

On Certain Subclass of p-valent Meromorphically Starlike Functions With Alternating Coefficients

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Abstract

A certain subclass $B_m(p, \alpha, \lambda, \ell, A, B)$ consisting of meromorphic p-valent functions with alternating coefficient in $U^* = \{z : z \in C : 0 < |z| < 1\}$ is introduced. In this paper we obtain coefficient inequalities, distortion theorem, closure theorems and class preserving integral operators for functions in the class $B_m(p, \alpha, \lambda, \ell, A, B)$ are obtained .

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1. Introduction

Let $\Sigma(p)$ denote the class of functions of the form :

$$f(z) = \frac{a_{-p}}{z^p} + \sum_{k=1}^{\infty} a_k z^k \quad (a_{-p} \neq 0; \ p \in N := \{1, 3, 5, \dots\}), \quad (1.1)$$

which are regular in the punctured disc $U^* = \{z : z \in C : 0 < |z| < 1\} = U \setminus \{0\}$, see[11].

Definition 1. Let f, g be analytic in U . Then g is said to be subordinate to f , written $g \prec f$, if there exists a schwarz function $w(z)$, which is analytic in

U , with $w(0) = 0$ and $|w(z)| < 1$ ($z \in U$) such that $g(z) = f(w(z))$ ($z \in U$). Hence $g(z) \prec f(z)$ ($z \in U$), then $g(0) = f(0)$ and $g(U) \subset f(U)$. In particular, if the function $f(z)$ is univalent in U , we have the following (e.g. [7]; [8]):
 $g(z) \prec f(z)$ ($z \in U$) if and only if $g(0) = f(0)$ and $g(U) \subset f(U)$.

Definition 2. For functions $f(z) \in \sum(p)$ given by (1.1) and $g(z) \in \sum(p)$ defined by

$$g(z) = \frac{b_{-p}}{z^p} + \sum_{k=1}^{\infty} b_k z^k \quad (b_k \geq 0, p \in N) \quad (1.2)$$

we define the convolution (or Hadamard product) of $f(z)$ and $g(z)$ by

$$(f * g)(z) = \frac{a_{-p} b_{-p}}{z^p} + \sum_{k=1}^{\infty} a_k b_k z^k \quad (p \in N, z \in U). \quad (1.3)$$

Now, using the integral operator $L_p^m(\lambda, \ell)$ ($\ell > 0; \lambda \geq 0; p \in N; m \in N_0; z \in U^*$) introduced by El-Ashwah [4], the for function $f(z) \in \sum(p)$ given by (1.1) as follows:

$$L_p^m(\lambda, \ell) f(z) = \frac{a_{-p}}{z^p} + \sum_{k=1}^{\infty} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^m a_k z^k. \quad (1.4)$$

It is easily verified from (1.4) that

$$\lambda z(L_p^{m+1}(\lambda, \ell) f(z))' = \ell L_p^m(\lambda, \ell) f(z) - (\ell + p\lambda) L_p^{m+1}(\lambda, \ell) f(z) \quad (\lambda > 0), \quad (1.5)$$

we note that:

- (i) $L_p^\alpha(1, 1) f(z) = p_p^\alpha f(z)$ (see Aqlan et al. [3]);
- (ii) $L_1^\alpha(1, \beta) f(z) = p_\beta^\alpha f(z) (a_{-p} = 1)$ (see Lashin [6]);
- (iii) $L(1, \gamma) f(z) = J(f(z)) (p = 1)$ (see Sh. Najafzadeh [9]);
- (iv) $L_p^m(1, \alpha) f(z) = J_{p, \alpha}^m f(z)$ (see El -Ashwah et. at [5]).

Let $B_m(p, \alpha, \lambda, \ell, A, B)$ denote the class of functions $f(z)$ in $\sum(p)$ that satisfy the condition :

$$\frac{\ell L_p^m(\lambda, \ell) f(z)}{\lambda L_p^{m+1}(\lambda, \ell) f(z)} - \left(\frac{\lambda p + \ell}{\lambda} \right) \prec - \frac{p + [pB + (A - B)(p - \alpha)] z}{1 + Bz}; z \in U^* \quad (1.6)$$

where $\prec$ denotes subordination, $0 \leq \alpha < p$, $-1 \leq A < B \leq 1$, $0 < B \leq 1$, $\lambda, \ell > 0$, $p \in N$ and $m \in N_0$.

By definition of subordination, the condition (1.6) is equivalent to

$$\frac{\ell L_p^m(\lambda, \ell)f(z)}{\lambda L_p^{m+1}(\lambda, \ell)f(z)} - \left(\frac{\lambda p + \ell}{\lambda} \right) = \frac{p + [pB + (A - B)(p - \alpha)] w(z)}{1 + Bw(z)}, \quad (1.7)$$

where $w(z) \in H = \{w \text{ regular}, w(0) = 0 \text{ and } |w(z)| < 1, z \in U\}$. It is easy to see that the condition (1.7) is equivalent to

$$\left| \frac{\frac{\ell L_p^m(\lambda, \ell)f(z)}{\lambda L_p^{m+1}(\lambda, \ell)f(z)} - \frac{\ell}{\lambda}}{B \left[\frac{\ell L_p^m(\lambda, \ell)f(z)}{\lambda L_p^{m+1}(\lambda, \ell)f(z)} - \left(\frac{\lambda p + \ell}{\lambda} \right) \right] + [pB + (A - B)(p - \alpha)]} \right| < 1 \quad (z \in U^*). \quad (1.8)$$

We note that:

(i) when $A = -1, B = 1$, we have $f(z) \in B_m(p, \alpha, \lambda, \ell)$ if

$$\operatorname{Re} \left\{ \frac{\ell L_p^m(\lambda, \ell)f(z)}{\lambda L_p^{m+1}(\lambda, \ell)f(z)} - \left(\frac{\lambda p + \ell}{\lambda} \right) \right\} < -\alpha,$$

(ii) when $\alpha = 0$

$$\frac{\ell L_p^m(\lambda, \ell)f(z)}{\lambda L_p^{m+1}(\lambda, \ell)f(z)} - \left(\frac{\lambda p + \ell}{\lambda} \right) \prec -\frac{1 + Az}{1 + Bz}.$$

Let $\sum_a(p)$ be the subclass of $\sum(p)$ consists of functions of the form:

$$f(z) = \frac{a_{-p}}{z^p} + \sum_{k=1}^{\infty} (-1)^{k-1} a_k z^k \quad (a_{-p} \neq 0; a_k \geq 0; p \in N), \quad (1.9)$$

that are regular and p-valent in U^*

Let us write

$$\sum_a^*(p, n, \alpha, \lambda, \ell, A, B) = B_m(p, \alpha, \lambda, \ell, A, B) \cap \sum_a(p). \quad (1.10)$$

In this paper coefficient inequalities, distortion theorem and closure theorems for the class $B_m^*(p, \alpha, \lambda, \ell, A, B)$ are obtained. Finally, the class preserving integral operators of the form

$$F(z) = (c - p + 1)z^{-c-1} \int_0^z t^c f(t) dt \quad (c > p - 1; p \in \mathbb{N}), \quad (1.11)$$

is considered. Techniques used are similar to those of Silverman [10] and Uralegaddi and Ganigi [11], Aouf and Darwish [1], Aouf and Hossen and Lashin [2].

2. Coefficient Inequalities

Theorem 1. Let $f(z) = \frac{a_{-p}}{z^p} + \sum_{k=1}^{\infty} a_k z^k$ be regular and p -valent in U^* . If

$$\begin{aligned} \sum_{k=1}^{\infty} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [(k+p)(1+B) + (A-B)(p-\alpha)] |a_k| \\ \leq (B-A)(p-\alpha) |a_{-p}|, \end{aligned} \quad (2.1)$$

then $f(z) \in B_m(p, \alpha, \lambda, \ell, A, B)$.

Proof: Suppose (2.1) holds for all admissible values of $p, m, \alpha, \lambda, \ell, A$ and B . It suffices to show that

$$\left| \frac{\frac{L_p^m(\lambda, \ell)f(z)}{L_p^{m+1}(\lambda, \ell)f(z)} - 1}{B \left[\frac{L_p^m(\lambda, \ell)f(z)}{L_p^{m+1}(\lambda, \ell)f(z)} - \left(\frac{\lambda}{\ell} p + 1 \right) \right] + \lambda/\ell [pB + (A-B)(p-\alpha)]} \right| < 1 \quad \text{for } |z| < 1$$

we have

$$\begin{aligned} & \left| \frac{\frac{L_p^m(\lambda, \ell)f(z)}{L_p^{m+1}(\lambda, \ell)f(z)} - 1}{B \left[\frac{L_p^m(\lambda, \ell)f(z)}{L_p^{m+1}(\lambda, \ell)f(z)} - \left(\frac{\lambda}{\ell} p + 1 \right) \right] + \lambda/\ell [pB + (A-B)(p-\alpha)]} \right| \\ &= \left| \frac{\sum_{k=1}^{\infty} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} (k+p) a_k z^{p+k}}{(A-B)(p-\alpha) a_{-p} + \sum_{k=1}^{\infty} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [B(k+p) + (A-B)(p-\alpha)] a_k z^{k+p}} \right| \end{aligned}$$

$$\leq \frac{\sum_{k=1}^{\infty} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} (k+p) |a_k|}{(B-A)(p-\alpha) |a_{-p}| - \sum_{k=1}^{\infty} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [B(k+p) + (A-B)(p-a)] |a_k|}.$$

The last expression is bounded above by (1.1) if

$$\begin{aligned} \sum_{k=1}^{\infty} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} (k+p) |a_k| &\leq (B-A)(p-a) |a_{-p}| - \\ \sum_{k=1}^{\infty} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [B(k+p) + (A-B)(p-\alpha)] |a_k| \end{aligned}$$

which is equivalent to

$$\sum_{k=1}^{\infty} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [(k+p)(1+B) + (A-B)(p-\alpha)] |a_k| \leq (B-A)(p-\alpha) |a_{-p}|.$$

This completes the proof of Theorem 1.

Theorem 2. Let $f(z) = \frac{a_{-p}}{z^p} + \sum_{k=1}^{\infty} (-1)^{k-1} a_k z^k$ ($a_{-p} \neq 0; a_k \geq 0, p \in N$) be regular and p -valent in U^* . Then $f(z) \in \sum_a^*(p, m, \alpha, \lambda, \ell, A, B)$ if and only if

$$\begin{aligned} \sum_{k=1}^{\infty} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [(k+p)(1+B) + (A-B)(p-\alpha)] a_k \\ \leq (B-A)(p-\alpha) |a_{-p}|. \end{aligned} \quad (2.2)$$

Proof. In view of Theorem 1, it is sufficient to show the "only if" part. Let us suppose that

$$f(z) = \frac{a_{-p}}{z^p} + \sum_{k=1}^{\infty} (-1)^{k-1} a_k z^k \quad (a_{-p} \neq 0; a_k \geq 0, p \in N)$$

is in $\sum_a^*(p, m, \alpha, \lambda, \ell, A, B)$. Then

$$\left| \frac{\frac{L_p^m(\lambda, \ell) f(z)}{L_p^{m+1}(\lambda, \ell) f(z)} - 1}{B \left[\frac{L_p^{m+1}(\lambda, \ell) f(z)}{L_p^{m+1}(\lambda, \ell) f(z)} - \left(\frac{\lambda}{\ell} p + 1 \right) \right] + \frac{\lambda}{\ell} [pB + (A-B)(p-\alpha)]} \right|$$

$$= \left| \frac{\sum_{k=1}^{\infty} (-1)^{k-1} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} (k+p)a_k z^{k+p}}{(B-A)(p-\alpha)a_{-p} - \sum_{k=1}^{\infty} (-1)^{k-1} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [B(k+p) + (A-B)(p-\alpha)]a_k z^{k+p}} \right| \leq 1$$

for all $z \in U^*$. Using the fact that $\operatorname{Re}(z) \leq |z|$ for all z , it follows that

$$\operatorname{Re} \left\{ \frac{\sum_{k=1}^{\infty} (-1)^{k-1} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} (k+p)a_k z^{k+p}}{(B-A)(p-\alpha)a_{-p} - \sum_{k=1}^{\infty} (-1)^{k-1} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [B(k+p) + (A-B)(p-\alpha)]a_k z^{k+p}} \right\} \leq 1 \quad (z \in U^*). \quad (2.3)$$

Now choose values of z on the real axis so that $\frac{L_p^m(\lambda, \ell)f(z)}{L_p^{m+1}(\lambda, \ell)f(z)}$ is real. Upon clearing the denominator in (2.3) and letting $z \rightarrow 1^-$ through real values, we obtain

$$\sum_{k=1}^{\infty} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} (k+p)a_k \leq (B-A)(p-\alpha) |a_{-p}| - \sum_{k=1}^{\infty} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [B(k+p) + (A-B)(p-\alpha)]a_k.$$

or

$$\sum_{k=1}^{\infty} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [(k+p)(1+B) + (A-B)(p-\alpha)]a_k \leq (B-A)(p-\alpha) |a_{-p}|.$$

Hence the result follows .

Corollary 1. If $f(z) = \frac{a_{-p}}{z^p} + \sum_{k=1}^{\infty} (-1)^{k-1} a_k z^k$ ($a_{-p} \neq 0; a_k \geq 0; p \in N$) is

in the class $\sum_a^*(p, \alpha, \lambda, \ell, A, B)$, then

$$a_k \leq \frac{(B-A)(p-\alpha) |a_{-p}|}{\left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [(k+p)(1+B) + (A-B)(p-\alpha)]} \quad (k \in N). \quad (2.4)$$

Equality holds for the functions of form

$$f_k(z) = \frac{a_{-p}}{z^p} + (-1)^{k-1} \frac{(B-A)(p-\alpha) |a_{-p}|}{\left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [(k+p)(1+B) + (A-B)(p-\alpha)]} z^k, \quad (2.5)$$

where $(k \in N \text{ and } p \in N)$.

3. Distortion Theorems

Theorem 3. Let $f(z) = \frac{a_{-p}}{z^p} + \sum_{k=1}^{\infty} (-1)^{k-1} a_k z^k$ ($a_{-p} \neq 0; a_k \geq 0; p \in N$) is

in the class $\sum_a^*(p, m, \alpha, \lambda, \ell, A, B)$, then for $0 < |z| = r < 1$,

$$\begin{aligned} & \frac{|a_{-p}|}{r^p} - \frac{(B-A)(p-\alpha) |a_{-p}|}{\left[\frac{\ell}{\ell + \lambda(1+p)} \right]^{m+1} [(1+p)(1+B) + (A-B)(p-\alpha)]} r \leq |f(z)| \\ & \leq \frac{|a_{-p}|}{r^p} + \frac{(B-A)(p-\alpha) |a_{-p}|}{\left[\frac{\ell}{\ell + \lambda(1+p)} \right]^{m+1} [(1+p)(1+B) + (A-B)(p-\alpha)]} r \end{aligned} \quad (3.1)$$

with equality for the function

$$f_1(z) = \frac{a_{-p}}{z^p} + \frac{(B-A)(p-\alpha) |a_{-p}|}{\left[\frac{\ell}{\ell + \lambda(1+p)} \right]^{m+1} [(1+p)(1+B) + (A-B)(p-\alpha)]} z \quad \text{at } z = r, ir. \quad (3.2)$$

Proof. Suppose that $f(z)$ in the class $\sum_a^*(p, m, \alpha, \lambda, \ell, A, B)$. In view of Theorem 2, we have

$$\begin{aligned} & \left[\frac{\ell}{\ell + \lambda(1+p)} \right]^{m+1} [(1+p)(1+B) + (A-B)(p-\alpha)] \sum_{k=1}^{\infty} a_k \\ & \leq \sum_{k=1}^{\infty} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [(1+p)(1+B) + (A-B)(p-\alpha)] a_k \\ & \leq (B-A)(p-\alpha) |a_{-p}|, \end{aligned}$$

which evidently yields

$$\sum_{k=1}^{\infty} a_k \leq \frac{(B-A)(p-\alpha) |a_{-p}|}{\left[\frac{\ell}{\ell + \lambda(1+p)} \right]^{m+1} [(1+p)(1+B) + (A-B)(p-\alpha)]} . \quad (3.3)$$

Consequently, we obtain

$$\begin{aligned} |f(z)| &\leq \frac{|a_{-p}|}{r^p} + \sum_{k=1}^{\infty} a_k r^k \leq \frac{|a_{-p}|}{r^p} + r \sum_{k=1}^{\infty} a_k \\ &\leq \frac{|a_{-p}|}{r^p} + \frac{(B-A)(p-\alpha) |a_{-p}|}{\left[\frac{\ell}{\ell + \lambda(1+p)} \right]^{m+1} [(1+p)(1+B) + (A-B)(p-\alpha)]} r, \end{aligned}$$

by (3.3). This gives the right hand inequality of (3.1). Also

$$\begin{aligned} |f(z)| &\geq \frac{|a_{-p}|}{r^p} - \sum_{k=1}^{\infty} a_k r^k \geq \frac{|a_{-p}|}{r^p} - r \sum_{k=1}^{\infty} a_k \\ &\geq \frac{|a_{-p}|}{r^p} - \frac{(B-A)(p-\alpha) |a_{-p}|}{\left[\frac{\ell}{\ell + \lambda(1+p)} \right]^{m+1} [(1+p)(1+B) + (A-B)(p-\alpha)]} r, \end{aligned}$$

which gives the left hand side of (3.1). It can be easily seen that the function $f(z)$ defined by (3.2) is an extremal function for the theorem.

Theorem 4. If $f(z) = \frac{a_{-p}}{z^p} + \sum_{k=1}^{\infty} (-1)^{k-1} a_k z^k$ ($a_{-p} \neq 0; a_k \geq 0, p \in N$) is

in the class $\sum_a^*(p, m, \alpha, \lambda, \ell, A, B)$, then for $0 < |z| = r < 1$,

$$\begin{aligned} \frac{p |a_{-p}|}{r^{p+1}} - \frac{(B-A)(p-\alpha) |a_{-p}|}{\left[\frac{\ell}{\ell + \lambda(1+p)} \right]^{m+1} [(1+p)(1+B) + (A-B)(p-\alpha)]} &\leq |f'(z)| \\ &\leq \frac{p |a_{-p}|}{r^{p+1}} + \frac{(B-A)(p-\alpha) |a_{-p}|}{\left[\frac{\ell}{\ell + \lambda(1+p)} \right]^{m+1} [(1+p)(1+B) + (A-B)(p-\alpha)]} . \quad (3.4) \end{aligned}$$

The result is sharp, the extremal function being of the form (3.2) .

Proof. From Theorem 2, we have

$$\begin{aligned}
& \left[\frac{\ell}{\ell + \lambda(1+p)} \right]^{m+1} [(1+p)(1+B) + (A-B)(p-\alpha)] \sum_{k=1}^{\infty} k a_k \\
& \leq \sum_{k=1}^{\infty} \left[\frac{\ell}{\ell + \lambda(1+p)} \right]^{m+1} [(1+p)(1+B) + (A-B)(p-\alpha)] a_k \\
& \leq (B-A)(p-\alpha) |a_{-p}|,
\end{aligned}$$

which evidently yields

$$\sum_{k=1}^{\infty} k a_k \leq \frac{(B-A)(p-\alpha) |a_{-p}|}{\left[\frac{\ell}{\ell + \lambda(1+p)} \right]^{m+1} [(1+p)(1+B) + (A-B)(p-\alpha)]} \quad (3.5)$$

Consequently, we obtain

$$\begin{aligned}
|f'(z)| & \leq \frac{p |a_{-p}|}{r^{p+1}} + \sum_{k=1}^{\infty} k a_k r^{k-1} \leq \frac{p |a_{-p}|}{r^{p+1}} + \sum_{k=1}^{\infty} k a_k \\
& \leq \frac{p |a_{-p}|}{r^{p+1}} + \frac{(B-A)(p-\alpha) |a_{-p}|}{\left[\frac{\ell}{\ell + \lambda(1+p)} \right]^{m+1} [(1+p)(1+B) + (A-B)(p-\alpha)]},
\end{aligned}$$

by (3.5). Also

$$\begin{aligned}
|f'(z)| & \geq \frac{p |a_{-p}|}{r^{p+1}} - \sum_{k=1}^{\infty} k a_k r^{k-1} \geq \frac{p |a_{-p}|}{r^{p+1}} - \sum_{k=1}^{\infty} k a_k \\
& \geq \frac{p |a_{-p}|}{r^{p+1}} - \frac{(B-A)(p-\alpha) |a_{-p}|}{\left[\frac{\ell}{\ell + \lambda(1+p)} \right]^{m+1} [(1+p)(1+B) + (A-B)(p-\alpha)]}. \quad (3.6)
\end{aligned}$$

This completes the proof of Theorem 4.

4. Closure Theorems.

Let the functions $f_j(z)$ be defined for $j=1,2,\dots,m$, by

$$f_j(z) = \frac{a_{-p,j}}{z^p} + \sum_{k=1}^{\infty} (-1)^{k-1} a_{k,j} z^k \quad (a_{-p,j} > 0; a_{k,j} \geq 0; p \in N) \quad \text{for } z \in U^*. \quad (4.1)$$

Theorem 5. Let the function $f_j(z)$ be defined by (4.1) be in the class $\sum_a^*(p, \alpha, \lambda, \ell, A, B)$ for every $j = 1, 2, \dots, m$. Then the function $F(z)$ defined by

$$F(z) = \frac{b_{-p}}{z^p} + \sum_{k=1}^{\infty} (-1)^{k-1} b_k z^k (b_{-p} > 0; b_k \geq 0; p \in N), \quad (4.2)$$

is a member of the class $\sum_a^*(p, \alpha, \lambda, \ell, A, B)$, where

$$b_{-p} = \frac{1}{m} \sum_{j=1}^m a_{-p,j} \quad \text{and} \quad b_k = \frac{1}{m} \sum_{j=1}^m a_{k,j} \quad (k \in N). \quad (4.3)$$

Proof. Since $f_j(z) \in \sum_a^*(p, \alpha, \lambda, \ell, A, B)$, it follows from Theorem 2 that

$$\sum_{k=1}^{\infty} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [(k+p)(1+B) + (A-B)(p-\alpha)] a_{k,j} \leq (B-A)(p-\alpha) |a_{-p,j}| \quad (4.4)$$

for every $j = 1, 2, \dots, m$. Hence,

$$\begin{aligned} & \sum_{k=1}^{\infty} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [(k+p)(1+B) + (A-B)(p-\alpha)] b_k \\ &= \sum_{k=1}^{\infty} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [(k+p)(1+B) + (A-B)(p-\alpha)] \left\{ \frac{1}{m} \sum_{j=1}^m a_{k,j} \right\} \\ &= \frac{1}{m} \sum_{j=1}^m \left(\sum_{k=1}^{\infty} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [(k+p)(1+B) + (A-B)(p-\alpha)] a_{k,j} \right) \\ &\leq (B-A)(p-\alpha) \left(\frac{1}{m} \sum_{j=1}^m a_{-p,j} \right) = (B-A)(p-\alpha) b_{-p}, \end{aligned}$$

which (in view of Theorem 2) implies that $F(z) \in \sum_a^*(p, \alpha, \lambda, \ell, A, B)$.

Theorem 6. The class $\sum_a^*(p, \alpha, \lambda, \ell, A, B)$ is closed under convex linear combination.

Proof. Let the functions $f_j(z)$ ($j = 1, 2$) defined by (4.1) be in the class $\sum_a^*(p, \alpha, \lambda, \ell, A, B)$, it is sufficient to prove that the function

$$H(z) = tf_1(z) + (1-t)f_2(z) \quad (0 \leq t \leq 1) \quad (4.5)$$

is also in the class $\sum_a^*(p, \alpha, \lambda, \ell, A, B)$. Since for $0 \leq t \leq 1$,

$$H(z) = \frac{ta_{-p,1} + (1-t)a_{-p,2}}{z^p} + \sum_{k=1}^{\infty} (-1)^{k-1} \{ta_{k,1} + (1-t)a_{k,2}\} z^k, \quad (4.6)$$

we observe that

$$\begin{aligned} & \sum_{k=1}^{\infty} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [(k+p)(1+B) + (A-B)(p-\alpha)] \{ta_{k,1} + (1-t)a_{k,2}\} \\ &= t \sum_{k=1}^{\infty} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [(k+p)(1+B) + (A-B)(p-\alpha)] a_{k,1} + \\ & (1-t) \sum_{k=1}^{\infty} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [(k+p)(1+B) + (A-B)(p-\alpha)] a_{k,2} \\ &\leq (B-A)(p-\alpha) \{ta_{-p,1} + (1-t)a_{-p,2}\}, \end{aligned} \quad (4.7)$$

which the aid of Theorem 2, hence $H(z) \in \sum_a^*(p, \alpha, \lambda, \ell, A, B)$. This completes the proof of Theorem 5.

Theorem 7. Let

$$f_0(z) = \frac{a_{-p}}{z^p} \quad (4.8)$$

and

$$f_k(z) = \frac{a_{-p}}{z^p} + (-1)^{k-1} \frac{(B-A)(p-\alpha)|a_{-p}|}{\left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} \{(k+p)(1+B) + (A-B)(p-\alpha)\}} z^k \quad (k \in N). \quad (4.9)$$

Then $f(z) \in \sum_a^*(p, \alpha, \lambda, \ell, A, B)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=0}^{\infty} \mu_k f_k(z), \text{ where } \mu_k \geq 0 \ (k \geq 0) \text{ and } \sum_{k=0}^{\infty} \mu_k = 1. \quad (4.10)$$

Proof. Suppose that

$$f(z) = \sum_{k=0}^{\infty} \mu_k f_k(z), \text{ where } \mu_k \geq 0 \ (k \geq 0) \text{ and } \sum_{k=0}^{\infty} \mu_k = 1. \text{ Then}$$

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} \mu_k f_k(z) \\ &= \mu_0 f_0(z) + \sum_{k=1}^{\infty} \mu_k f_k(z) \\ &= \frac{a_{-p}}{z^p} + \sum_{k=1}^{\infty} \mu_k (-1)^{k-1} \frac{(B-A)(p-\alpha) |a_{-p}|}{\left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [(k+p)(1+B) + (A-B)(p-\alpha)]} z^k \ (k \in N). \end{aligned} \quad (4.11)$$

Since

$$\begin{aligned} &\sum_{k=1}^{\infty} \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} \{ (k+p)(1+B) + (A-B)(p-\alpha) \} \cdot \\ &\quad \frac{(B-A)(p-\alpha) |a_{-p}|}{\left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} \{ (k+p)(1+B) + (A-B)(p-\alpha) \}} \mu_k \\ &= (B-A)(p-\alpha) |a_{-p}| \sum_{k=1}^{\infty} \mu_k \\ &= (B-A)(p-\alpha) |a_{-p}| (1 - \mu_0) \\ &\leq (B-A)(p-\alpha) |a_{-p}|, \end{aligned} \quad (4.12)$$

we have $f(z) \in \sum_a^*(p, \alpha, \lambda, \ell, A, B)$, by Theorem 2 .

Conversely, suppose that the function $f(z)$ defined by (1.9) belongs to the class $\sum_a^*(p, \alpha, \lambda, \ell, A, B)$. Since

$$a_k \leq \frac{(B-A)(p-\alpha)|a_{-p}|}{\left[\frac{\ell}{\ell + \lambda(k+p)}\right]^{m+1} \{(k+p)(1+B) + (A-B)(p-\alpha)\}} (k \in N) \quad (4.13)$$

by Corollary 1, setting

$$\mu_k = \frac{\left[\frac{\ell}{\ell + \lambda(k+p)}\right]^{m+1} \{(k+p)(1+B) + (A-B)(p-\alpha)\}}{1/\ell(A-B)(p-\alpha)|a_{-p}|} a_k \quad (k \in N) \quad (4.14)$$

and

$$\mu_0 = 1 - \sum_{k=1}^{\infty} \mu_k f_k(z), \quad (4.15)$$

it follows that

$$f(z) = \sum_{k=0}^{\infty} \mu_k f_k(z).$$

This completes the proof of Theorem 6.

5. Integral Operators

Theorem 8. If $f(z) = \frac{a_{-p}}{z^p} + \sum_{k=1}^{\infty} (-1)^{k-1} a_k z^k$ ($a_{-p} \neq 0; a_k \geq 0, p \in N$) is in the class $\sum_a^*(p, m, \alpha, \lambda, \ell, A, B)$, then

$$\begin{aligned} F(z) &= (c-p+1)z^{-c-1} \int_0^z t^c f(t) dt \\ &= \frac{a_{-p}}{z^p} + \sum_{k=1}^{\infty} (-1)^{k-1} \left(\frac{c-p+1}{c+k+1} \right) a_k z^k, \end{aligned} \quad (5.1)$$

$c > p-1$, belongs to the class $\sum_a^*(p, m, \gamma, (p, \alpha, c, A, B), \lambda, \ell, A, B)$, where

$$\gamma(p, \alpha, c, A, B) = p - \frac{(c-p+1)(1+p)(1+B)(p-\alpha)}{(c+p+1)[(1+p)(1+B) + (A-B)(p-\alpha)] - (c-p+1)(A-B)(p-\alpha)}. \quad (5.2)$$

The result is sharp for the function $f(z)$ given by

$$f(z) = \frac{a_{-p}}{z^p} + \frac{(B-A)(p-\alpha)|a_{-p}|}{\left[\frac{\ell}{\ell + \lambda(1+p)} \right]^{m+1} [(1+p)(1+B)+(A-B)(p-\alpha)]} z. \quad (5.3)$$

Proof. Suppose $f(z) = \frac{a_{-p}}{z^p} + \sum_{k=1}^{\infty} (-1)^{k-1} a_k z^k \in \sum_a^*(p, m, \alpha, \lambda, \ell, A, B)$, in view of Theorem 2, we shall find the largest value of γ for which

$$\sum_{k=1}^{\infty} \frac{\left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [(k+p)(1+B)+(A-B)(p-\gamma)]}{(B-A)(p-\alpha)|a_{-p}|} \left(\frac{c-p+1}{c+k+1} \right) a_k \leq 1.$$

It is sufficient to find the range values of γ for which

$$\frac{(c-p+1)[(k+p)(1+B)+(A-B)(p-\gamma)]}{(c+k+1)(p-\gamma)} \leq \frac{[(k+p)(1+B)+(A-B)(p-\alpha)]}{(p-\alpha)} \quad \text{for each } k.$$

Solving the above inequality for γ , we obtain

$$\gamma \leq p - \frac{(c-p+1)(k+p)(1+B)(p-\alpha)}{(c+k+1)[(k+p)(1+B)+(A-B)(p-\alpha)] - (c-p+1)(A-B)(p-\alpha)}.$$

For each $p, \alpha, \lambda, \ell, A, B$ and c fixed let

$$F(k) = p - \frac{(c-p+1)(k+p)(1+B)(p-\alpha)}{(c+k+1)[(k+p)(1+B)+(A-B)(p-\alpha)] - (c-p+1)(A-B)(p-\alpha)}.$$

Then $F(k+1) - F(k) = \frac{D}{G} > 0$ for each k , where

$$D = (c-p+1)(1+B^2)(p-\alpha)(k+p)(k+p+1)$$

and

$$G = \{(c+k+2)[(k+p+1)(1+B)+(A-B)(p-\alpha)] - (c-p+1)(A-B)(p-\alpha)\}.$$

$$\cdot \{(c+k+1)[(k+p)(1+B)+(A-B)(p-\alpha)] - (c-p+1)(A-B)(p-\alpha)\}.$$

Hence $F(k)$ is an increasing function of k . Since

$$F(1) = p - \frac{(c-p+1)(1+p)(1+B)(p-\alpha)}{(c+2)[(1+p)(1+B) + (A-B)(p-\alpha)] - (c-p+1)(A-B)(p-\alpha)},$$

the result follows.

Remark: (i) Putting $a_{-p} = 1, p = 1, m = 0, A = -1$ and $B = 1$ in the above results, we have the results obtained by Uralegaddi and Ganigi [10];

(ii) Putting $a_{-p} = 1, p = 1, A = -1$ and $B = 1$ in the above results, we have the results obtained by Aouf and Darwish [1];

(iii) Putting $a_{-p} = 1$ and $p = 1$ in the above results, we have the results obtained by Aouf et. al. [2].

6. Convolution Properties

Theorem 9. If $f(z)$ and $g(z)$ belong to the class $B_m(p, \alpha, \lambda, \ell, A, B)$, then

$$T(z) = \frac{a_{-p}b_{-p}}{z^p} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2)z^k, \quad (6.1)$$

is in the class $B_m(p, \alpha, \lambda, \ell, A, B)$ such that $A_1 < -\mu^2 + B_1(1 - \mu^2)$, where

$$\mu = \frac{\sqrt{2(p-\alpha)(k+p)(|a_{-p}||b_{-p}|)(B-A)}}{\sqrt{2|a_{-p}||b_{-p}|(B-A)(p-\alpha) + \sqrt{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}[(k+p)(1+B) + (A-B)(p-\alpha)]}}}$$

Proof. Since $f, g \in B_m(p, \alpha, \lambda, \ell, A, B)$. Theorem 2 yields

$$\sum_{k=1}^{\infty} \left(\frac{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1} [(k+p)(1+B) + (A-B)(p-\alpha)] a_k}{(B-A)(p-\alpha) |a_{-p}|} \right)^2 \leq 1,$$

and

$$\sum_{k=1}^{\infty} \left(\frac{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1} [(k+p)(1+B) + (A-B)(p-\alpha)] b_k}{(B-A)(p-\alpha) |b_{-p}|} \right)^2 \leq 1,$$

we obtain from the last two inequalities

$$\sum_{k=1}^{\infty} \frac{1}{2} \left(\frac{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1} [(k+p)(1+B) + (A-B)(p-\alpha)]}{(B-A)(p-\alpha) |a_{-p}| |b_{-p}|} \right)^2 (a_k^2 + b_k^2) \leq 1, \quad (6.2)$$

However $T(z) \in B_m(p, \alpha, \lambda, \ell, A, B)$ if and only if

$$\sum_{k=1}^{\infty} \left(\frac{\left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [(k+p)(1+B_1) + (A_1 - B_1)(p-\alpha)]}{(B_1 - A_1)(p-\alpha) |a_{-p}| |b_{-p}|} \right) (a_k^2 + b_k^2) \leq 1. \quad (6.3)$$

where $-1 \leq A_1 < B_1 \leq 1$, $\lambda \geq 0$, $\ell > 0$, but (6.2) implies (6.3) if

$$\frac{\left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [(k+p)(1+B_1) + (A_1 - B_1)(p-\alpha)]}{(B_1 - A_1)(p-\alpha) |a_{-p}| |b_{-p}|} < \frac{1}{2} \left(\frac{\left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [(k+p)(1+B) + (A-B)(p-\alpha)]}{(B-A)(p-\alpha) |a_{-p}| |b_{-p}|} \right)^2.$$

Hence, if

$$\frac{1 + B_1}{B_1 - A_1} < \frac{2(B-A)^2(p-\alpha)^2 |a_{-p}| |b_{-p}| + \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [(k+p)(1+B) + (A-B)(p-\alpha)]^2}{2(B-A)^2(p-\alpha)(k+p) |a_{-p}| |b_{-p}|}.$$

This is equivalent to

$$\frac{B_1 - A_1}{1 + B_1} > \frac{2(B-A)^2(p-\alpha)(k+p) |a_{-p}| |b_{-p}|}{2(B-A)^2(p-\alpha)^2 |a_{-p}| |b_{-p}| + \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [(k+p)(1+B) + (A-B)(p-\alpha)]^2} = \mu^2. \quad (6.4)$$

Hence we get $A_1 < -\mu^2 + B_1(1 - \mu^2)$.

Theorem 10. Let $f(z)$ and $g(z)$ belong to the class $B_m(p, \alpha, \lambda, \ell, A, B)$. Then the convolution (or Hadamard product) of two functions f and g belong to the class that is, $(f * g)(z) \in B_m(p, \alpha, \lambda, \ell, A, B)$, where $A_1 < -v + B_1(1 - v)$ and

$$v = \frac{(B-A)^2(p-\alpha) |a_{-p}| |b_{-p}|}{(B-A)^2(p-\alpha)^2 |a_{-p}| |b_{-p}| + \left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [(k+p)(1+B) + (A-B)(p-\alpha)]^2}.$$

Proof. Since $f, g \in B_m(p, \alpha, \lambda, \ell, A, B)$, by using the Cauchy-Schwarz inequality and Theorem 2, we obtain

$$\begin{aligned} & \sum_{k=1}^{\infty} \frac{\left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [(k+p)(1+B) + (A-B)(p-\alpha)]}{(B-A)(p-\alpha) |a_{-p}| |b_{-p}|} \\ & \leq \left(\sum_{k=1}^{\infty} \frac{\left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [(k+p)(1+B) + (A-B)(p-\alpha)]}{(B-A)(p-\alpha) |a_{-p}|} a_k \right)^{1/2} \left(\sum_{k=1}^{\infty} \frac{\left[\frac{\ell}{\ell + \lambda(k+p)} \right]^{m+1} [(k+p)(1+B) + (A-B)(p-\alpha)]}{(B-A)(p-\alpha) |b_{-p}|} b_k \right)^{1/2} \\ & \leq 1, \end{aligned} \quad (6.5)$$

we must find the values of A_1, B_1 so that

$$\sum_{k=1}^{\infty} \frac{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1} [(k+p)(1+B_1) + (A_1 - B_1)(p-\alpha)]}{(B_1 - A_1)(p-\alpha) |a_{-p}| |b_{-p}|} a_k b_k < 1. \quad (6.6)$$

There for, by (6.5), (6.6) holds true if

$$\sqrt{a_k b_k} \leq \frac{(B_1 - A_1)[(k+p)(1+B) + (A-B)(p-\alpha)]}{(B-A)[(k+p)(1+B_1) + (A_1 - B_1)(p-\alpha)]}, k \geq m, m \geq p, a_k \neq 0, b_k \neq 0. \quad (6.7)$$

By (6.5), we have

$$\sqrt{a_k b_k} < \frac{(B-A)(p-\alpha)|a_{-p}|}{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1} [(k+p)(1+B) + (A-B)(p-\alpha)]}, \text{ therefor (6.7) holds true}$$

if

$$\frac{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1} [(k+p)(1+B_1) + (A_1 - B_1)(p-\alpha)]}{(B_1 - A_1)(p-\alpha) |a_{-p}| |b_{-p}|} \leq \left[\frac{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1} [(k+p)(1+B) + (A-B)(p-\alpha)]}{(B-A)(p-\alpha) |a_{-p}| |b_{-p}|} \right]^2,$$

which is equivalent to

$$\frac{1 + B_1}{B_1 - A_1} < \frac{(B-A)^2(p-\alpha)^2 |a_{-p}| |b_{-p}| + \left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1} [(k+p)(1+B) + (A-B)(p-\alpha)]^2}{(B-A)^2(p-\alpha) |a_{-p}| |b_{-p}|}.$$

Alternatively, we can write

$$\frac{B_1 - A_1}{1 + B_1} > \frac{(B-A)^2(p-\alpha) |a_{-p}| |b_{-p}|}{(B-A)^2(p-\alpha)^2 |a_{-p}| |b_{-p}| + \left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1} [(k+p)(1+B) + (A-B)(p-\alpha)]^2} = v.$$

Hence we get $A_1 < -v + B_1(1 - v)$.

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