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FOUR DIMENSIONAL TAUBERIAN THEOREMS VIA BLOCK DOMINATED MATRICES

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Abstract. The goal of this paper is to present simple multidimmensional conditions to ensure that $T_{m,n} = \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}x_{k,l}$ is included by P-convergence.

1. Introduction

The goal of this paper is to present simple conditions to ensure that $T_{m,n} = \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}x_{k,l}$ is included by P-convergence. That is

$$P - \lim_{k,l} x_{k,l} = P - \lim_{k,l} T_{k,l}$$

whenever $P - \lim_{k,l} T_{k,l}$ exists. The method of proof to be used here is suggested by a argument employed by R. P. Agnew in [1]. R. P. Agnew uses two dimensional matrix transformation and single dimension sequences. In this paper the author will present a multidimensional analog of R. P. Agnew results.

2. Definitions, Notations and Preliminary Results

Definition 2.1. (Pringsheim, [6]). A double sequence $x = [x_{k,l}]$ has **Pringsheim limit** L (denoted by P-lim $x = L$) provided that given $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > N$. We shall describe such an x more briefly as "**P-convergent**".

Definition 2.2. (Pringsheim, [6]). A double sequence x is called **definitely divergent**, if for every (arbitrarily large) $G > 0$ there exist two natural numbers n_1 and n_2 such that $|x_{n,k}| > G$ for $n \geq n_1, k \geq n_2$.

Definition 2.3. (Patterson, [4]). The double sequence $[y]$ is a double **subsequence** of the sequence $[x]$ provided that there exist two increasing double index sequences $\{n_j\}$

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and $\{k_j\}$ such that if $z_j = x_{n_j,k_j}$, then y is formed by

$$\begin{array}{cccc} z_1 & z_2 & z_5 & z_{10} \\ z_4 & z_3 & z_6 & - \\ z_9 & z_8 & z_7 & - \\ - & - & - & -. \end{array}$$

Definition 2.4. (Patterson [4]). A number λ is called a **Pringsheim limit point** of the double sequence $[x]$ provided that there exists a subsequence $[y]$ of $[x]$ that has Pringsheim limit λ : P-lim$[y] = \lambda$.

With the definition of P-convergence and the notion of subsequences presented in [4] it is clear that a double sequence $[x]$ is *divergent* in the Pringsheim sense (P-divergent) provided that $[x]$ is not P-convergent. This is equivalent to the following: a double sequence $[x]$ is P-divergent if and only if either $[x]$ contains two subsequences with distinct finite limit points or $[x]$ contains an unbounded subsequence. Observe that, if $[x]$ contains an unbounded subsequence then $[x]$ also contains a definite divergent subsequence. In 1926 Robison presented a four dimensional analog of regularity for double sequences in which he added an additional assumption of boundedness: A four dimensional matrix A is said to be RH-*regular* if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit. The necessary and sufficient conditions for RH-regularity of $(Ax)_{m,n} = \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}x_{k,l}$ are as follows:

RH_1: P-$\lim_{m,n} a_{m,n,k,l} = 0$ for each k and l;
RH_2: P-$\lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} = 1$;
RH_3: P-$\lim_{m,n} \sum_{k=1}^{\infty} |a_{m,n,k,l}| = 0$ for each l;
RH_4: P-$\lim_{m,n} \sum_{l=1}^{\infty} |a_{m,n,k,l}| = 0$ for each k;
RH_5: $\sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}|$ is P-convergent; and
RH_6: there exist finite positive integers A and B such that
$\sum_{k,l>B} |a_{m,n,k,l}| < A$.

The following definition for *Pringsheim limit superior and inferior* is presented [5]: Let $[x] = \{x_{k,l}\}$ be a double sequence of real numbers and for each n, let $\alpha_n = \sup_n\{x_{k,l} : k,l \geq n\}$. The *Pringsheim limit superior* of $[x]$ is defined as follows:

(1) if $\alpha = +\infty$ for each n, then P-$\limsup[x] := +\infty$;

(2) if $\alpha < \infty$ for some n, then P-$\limsup[x] := \inf_n\{\alpha_n\}$.

Similarly, let $\beta_n = \inf_n\{x_{k,l} : k,l \geq n\}$ then the *Pringsheim limit inferior* of $[x]$ is defined as follows:

(1) if $\beta_n = -\infty$ for each n, then P-$\liminf[x] := -\infty$;

(2) if $\beta_n > -\infty$ for some n, then P-$\liminf[x] := \sup_n\{\beta_n\}$.

3. Main Results

Definition 3.1. A four dimensional matrix A is said to $\{B^A_{m,n}(\zeta,\eta)\}$-dominated if ζ & η are nonnegative interges and

$$P-\liminf_{m,n}\left\{\Big|\sum_{(k,l)\in B^A_{m,n}(\zeta,\eta)} a_{m,n,k,l}\Big| - \sum_{(k,l)\notin B^A_{m,n}(\zeta,\eta)} \Big|a_{m,n,k,l}\Big|\right\} > 0$$

where

$$B^A_{m,n}(\zeta,\eta) = \{(k,l) : A = [a_{m,n,k,l}], m-\zeta < k \le m+\zeta,\ \&\ n-\eta < l \le n+\eta\}.$$

Theorem 3.1. *If A is an RH-regular real matrix that is $\{B^A_{m,n}(\zeta,\eta)\}$-dominated and $[x]$ is a bounded double sequence such that $[Ax]$ is P-convergent then $[x]$ is P-convergent.*

Proof. Suppose $[x]$ is a bounded real double sequence which is divergent in the Pringsheim sense. We shall use this sequence to show that there does not exists r real such that $P-\lim(Ax)_{m,n} = r$. Let $R = P-\limsup_{k,l}|x_{k,l}-r|$. The regularity condition RH_2 implies the following:

$$(Ax)_{m,n} - r = o(1) + \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}(x_{k,l}-r).$$

Suppose $0 < \epsilon < R$ and choose k_0 and l_0 large such that for $k > k_0$ and $l > l_0$ implies $|x_{k,l}-r| < R+\epsilon$. This yields the following:

$$\begin{aligned}
|(Ax)_{m,n} - r| &\ge o(1) + \Big|\sum_{(k,l)\in B^A_{m,n}(\zeta,\eta)} a_{m,n,k,l}(x_{k,l}-r)\Big| \\
&\quad - \sum_{(k,l)\notin B^A_{m,n}(\zeta,\eta)} \Big|a_{m,n,k,l}\Big|\Big|x_{k,l}-r\Big| \\
&\ge o(1) + \Big|\sum_{(k,l)\in B^A_{m,n}(\zeta,\eta)} a_{m,n,k,l}(x_{k,l}-r)\Big| \\
&\quad - \sum_{(k,l)\notin B^A_{m,n}(\zeta,\eta)\&k\ge k_0,l\ge l_0} \Big|a_{m,n,k,l}\Big|\Big|x_{k,l}-r\Big| \\
&> o(1) + \Big|\sum_{(k,l)\in B^A_{m,n}(\zeta,\eta)} a_{m,n,k,l}(x_{k,l}-r)\Big| \\
&\quad -(R+\epsilon) \sum_{(k,l)\notin B^A_{m,n}(\zeta,\eta)\&k\ge k_0,l\ge l_0} \Big|a_{m,n,k,l}\Big| \\
&\ge o(1) + \Big|\sum_{(k,l)\in B^A_{m,n}(\zeta,\eta)} a_{m,n,k,l}(x_{k,l}-r)\Big| \\
&\quad -R \sum_{(k,l)\notin B^A_{m,n}(\zeta,\eta)\&k\ge k_0,l\ge l_0} \Big|a_{m,n,k,l}\Big| - \epsilon\|A\|
\end{aligned}$$

where $\|A\| = \sup_{m,n} \sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}|$. We can now assume that there are infinitely many m and n such that for $(k,l) \in B_{m,n}^A(\zeta,\eta)$ we obtain the following $|x_{k,l} - r| > R - \epsilon$. Thus

$$\begin{aligned}\left|(Ax)_{m,n} - r\right| &> o(1) + \Big|\sum_{(k,l)\in B_{m,n}^A(\zeta,\eta)} a_{m,n,k,l}(x_{k,l} - r)\Big| \\ &\quad -R \sum_{(k,l)\notin B_{m,n}^A(\zeta,\eta)\&k\geq k_0,l\geq l_0} \left|a_{m,n,k,l}\right| - \epsilon\|A\| \\ &\geq o(1) + (R-\epsilon)\Big|\sum_{(k,l)\in B_{m,n}^A(\zeta,\eta)} a_{m,n,k,l}\Big| \\ &\quad -R \sum_{(k,l)\notin B_{m,n}^A(\zeta,\eta)\&k\geq k_0,l\geq l_0} \left|a_{m,n,k,l}\right| - \epsilon\|A\|\end{aligned}$$

$$\begin{aligned}\left|(Ax)_{m,n} - r\right| &> o(1) - \epsilon\left\{\|A\| + \Big|\sum_{(k,l)\in B_{m,n}^A(\zeta,\eta)} a_{m,n,k,l}\Big|\right\} \\ &\quad +R\left\{\Big|\sum_{(k,l)\in B_{m,n}^A(\zeta,\eta)} a_{m,n,k,l}\Big| - \sum_{(k,l)\notin B_{m,n}^A(\zeta,\eta)} \left|a_{m,n,k,l}\right|\right\} \\ &\geq o(1) - 2\epsilon\|A\| + R\left\{\Big|\sum_{(k,l)\in B_{m,n}^A(\zeta,\eta)} a_{m,n,k,l}\Big| - \sum_{(k,l)\notin B_{m,n}^A(\zeta,\eta)} \left|a_{m,n,k,l}\right|\right\}.\end{aligned}$$

Since ϵ is an arbitrarily small positive number, we obtain the following

$$P - \liminf_{m,n} \left|(Ax)_{m,n} - r\right| > 0.$$

Thus $[Ax]$ diverges in the Pringsheim sense.

Theorem 3.2. *Suppose A is a four dimensional summability matrix method and $[x]$ is unbounded double sequence for which $(Ax)_{m,n} = \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}x_{k,l}$ exist for each m and n and $[x]$ contains an unbounded subsequence (i.e. unbounded in the Pringsheim sense). Then $(Ax)_{m,n}$ is unbounded in the Pringsheim sense.*

Proof. Suppose $[x]$ is unbounded in the Pringsheim sense then there are infinity many pairs (m,n) in $B_{m,n}^A(\zeta,\eta)$ such that

$$x_{k,l} \leq x_{m,n} \text{ for } (k,l) \notin B_{m,n}^A(\zeta,\eta).$$

For pairs of (m,n) satisfying the above condition we obtain the following:

$$\left|(Ax)_{m,n}\right| = \Big|\sum_{(k,l)\in B_{m,n}^A(\zeta,\eta)} a_{m,n,k,l} + \sum_{(k,l)\notin B_{m,n}^A(\zeta,\eta)} a_{m,n,k,l}\Big|$$

$$\geq \Big| \sum_{(k,l)\in B^A_{m,n}(\zeta,\eta)} a_{m,n,k,l}x_{k,l}\Big| - \sum_{(k,l)\notin B^A_{m,n}(\zeta,\eta)} \Big|a_{m,n,k,l}\Big| \Big|x_{k,l}\Big|$$

$$\geq \left[\Big| \sum_{(k,l)\in B^A_{m,n}(\zeta,\eta)} a_{m,n,k,l}\Big| - \sum_{(k,l)\notin B^A_{m,n}(\zeta,\eta)} \Big|a_{m,n,k,l}\Big|\right] \Big|x_{m,n}\Big|.$$

Hence $P - \limsup_{m,n} |(Ax)_{m,n}| = \infty$. Thus $(Ax)_{m,n}$ is unbounded in the Pringsheim sense.

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