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A GENERALIZATION OF AN IDENTITY INVOLVING THE INVERSES OF BINOMIAL COEFFICIENTS

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Abstract. By applying the integral expression of the inverses of binomial coefficients, the authors generalize the Sury's identity.

1. Introduction

As usual, the binomial coefficients are defined by

$$\binom{n}{m} = \begin{cases} \dfrac{n!}{m!(n-m)!} & \text{if } n \geq m, \\ 0 & \text{if } n < m, \end{cases}$$

where n and m are nonnegative integers. Sury [3] used

$$\binom{n}{m}^{-1} = (n+1)\int_0^1 t^m(1-t)^{n-m}dt \tag{1}$$

to obtain that

$$\sum_{m=0}^{n} \frac{1}{\binom{n}{m}} = \frac{n+1}{2^n}\sum_{m=0}^{n}\frac{2^m}{m+1} = \frac{n+1}{2^n}\sum_{j\ odd}\frac{1}{j}\binom{n+1}{j}. \tag{2}$$

It's necessary to point out that the first equation of identity (2) is identity (2.25) in Could's collection [1]. Using the integral identity (1), many results have been obtained for the finite and infinite sums related to the reciprocals of binomial coefficients. See [2, 3, 4, 5, 6, 7]. And in [7], Sury, Wang and Zhao generalized identity (2) and obtained a polynomial version.

The purpose of this paper is to extend the identity (2) much further and obtain some interesting results.

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2. Main results

Theorem 2.1. *Let n, k, and r be nonnegative integers, then in the ring $Q[T]$ of rational polynomials, the identity*

$$\sum_{m=k}^{n}\frac{T^m(1-T)^{n-m}}{\binom{n+r}{m}} = (n+r+1)\sum_{i=0}^{n-k}\frac{T^{n-i}(1-T)^{n-k+1}}{(r+k+i+1)\binom{r+k+i}{k}} + (n+r+1)\sum_{i=0}^{r}\binom{r}{i}(-1)^i\sum_{p=k}^{n}\frac{T^{n+1}(1-T)^{n-p}}{i+p+1} \tag{3}$$

holds for $k \le n$. An equivalent form is that for $\lambda \neq -1$,

$$\sum_{m=k}^{n}\frac{\lambda^m}{\binom{n+r}{m}} = (n+r+1)\sum_{i=0}^{n-k}\frac{\lambda^{k+i}}{(\lambda+1)^{i+1}}\sum_{p=0}^{n+r-k-i}\binom{n+r-k-i}{p}(-1)^p\frac{1}{p+k+1} + (n+r+1)\frac{\lambda^{n+1}}{(\lambda+1)^{n+2}}\sum_{i=0}^{r}\binom{r}{i}(-1)^i\sum_{p=k}^{n}\frac{(\lambda+1)^{p+1}}{i+p+1}. \tag{4}$$

Moreover,

$$\sum_{m=k}^{n}\frac{\lambda^m}{\binom{n+r}{m}} = (n+r+1)\sum_{i=0}^{n-k}\lambda^{n-i}\sum_{p=0}^{i+r}\binom{i+r}{p}(-1)^p\frac{1}{n+p-i+1}. \tag{5}$$

Proof. For a fixed real number λ, letting $I_{n,k,r}(\lambda) = \sum_{m=k}^{n}\frac{\lambda^m}{\binom{n+r}{m}}$, it follows from identity (1) that

$$I_{n,k,r}(\lambda) = (n+r+1)\int_0^1 (1-t)^{n+r}\sum_{m=k}^{n}\left(\frac{\lambda t}{1-t}\right)^m dt = (n+r+1)\int_0^1 (1-t)^r(\lambda t)^k\frac{(\lambda t)^{n-k+1}-(1-t)^{n-k+1}}{(\lambda+1)t-1}dt.$$

Putting $s = (\lambda+1)t-1$, we get

$$I_{n,k,r}(\lambda) = (n+r+1)\frac{\lambda^k}{(\lambda+1)^{k+r+1}}\int_{-1}^{\lambda}(\lambda-s)^r(s+1)^k\frac{\lambda^{n+1-k}(s+1)^{n+1-k}-(\lambda-s)^{n+1-k}}{s(\lambda+1)^{n+1-k}}ds. \tag{6}$$

Then we have that $I_{n,k,r} = (n+r+1)(I_1+I_2)$, where

$$I_1 = \frac{\lambda^k}{(\lambda+1)^{n+r+2}}\int_{-1}^{\lambda}(\lambda-s)^r\frac{(s+1)^k}{s}(\lambda^{n+1-k}-(\lambda-s)^{n+1-k})ds$$

and

$$I_2 = \frac{\lambda^{n+1}}{(\lambda+1)^{n+r+2}}\int_{-1}^{\lambda}(\lambda-s)^r\frac{(\lambda+1)^{n+1}-(s+1)^k}{s}ds.$$

By writting

$$\frac{(\lambda+1)^{n+1}-(s+1)^k}{s} = \sum_{p=k}^{n}(s+1)^p$$

and interchanging the order of the summation and the integration, we have

$$I_2 = \frac{\lambda^{n+1}}{(\lambda+1)^{n+2}} \sum_{i=0}^{r} \binom{r}{i} (-1)^i \sum_{p=k}^{n} \frac{(\lambda+1)^{p+1}}{i+p+1}.$$

Similarly,

$$I_1 = \sum_{i=0}^{n-k} \frac{\lambda^{n-i}}{(\lambda+1)^{n+1-k-i}} \sum_{p=0}^{i+r} \binom{i+r}{p} (-1)^p \frac{1}{p+k+1}.$$

The above manipulation are valid when λ is any real number different from -1. So, we have, for $\lambda \neq -1$,

$$\begin{aligned} I_{n,k,r}(\lambda) = (n+r+1) &\sum_{i=0}^{n-k} \frac{\lambda^{k+i}}{(\lambda+1)^{i+1}} \sum_{p=0}^{n+r-k-i} \binom{n+r-k-i}{p} (-1)^p \frac{1}{p+k+1} \\ &+ (n+r+1) \frac{\lambda^{n+1}}{(\lambda+1)^{n+2}} \sum_{i=0}^{r} \binom{r}{i} (-1)^i \sum_{p=k}^{n} \frac{(\lambda+1)^{p+1}}{i+p+1}. \end{aligned}$$

This proves (4). Now, using (4) with λ replaced by $\frac{\theta}{1-\theta}$ for $\theta \neq 1$ (because $\frac{\theta}{1-\theta}$ can takes all values except -1), and multiplying both sides with $(1-\theta)^n$, then we have

$$\begin{aligned} \sum_{m=k}^{n} \frac{\theta^m (1-\theta)^{n-m}}{\binom{n+r}{m}} &= (n+r+1) \sum_{i=0}^{n-k} \theta^{n-i} (1-\theta)^{n-k+1} \sum_{p=0}^{r+i} \binom{r+i}{p} (-1)^p \frac{1}{p+k+1} \\ &\quad + (n+r+1)\theta^{n+1} \sum_{i=0}^{r} \binom{r}{i} (-1)^i \sum_{p=k}^{n} \frac{(1-\theta)^{n-p}}{i+p+1}. \end{aligned} \tag{7}$$

Now, if we compare the coefficients of θ^k on both sides of (7), we can obtain

$$\frac{1}{\binom{n+r}{k}} = (n+r+1) \sum_{p=0}^{k} \binom{k}{p} (-1)^{k-p} \frac{1}{n+r-p+1}. \tag{8}$$

In terms of $n+r = s+k$, then we have

$$\sum_{p=0}^{k} \binom{k}{p} (-1)^p \frac{1}{s+p+1} = \frac{1}{(s+k+1)\binom{s+k}{k}}. \tag{9}$$

To show that (7) is a polynomial identity over Q, it is means that both sides of (7) have to coincide in $Q[T]$. Using (9) in (7), we can get the desired result (3). From (6), we can also obtain (5), the proof of which is not detailed here.

Corollary 2.2. *For any nonnegative integer n, k, and r, then*

$$\sum_{m=k}^{n} \frac{(-1)^{n-m}}{\binom{n+r}{m}} = (n+r+1) \sum_{i=0}^{n-k} (-1)^i \frac{\binom{n-k+1}{i}}{\binom{k+r+i}{k}(k+r+i+1)}. \tag{10}$$

Proof. Comparing the coefficients of θ^n on both sides of (7), we can obtain the desired result.

Theorem 2.3. *Let n and r be nonnegative integers. Then*

$$\sum_{m=0}^{n} \frac{1}{\binom{n+r}{m}} = \sum_{m=0}^{n} \frac{1}{\binom{n+r}{m+r}} = \frac{n+r+1}{2^{n+1}} \sum_{m=0}^{n} \frac{2^m}{m+r+1}\left(1+\binom{m+r}{m}^{-1}\right) \tag{11}$$

$$= \frac{n+r+1}{2^{n+r}} \sum_{\substack{j\ odd,\\ i\ even}} \binom{n+1}{j}\binom{r}{i}\frac{1}{i+j}. \tag{12}$$

Proof. The particular case $k=0$, $\lambda=1$ of (4) in theorem 2.1 gives (11). Now, let us to see the second identity, it follows from identity (1) that

$$\begin{aligned}
\sum_{m=0}^{n} \frac{1}{\binom{n+r}{m}} &= \sum_{m=0}^{n} \frac{1}{\binom{n+r}{n-m}} = \sum_{m=0}^{n} \frac{1}{\binom{n+r}{m+r}} = (n+r+1)\sum_{m=0}^{n}\int_0^1 t^{m+r}(1-t)^{n-m}dt \\
&= (n+r+1)\int_0^1 t^r(1-t)^n \sum_{m=0}^{n} t^m(1-t)^{-m}dt \\
&= (n+r+1)\int_0^1 t^r \frac{t^{n+1}-(1-t)^{n+1}}{2t-1}dt \\
&= \frac{n+r+1}{2}\int_{-1}^1 \left(\frac{1+s}{2}\right)^r \frac{(\frac{1+s}{2})^{n+1}-(\frac{1-s}{2})^{n+1}}{s}ds \\
&= \frac{n+r+1}{2^{n+r+2}}\int_{-1}^1 (1+s)^r \frac{(1+s)^{n+1}-(1-s)^{n+1}}{s}ds \\
&= \frac{n+r+1}{2^{n+r+2}}\sum_{j=0}^{n+1}\binom{n+1}{j}\left(1-(-1)^j\right)\int_{-1}^1 (1+s)^r s^{j-1}ds \\
&= \frac{n+r+1}{2^{n+r+2}}\sum_{j=0}^{n+1}\binom{n+1}{j}\left(1-(-1)^j\right)\int_{-1}^1 \sum_{i=0}^{r}\binom{r}{i} s^{i+j-1}ds \\
&= \frac{n+r+1}{2^{n+r+2}}\sum_{j=0}^{n+1}\left(1-(-1)^j\right)\binom{n+1}{j}\sum_{i=0}^{r}\binom{r}{i}\frac{1+(-1)^{i+j+1}}{i+j} \\
&= \frac{n+r+1}{2^{n+r+1}}\sum_{j\ odd}\binom{n+1}{j}\sum_{i=0}^{r}\binom{r}{i}\frac{1+(-1)^i}{i+j} \\
&= \frac{n+r+1}{2^{n+r}}\sum_{j\ odd}\sum_{i\ even}\binom{n+1}{j}\binom{r}{i}\frac{1}{i+j} \\
&= \frac{n+r+1}{2^{n+r}}\sum_{\substack{j\ odd,\\ i\ even}}\binom{n+1}{j}\binom{r}{i}\frac{1}{i+j}.
\end{aligned}$$

This proves identity (12).

Then in theorem 2.3, when $r=0$, we obtain (2) directly, and when $r=1$, $r=2$ or $r=n$, we give the corresponding results as the following corollary.

Corollary 2.4. *For any nonnegative integer n, then we have*

$$\sum_{m=0}^{n}\frac{1}{\binom{n+1}{m}}=\frac{n+2}{2^{n+1}}\sum_{m=0}^{n}\frac{2^m}{m+1}=\frac{n+2}{2^{n+1}}\sum_{j\ odd}\frac{1}{j}\binom{n+1}{j}, \tag{13}$$

$$\begin{aligned}\sum_{m=0}^{n}\frac{1}{\binom{n+2}{m}}&=\frac{n+3}{2^{n+1}}\sum_{m=0}^{n}2^m\frac{(m+2)(m+1)+2!}{(m+3)(m+2)(m+1)}\\&=\frac{n+3}{2^{n+1}}\sum_{j\ odd}\frac{j+1}{j(j+2)}\binom{n+1}{j},\end{aligned} \tag{14}$$

$$\begin{aligned}\sum_{m=0}^{n}\frac{1}{\binom{2n}{m}}&=\frac{2n+1}{2^{n+1}}\sum_{m=0}^{n}\frac{2^m}{m+n+1}\left(1+\binom{m+n}{m}^{-1}\right)\\&=\frac{2n+1}{2^{2n}}\sum_{\substack{j\ odd,\\ i\ even}}\binom{n+1}{j}\binom{n}{i}\frac{1}{i+j}.\end{aligned} \tag{15}$$

From (11) and (12), we can also obtain the following corollary.

Corollary 2.5. *For any nonnegative integer n and r, we have*

$$\sum_{m=0}^{n}\frac{2^{m+r-1}}{m+r+1}\left(1+\binom{m+r}{m}^{-1}\right)=\sum_{\substack{j\ odd,\\ i\ even}}\binom{n+1}{j}\binom{r}{i}\frac{1}{i+j}. \tag{16}$$

For example, when $r=0$ or $r=1$, we have

$$\sum_{m=0}^{n}\frac{2^m}{m+1}=\sum_{j\ odd}\frac{1}{j}\binom{n+1}{j}. \tag{17}$$

When $r=2$, we obtain

$$\sum_{m=0}^{n}\frac{2^m}{m+3}\left(1+\binom{m+2}{m}^{-1}\right)=\sum_{j\ odd}\frac{j+1}{j(j+2)}\binom{n+1}{j}. \tag{18}$$

When $r=(k-1)n$, where $k\in N^+$, we obtained a rather general theorem.

Theorem 2.6. *For any nonnegative integer n, and $k\in N^+$, then*

$$\sum_{m=0}^{n}\frac{1}{\binom{kn}{m}}=\frac{kn+1}{2^{n+1}}\sum_{m=0}^{n}\frac{2^m}{m+(k-1)n+1}\left(1+\binom{m+kn-n}{m}^{-1}\right)$$

$$= \frac{kn+1}{2^{kn}} \sum_{\substack{j\ odd,\\ i\ even}} \binom{n+1}{j}\binom{kn-n}{i}\frac{1}{i+j}. \tag{19}$$

Proof. When $r=(k-1)n$, substituting it into (11) and (12), we can immediately obtain the desired results.

When $k=1$, from (19) we can also obtain the (2). Next, we use another approach to evaluate the sums $\sum_{m=0}^{n}\frac{1}{\binom{n+1}{m}}$ and $\sum_{m=0}^{n}\frac{1}{\binom{n+2}{m}}$.

Theorem 2.7. *For any nonnegative integer n, then*

$$\sum_{m=0}^{n}\frac{1}{\binom{n+1}{m}} = \frac{n+2}{2^{n+1}}\sum_{j\ odd}\frac{1}{j}\binom{n+2}{j} - 1, \tag{20}$$

$$\sum_{m=0}^{n}\frac{1}{\binom{n+2}{m}} = \frac{n+3}{2^{n+2}}\sum_{m=0}^{n}\frac{2^m}{m+1} + \frac{1}{2} - \frac{1}{2(n+2)} \tag{21}$$

$$= \frac{n+3}{2^{n+2}}\sum_{j\ odd}\frac{1}{j}\binom{n+3}{j} - \frac{1}{n+2} - 1. \tag{22}$$

Proof. First, we have

$$\sum_{m=0}^{n}\frac{1}{\binom{n+1}{m}} = \sum_{m=0}^{n+1}\frac{1}{\binom{n+1}{m}} - \frac{1}{\binom{n+1}{n+1}} = \frac{n+2}{2^{n+1}}\sum_{j\ odd}\frac{1}{j}\binom{n+2}{j} - 1$$

and

$$\begin{aligned}
\sum_{m=0}^{n}\frac{1}{\binom{n+2}{m}} &= \sum_{m=0}^{n+2}\frac{1}{\binom{n+2}{m}} - \frac{1}{\binom{n+2}{n+1}} - \frac{1}{\binom{n+2}{n+2}} = \frac{n+3}{2^{n+2}}\sum_{m=0}^{n+2}\frac{2^m}{m+1} - \frac{1}{n+2} - 1\\
&= \frac{n+3}{2^{n+2}}\Big(\frac{2^{n+2}}{n+3} + \frac{2^{n+1}}{n+2} + \sum_{m=0}^{n}\frac{2^m}{m+1}\Big) - \frac{1}{n+2} - 1\\
&= \frac{n+3}{2^{n+2}}\sum_{m=0}^{n}\frac{2^m}{m+1} + \frac{1}{2} - \frac{1}{2(n+2)}.
\end{aligned}$$

On the other hand ,

$$\sum_{m=0}^{n}\frac{1}{\binom{n+2}{m}} = \sum_{m=0}^{n+2}\frac{1}{\binom{n+2}{m}} - \frac{1}{\binom{n+2}{n+1}} - \frac{1}{\binom{n+2}{n+2}} = \frac{n+3}{2^{n+2}}\sum_{j\ odd}\frac{1}{j}\binom{n+3}{j} - \frac{1}{n+2} - 1,$$

which conclude the result.

From (13), (14), (20), (21) and (22), after some computation, we can obtain the following result.

Theorem 2.8. *For any nonnegative integer n, then*

$$\sum_{m=0}^{n} 2^m \frac{m^2+m+2}{(m+3)(m+2)(m+1)} = \frac{(n+1)2^{n+1}}{(n+2)(n+3)}. \tag{23}$$

$$\sum_{\substack{j\ odd,\\ j\le n+2}} \frac{1}{j}\binom{n+1}{j-1} = \frac{2^{n+1}}{n+2}. \tag{24}$$

$$\sum_{\substack{j\ odd,\\ j\le n+3}} \frac{1}{j}\left(\binom{n+3}{j} - \frac{2(j+1)}{j+2}\binom{n+1}{j}\right) = \frac{2^{n+2}}{n+2}. \tag{25}$$

Proof. From the first equation of (14) and (21), we can get (23). From the second equation of (13) and (20), we obtain (24) easily. From the second equation of (14) and (22), we obtain (25).

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