.1), we get

$$g_j'(\xi) = \rho_j^2 f_j'(z_j + \rho_j \xi) \to g'(\xi) \tag{3.2}$$

also locally uniformly with respect to the spherical metric.

Thus

$$g_j'(\xi) + a g_{n_j}(\xi) - \rho_j^2 b = \rho_j^2 (f_j'(z_j + \rho_j \xi) + a f_j^2(z_j + \rho_j \xi) - b) \to g'(\xi) + a g^2(\xi) \tag{3.3}$$

also locally uniformly with respect to the spherical metric.

We claim that $g'(\xi)+ag^2$ has at most 1 zero ignoring multiplicity.

If $g'(\xi)+ag^2 \equiv 0$, then $g(\xi) \equiv \frac{1}{az+c}$, this contradicts the fact that the poles of $g(\xi)$ are of multiplicity at least 3. So $g'(\xi)+ag^2 \not\equiv 0$.

Suppose that $g'(\xi)+ag^2$ has two distinct zeros ξ_0 and ξ_0^* and choose $\delta(>0)$ small enough such that $D(\xi_0,\delta)\cap D(\xi_0^*,\delta)=\emptyset$ where $D(\xi_0,\delta)=\{\xi||\xi-\xi_0|<\delta\}$ and $D(\xi_0^*,\delta)=\{\xi||\xi-\xi_0^*|<\delta\}$.

From (3.3), by Hurwitz's theorem, there exist points $\xi_j\in D(\xi_0,\delta)$, $\xi_j^*\in D(\xi_0^*,\delta)$ such that for sufficiently large j

$$f_j'(z_j+\rho_j\xi_j)+af_j^2(z_j+\rho_j\xi_j)-b=0.$$
$$f_j'(z_j+\rho_j\xi_j^*)+af_j^2(z_j+\rho_j\xi_j^*)-b=0.$$

Since $z_j\to 0$ and $\rho_j\to 0^+$, we have $z_j+\rho_j\xi_j\in D(\xi_0,\delta)$ and $z_j+\rho_j\xi_j^*\in D(\xi_0^*,\delta)$ for sufficiently large j, so $f_j'(z)+af_j^2-b$ has two distinct zeros, which contradicts the fact that $f_j'(z)+af_j^2-b$ has at most 1 zero.

However, by Lemma 5, there does not exist non-constant meromorphic functions satisfying above properties such that our claim holds. This contradiction shows that $\mathscr{F}$ is normal in $\mathscr{D}$ and hence Theorem 1 is proved.

Proof of Theorem 2. Suppose that $\mathscr{F}$ is not normal in $\mathscr{D}$. Then there exists at least one point z_0 such that $\mathscr{F}$ is not normal at the point $z_0\in\mathscr{D}$. Without loss of generality we assume that $z_0=0$. By Zalcman's lemma, there exist:

(a) points z_n, $z_n\to z_0$;

(b) functions $f_n\in\mathscr{F}$; and

(c) positive numbers $\rho_n\to 0^+$

such that

$$g_j(\xi)=\rho_j f_j(z_j+\rho_j\xi)\to g(\xi) \tag{3.4}$$

spherically uniformly on compact subsets of $\mathbb{C}$, where $g(\xi)$ is a non-constant meromorphic function in $\mathbb{C}$ and the poles of $g(\xi)$ are of multiplicity at least 3.

Proceeding as in the proof of Theorem 1, we also have (3.3).

If $g'(\xi)+ag^2 \equiv 0$, then $g(\xi) \equiv \frac{1}{az+c}$, this contradicts the fact that the poles of $g(\xi)$ are of multiplicity at least 3. So $g'(\xi)+ag^2 \not\equiv 0$.

Since g is a non-constant meromorphic function, by Lemma 5, we deduce that $g'(\xi)+ag^2$ has at least two distinct zeros.

We claim that $g'(\xi)+ag^2(\xi)$ has just a unique zero.

Suppose that there exist two distinct zeros ξ_0 and ξ_0^* and choose $\delta(>0)$ small enough such that $D(\xi_0,\delta)\cap D(\xi_0^*,\delta)=\emptyset$ where $D(\xi_0,\delta)=\{\xi||\xi-\xi_0|<\delta\}$ and $D(\xi_0^*,\delta)=\{\xi||\xi-\xi_0^*|<\delta\}$.

From (3.4), by Hurwitz's theorem, there exist points $\xi_j\in D(\xi_0,\delta)$, $\xi_j^*\in D(\xi_0^*,\delta)$ such that for sufficiently large j

$$f_j'(z_j+\rho_j\xi_j)+af_j^2(z_j+\rho_j\xi_j)-b=0.$$
$$f_j'(z_j+\rho_j\xi_j^*)+af_j^2(z_j+\rho_j\xi_j^*)-b=0.$$

By the assumption that $f'+af^2$ and $g'+ag^2$ share b in $\mathscr{D}$ for each pair f and g in $\mathscr{F}$, we know that for any integer m

$$f_m'(z_j+\rho_j\xi_j)+af_m^2(z_j+\rho_j\xi_j)-b=0.$$
$$f_m'(z_j+\rho_j\xi_j^*)+af_m^2(z_j+\rho_j\xi_j^*)-b=0.$$

We fix m and note that $z_j+\rho_j\xi_j\to 0$, $z_j+\rho_j\xi_j^*\to 0$ if $j\to\infty$. From this we deduce

$$f_m'(0)+af_m^2(0)-b=0.$$

Since the zeros of $f_m'(z)+af_m^2(z)-b$ have no accumulation point, for sufficiently large j, we have

$$z_j+\rho_j\xi_j=0,\ z_j+\rho_j\xi_j^*=0.$$

Hence

$$\xi_j=-\frac{z_j}{\rho_j},\ \xi_j^*=-\frac{z_j}{\rho_j}.$$

This contradicts the fact that $\xi_j\in D(\xi_0,\delta)$, $\xi_j^*\in D(\xi_0^*,\delta)$ and $D(\xi_0,\delta)\cap D(\xi_0^*,\delta)=\emptyset$. So $g'(\xi)+ag^2(\xi)$ has just a unique zero. This contradicts the fact that $g'(\xi)+ag^2(\xi)$ has at least two distinct zeros.

This proves the theorem.

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References

[1] J. Schiff, Normal Families, Springer-Verlag, Berlin, 1993.

[2] C. C. Yang and H. X. Yi, Uniqueness Theory of Meromorphic Functions, Science Press, Beijing, 1995 (In Chinese); Kluwer Academic Publishers, The Netherlands, 2003.

[3] W. K. Hayman, *Picard values of meromorphic functions and their derivatives*, Ann. of Math., **70**(1959), 9–42.

[4] W. K. Hayman, Research Problems in Function Theory, Athlone Press, London, 1967.

[5] S. Li, *On normality criterion of a class of the functions*, J. Fujian Normal Univ., **2** (1984), 156–158 (in Chinese).

[6] X. Li, *Proof of Hayman's conjecture on normal families*, Sci. China Ser. A, **28** (1985), 596–603.

[7] J. Langley, *On normal families and a result of Drasin*, Proc. Roy. Soc. Edinburgh Sect. A, **98** (1984), 385–393.

[8] X. Pang, *On normal criterion of meromorphic functions*, Sci. China Ser. A, **33** (1990), 521–527.

[9] H. Chen and M. Fang, *On the value distribution of* $f^n f'$, Sci. China Ser. A, **38** (1995), 789–798.

[10] L. Zalcman, On some questions of Hayman, unpublished manuscript, 1994.

[11] D. Drasin, *Normal families and Nevanlinna theory*, Acta Math., **122** (1969), 231–263.

[12] Y. Ye, *A new criterion and its application*, Chinese Ann. Math. Ser. A, **12** (suppl.) (1991), 44–49 (in Chinese).

[13] M. L. Fang and W. J. Yuan, *On the normality for families of meromorphic functions*, Indian J. Math., **43**(2001), 341–350.

[14] Q. Zhang, *Normal families of meromorphic functions concerning shared values*, J. Math. Anal. Appl., **338**(2008), 545–551.

[15] L. Zalcman, *Normal families: New perspectives*, Bull. Amer. Math. Soc., **35** (1998), 215–230.

[16] W. J. Yuan and M. L. Fang, *Further results of the normality for meromorphic function families*, J. Guangzhou University (Natural Science Edition), **1**(2002), 1–9.

[17] Y. F. Wang and M. L. Fang, *Picard values and normal families of meromorphic functions with multiple zeros*, Acta Math. N. S., **14**(1998), 17–26. .

[18] N. Steinmetz, Rational iteration, De Gruyter Studies in Mathematics 16, Walter de Gruyter, Berlin, New Youk, 1993.

[19] J. M. Chang, *Normality of meromorphic functions whose derivatives have* 1*-points*, Arch. Math., **94**(2010), 555–564.

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