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ON CERTAIN SUBCLASS OF P-VALENT MEROMORPHICALLY STARLIKE FUNCTIONS WITH ALTERNATING COEFFICIENTS

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Abstract. A certain subclass $B_m(p,\alpha,\lambda,\ell,A,B)$ consisting of meromorphic p-valent functions with alternating coefficient in $U^* = \{z : z \in C : 0 < |z| < 1\}$ is introduced. In this paper we obtain coefficient inequalities, distortion theorem, closure theorems and class preserving integral operators for functions in the class $B_m(p,\alpha,\lambda,\ell,A,B)$ are obtained.

1. Introduction

Let $\Sigma(p)$ denote the class of functions of the form:

$$f(z) = \frac{a_{-p}}{z^p} + \sum_{k=1}^{\infty} a_k z^k \quad (a_{-p} \neq 0;\ p \in \mathbb{N} := \{1,3,5,\ldots\}), \tag{1.1}$$

which are regular in the punctured disc $U^* = \{z : z \in C : 0 < |z| < 1\} = U\backslash\{0\}$, see [11].

Definition 1. Let f, g be analytic in U. Then g is said to be subordinate to f, written $g \prec f$, if there exists a Schwarz function $w(z)$, which is analytic in U, with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $g(z) = f(w(z))(z \in U)$. Hence $g(z) \prec f(z)$ $(z \in U)$, then $g(0) = f(0)$ and $g(U) \subset f(U)$.In particular, if the function $f(z)$ is univalent in U, we have the following (e.g. [7]; [8]):

$g(z) \prec f(z)$ $(z \in U)$ if and only if $g(0) = f(0)$ and $g(U) \subset f(U)$.

Definition 2. For functions $f(z) \in \sum(p)$ given by (1.1) and $g(z) \in \sum(p)$ defined by

$$g(z) = \frac{b_{-p}}{z^p} + \sum_{k=1}^{\infty} b_k z^k \quad (b_k \geq 0,\ p \in \mathbb{N}), \tag{1.2}$$

we define the convolution (or Hadamard product) of $f(z)$ and $g(z)$ by

$$(f * g)(z) = \frac{a_{-p} b_{-p}}{z^p} + \sum_{k=1}^{\infty} a_k b_k z^k \quad (p \in \mathbb{N},\ z \in U). \tag{1.3}$$

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Now, using the integral operator $L_p^m(\lambda,\ell)$ $(\ell > 0;\ \lambda \geq 0;\ p \in \mathbb{N};\ m \in N_0 = N \cup \{0\} = \{0,1,2,\ldots\},\ z \in U^*)$ introduced by El-Ashwah [4], the for function $f(z) \in \sum(p)$ given by (1.1) as follows:

$$L_p^m(\lambda,\ell)f(z) = \frac{a_{-p}}{z^p} + \sum_{k=1}^{\infty}\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^m a_k z^k. \tag{1.4}$$

It is easily verified from (1.4) that

$$\lambda z(L_p^{m+1}(\lambda,\ell))f(z))' = \ell L_p^m(\lambda,\ell)f(z) - (\ell+p\lambda)L_p^{m+1}(\lambda,\ell)f(z)(\lambda>0), \tag{1.5}$$

we note that:

(i) $L_p^{\alpha}(1,1)f(z) = p_p^{\alpha}f(z)$ (see Aqlan et al. [3]);

(ii) $L_1^{\alpha}(1,\beta)f(z) = p_{\beta}^{\alpha}f(z)$ $(a_{-p}=1)$ (see Lashin [6]);

(iii) $L(1,\gamma)f(z) = J(f(z))$ $(p=1)$ (see Sh. Najafzadeh [9]);

(iv) $L_p^m(1,\alpha)f(z) = J_{p,\alpha}^m f(z)$ (see El -Ashwah et. at [5]).

Let $B_m(p,\alpha,\lambda,\ell,A,B)$ denote the class of functions $f(z)$ in $\sum(p)$ that satisfy the condition :

$$\frac{\ell L_p^m(\lambda,\ell)f(z)}{\lambda L_p^{m+1}(\lambda,\ell)f(z)} - \left(\frac{\lambda p+\ell}{\lambda}\right) \prec -\frac{p+\left[pB+(A-B)(p-\alpha)\right]z}{1+Bz}, \quad z\in U^* \tag{1.6}$$

where $\prec$ denotes subordination, $0 \leq \alpha < p$, $-1 \leq A < B \leq 1$, $0 < B \leq 1$, $\lambda,\ell > 0$, $p \in \mathbb{N}$ and $m \in \mathbb{N}_0$.

By definition of subordination, the condition (1.6) is equivalent to

$$\frac{\ell L_p^m(\lambda,\ell)f(z)}{\lambda L_p^{m+1}(\lambda,\ell)f(z)} - \left(\frac{\lambda p+\ell}{\lambda}\right) = \frac{p+\left[pB+(A-B)(p-\alpha)\right]w(z)}{1+Bw(z)}, \tag{1.7}$$

where $w(z) \in H = \{w \text{ regular},\ w(0)=0 \text{ and } |w(z)|<1,\ z\in U\}$. It is easy to see that the condition (1.7) is equivalent to

$$\left|\frac{\dfrac{\ell L_p^m(\lambda,\ell)f(z)}{\lambda L_p^{m+1}(\lambda,\ell)f(z)} - \dfrac{\ell}{\lambda}}{B\left[\dfrac{\ell L_p^m(\lambda,\ell)f(z)}{\lambda L_p^{m+1}(\lambda,\ell)f(z)} - \left(\dfrac{\lambda p+\ell}{\lambda}\right)\right] + \left[pB+(A-B)(p-\alpha)\right]}\right| < 1 \quad (z\in U^*). \tag{1.8}$$

We note that:

(i) when $A=-1, B=1$, we have $f(z)\in B_m(p,\alpha,\lambda,\ell)$ if

$$\operatorname{Re}\left\{\frac{\ell L_p^m(\lambda,\ell)f(z)}{\lambda L_p^{m+1}(\lambda,\ell)f(z)} - \left(\frac{\lambda p+\ell}{\lambda}\right)\right\} < -\alpha,$$

(ii) when $\alpha = 0$

$$\frac{\ell L_p^m(\lambda,\ell)f(z)}{\lambda L_p^{m+1}(\lambda,\ell)f(z)} - \left(\frac{\lambda p+\ell}{\lambda}\right) \prec -\frac{1+Az}{1+Bz}.$$

Let $\sum_a(p)$ be the subclass of $\sum(p)$ consists of functions of the form:

$$f(z) = \frac{a_{-p}}{z^p} + \sum_{k=1}^{\infty}(-1)^{k-1}a_k z^k \quad (a_{-p} \neq 0;\ a_k \geq 0;\ p \in \mathbb{N}), \tag{1.9}$$

that are regular and p-valent in U^*.

Let us write

$$\sum_a^*(p,m,\alpha,\lambda,\ell,A,B) = B_m(p,\alpha,\lambda,\ell,A,B) \cap \sum_a(p). \tag{1.10}$$

In this paper coefficient inequalities, distortion theorem and closure theorems for the class $B_m^*(p,\alpha,\lambda,\ell,A,B)$ are obtained.

Finally, the class preserving integral operators of the form

$$F(z) = (c-p+1)z^{-c-1}\int_0^z t^c f(t)dt \quad (c > p-1;\ p \in N_0), \tag{1.11}$$

is considered. Techinques used are similar to those of Silvermen [10] and Uralegaddi and Ganigi [11], Aouf and Darwish [1], Aouf et al. [2].

2. Coefficient inequalities

Theorem 1. *Let* $f(z) = \frac{a_{-p}}{z^p} + \sum_{k=1}^{\infty} a_k z^k$ *be regular and p-valent in* U^*. *If*

$$\sum_{k=1}^{\infty}\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]|a_k|$$
$$\leq (B-A)(p-\alpha)\left|a_{-p}\right|, \tag{2.1}$$

then $f(z) \in B_m(p,\alpha,\lambda,\ell,A,B)$.

Proof. Suppose (2.1) holds for all admissible values of $p,m,\alpha,\lambda,\ell,A$ and B. It suffices show that

$$\left|\frac{\dfrac{L_p^m(\lambda,\ell)f(z)}{L_p^{m+1}(\lambda,\ell)f(z)}-1}{B\left[\dfrac{L_p^m(\lambda,\ell)f(z)}{L_p^{m+1}(\lambda,\ell)f(z)}-(\frac{\lambda}{\ell}p+1)\right]+\lambda/\ell\left[pB+(A-B)(p-\alpha)\right]}\right| < 1 \quad \text{for} \quad |z|<1$$

we have

$$\left|\frac{\dfrac{L_p^m(\lambda,\ell)f(z)}{L_p^{m+1}(\lambda,\ell)f(z)}-1}{B\left[\dfrac{L_p^m(\lambda,\ell)f(z)}{L_p^{m+1}(\lambda,\ell)f(z)}-(\frac{\lambda}{\ell}p+1)\right]+\lambda/\ell\left[pB+(A-B)(p-\alpha)\right]}\right|$$

$$= \left| \frac{\sum_{k=1}^{\infty} \left[\frac{\ell}{\ell+\lambda(k+p)} \right]^{m+1} (k+p) a_k z^{p+k}}{(A-B)(p-\alpha)a_{-p} + \sum_{k=1}^{\infty} \left[\frac{\ell}{\ell+\lambda(k+p)} \right]^{m+1} [B(k+p)+(A-B)(p-a)] a_k z^{k+p}} \right|$$

$$\leq \frac{\sum_{k=1}^{\infty} \left[\frac{\ell}{\ell+\lambda(k+p)} \right]^{m+1} (k+p) |a_k|}{(B-A)(p-\alpha)\left|a_{-p}\right| - \sum_{k=1}^{\infty} \left[\frac{\ell}{\ell+\lambda(k+p)} \right]^{m+1} [B(k+p)+(A-B)(p-a)] |a_k|}.$$

The last expression is bounded above by (1.1) if

$$\sum_{k=1}^{\infty} \left[\frac{\ell}{\ell+\lambda(k+p)} \right]^{m+1} (k+p) |a_k| \leq (B-A)(p-a)\left|a_{-p}\right| - \sum_{k=1}^{\infty} \left[\frac{\ell}{\ell+\lambda(k+p)} \right]^{m+1} \left[B(k+p)+(A-B)(p-\alpha)\right] |a_k|$$

which is equivalent to

$$\sum_{k=1}^{\infty} \left[\frac{\ell}{\ell+\lambda(k+p)} \right]^{m+1} \left[(k+p)(1+B)+(A-B)(p-\alpha)\right] |a_k| \leq (B-A)(p-\alpha)\left|a_{-p}\right|.$$

The completes the proof of Theorem 1.

Theorem 2. *Let* $f(z) = \frac{a_{-p}}{z^p} + \sum_{k=1}^{\infty}(-1)^{k-1} a_k z^k$ $(a_{-p} \neq 0;\ a_k \geq 0,\ p \in \mathbb{N})$ *be regular and p-valent in* U^*. *Then* $f(z) \in \sum_a^*(p,m,\alpha,\lambda,\ell,A,B)$ *if and only if*

$$\sum_{k=1}^{\infty} \left[\frac{\ell}{\ell+\lambda(k+p)} \right]^{m+1} \left[(k+p)(1+B)+(A-B)(p-\alpha)\right] a_k \leq (B-A)(p-\alpha)\left|a_{-p}\right|. \tag{2.2}$$

Proof. In view of Theorem 1, it is sufficient to shwo the "only if" part. Let us suppose that

$$f(z) = \frac{a_{-p}}{z^p} + \sum_{k=1}^{\infty}(-1)^{k-1} a_k z^k \quad (a_{-p} \neq 0;\ a_k \geq 0,\ p \in \mathbb{N}),$$

is in $\sum_a^*(p,m,\alpha,\lambda,\ell,A,B)$. Then

$$\left| \frac{\frac{L_p^m(\lambda,\ell)f(z)}{L_p^{m+1}(\lambda,\ell)f(z)} - 1}{B\left[\frac{L_p^{m+1}(\lambda,\ell)f(z)}{L_p^{m+1}(\lambda,\ell)f(z)} - (\frac{\lambda}{\ell}p+1)\right] + \frac{\lambda}{\ell}\left[pB+(A-B)(p-\alpha)\right]} \right|$$

$$= \left| \frac{\sum_{k=1}^{\infty}(-1)^{k-1} \left[\frac{\ell}{\ell+\lambda(k+p)} \right]^{m+1} (k+p) a_k z^{k+p}}{(B-A)(p-\alpha)a_{-p} - \sum_{k=1}^{\infty}(-1)^{k-1} \left[\frac{\ell}{\ell+\lambda(k+p)} \right]^{m+1} [B(k+p)+(A-B)(p-\alpha)] a_k z^{k+p}} \right| \leq 1$$

for all $z \in U^*$. Using the fact that $\mathrm{Re}(z) \le |z|$ for all z, it follows that

$$\mathrm{Re}\left\{\frac{\sum_{k=1}^{\infty}(-1)^{k-1}\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}(k+p)a_k z^{k+p}}{(B-A)(p-\alpha)a_{-p}-\sum_{k=1}^{\infty}(-1)^{k-1}\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[B(k+p)+(A-B)(p-\alpha)\right]a_k z^{k+p}}\right\} \le 1 \ (z \in U^*). \quad (2.3)$$

Now choose values of z on the real axis so that $\frac{L_p^m(\lambda,\ell)f(z)}{L_p^{m+1}(\lambda,\ell)f(z)}$ is real. Upon clearing the denominator in (2.3) and letting $z \to (-1)^+$ through real values, we obtain

$$\sum_{k=1}^{\infty}\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}(k+p)a_k \le (B-A)(p-\alpha)\left|a_{-p}\right| - \sum_{k=1}^{\infty}\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}[B(k+p)+(A-B)(p-\alpha)]a_k.$$

or

$$\sum_{k=1}^{\infty}\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}[(k+p)(1+B)+(A-B)(p-\alpha)]a_k \le (B-A)(p-\alpha)\left|a_{-p}\right|.$$

Hence the result follows.

Corollary 1. *If* $f(z) = \frac{a_{-p}}{z^p} + \sum_{k=1}^{\infty}(-1)^{k-1}a_k z^k \ (a_{-p} \ne 0;\ a_k \ge 0;\ p \in \mathbb{N})$ *is in the class* $\sum_a^*(p,\alpha,\lambda,\ell,A,B)$, *then*

$$a_k \le \frac{(B-A)(p-\alpha)\left|a_{-p}\right|}{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]} \ (k \in N). \quad (2.4)$$

Equality holds for the functions of form

$$f_k(z) = \frac{a_{-p}}{z^p} + (-1)^{k-1}\frac{(B-A)(p-\alpha)a_{-p}}{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]}z^k, \quad (2.5)$$

where $(k \in N$ *and* $p \in \mathbb{N})$.

3. Distortion theorems

Theorem 3. *Let* $f(z) = \frac{a_{-p}}{z^p} + \sum_{k=1}^{\infty}(-1)^{k-1}a_k z^k \ (a_{-p} \ne 0;\ a_k \ge 0;\ p \in \mathbb{N})$ *is in the class* $\sum_a^*(p,m,\alpha,\lambda,\ell,A,B)$, *then for* $0 < |z| = r < 1$,

$$\frac{\left|a_{-p}\right|}{r^p} - \frac{(B-A)(p-\alpha)\left|a_{-p}\right|}{\left[\frac{\ell}{\ell+\lambda(1+p)}\right]^{m+1}\left[(1+p)(1+B)+(A-B)(p-\alpha)\right]}r \le \left|f(z)\right|$$

$$\leq \frac{|a_{-p}|}{r^p} + \frac{(B-A)(p-\alpha)|a_{-p}|}{\left[\frac{\ell}{\ell+\lambda(1+p)}\right]^{m+1}\left[(1+p)(1+B)+(A-B)(p-\alpha)\right]} r \tag{3.1}$$

with equality for the function

$$f_1(z) = \frac{a_{-p}}{z^p} + \frac{(B-A)(p-\alpha)|a_{-p}|}{\left[\frac{\ell}{\ell+\lambda(1+p)}\right]^{m+1}\left[(1+p)(1+B)+(A-B)(p-\alpha)\right]} z \tag{3.2}$$

at $z = r,\ ir$.

Proof. Suppose that $f(z)$ in the class $\sum_a^*(p, m, \alpha, \lambda, \ell, A, B)$. In view of Theorem 2, we have

$$\begin{aligned}
&\left[\frac{\ell}{\ell+\lambda(1+p)}\right]^{m+1}\left[(1+p)(1+B)+(A-B)(p-\alpha)\right]\sum_{k=1}^{\infty} a_k \\
&\leq \sum_{k=1}^{\infty}\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(1+p)(1+B)+(A-B)(p-\alpha)\right] a_k \\
&\leq (B-A)(p-\alpha)|a_{-p}|,
\end{aligned}$$

which evidently yields

$$\sum_{k=1}^{\infty} a_k \leq \frac{(B-A)(p-\alpha)|a_{-p}|}{\left[\frac{\ell}{\ell+\lambda(1+p)}\right]^{m+1}\left[(1+p)(1+B)+(A-B)(p-\alpha)\right]}. \tag{3.3}$$

Consequently, we obtain

$$\begin{aligned}
|f(z)| &\leq \frac{|a_{-p}|}{r^p} + \sum_{k=1}^{\infty} a_k r^k \leq \frac{|a_{-p}|}{r^p} + r\sum_{k=1}^{\infty} a_k \\
&\leq \frac{|a_{-p}|}{r^p} + \frac{(B-A)(p-\alpha)|a_{-p}|}{\left[\frac{\ell}{\ell+\lambda(1+p)}\right]^{m+1}\left[(1+p)(1+B)+(A-B)(p-\alpha)\right]} r,
\end{aligned}$$

by (3.3). This gives the right hand inequality of (3.1). Also

$$\begin{aligned}
|f(z)| &\geq \frac{|a_{-p}|}{r^p} - \sum_{k=1}^{\infty} a_k r^k \geq \frac{|a_{-p}|}{r^p} - r\sum_{k=1}^{\infty} a_k \\
&\geq \frac{|a_{-p}|}{r^p} - \frac{(B-A)(p-\alpha)|a_{-p}|}{\left[\frac{\ell}{\ell+\lambda(1+p)}\right]^{m+1}\left[(1+p)(1+B)+(A-B)(p-\alpha)\right]} r,
\end{aligned}$$

which gives the left hand side of (3.1). It can be easily seen that the function $f(z)$ defined by (3.2) is an extremal function for the theorem.

Theorem 4. *If* $f(z) = \dfrac{a_{-p}}{z^p} + \sum_{k=1}^{\infty}(-1)^{k-1}a_k z^k$ $(a_{-p} \neq 0;\ a_k \geq 0,\ p \in \mathbb{N})$ *is in the class* $\sum_a^*(p, m, \alpha, \lambda, \ell, A, B)$, *then for* $0 < |z| = r < 1$,

$$\frac{p\left|a_{-p}\right|}{r^{p+1}} - \frac{(B-A)(p-\alpha)\left|a_{-p}\right|}{\left[\dfrac{\ell}{\ell+\lambda(1+p)}\right]^{m+1}\left[(1+p)(1+B)+(A-B)(p-\alpha)\right]} \leq \left|f'(z)\right|$$
$$\leq \frac{p\left|a_{-p}\right|}{r^{p+1}} + \frac{(B-A)(p-\alpha)\left|a_{-p}\right|}{\left[\dfrac{\ell}{\ell+\lambda(1+p)}\right]^{m+1}\left[(1+p)(1+B)+(A-B)(p-\alpha)\right]}. \tag{3.4}$$

The result is sharp, the extremal function being of the form (3.2).

Proof. From Theorem 2, we have

$$\begin{aligned}&\left[\frac{\ell}{\ell+\lambda(1+p)}\right]^{m+1}\left[(1+p)(1+B)+(A-B)(p-\alpha)\right]\sum_{k=1}^{\infty} k a_k \\ &\quad\leq \sum_{k=1}^{\infty}\left[\frac{\ell}{\ell+\lambda(1+p)}\right]^{m+1}\left[(1+p)(1+B)+(A-B)(p-\alpha)\right] a_k \\ &\quad\leq (B-A)(p-\alpha)\left|a_{-p}\right|,\end{aligned}$$

which evidently yields

$$\sum_{k=1}^{\infty} k a_k \leq \frac{(B-A)(p-\alpha)\left|a_{-p}\right|}{\left[\dfrac{\ell}{\ell+\lambda(1+p)}\right]^{m+1}\left[(1+p)(1+B)+(A-B)(p-\alpha)\right]}. \tag{3.5}$$

Consequently, we obtain

$$\begin{aligned}\left|f'(z)\right| &\leq \frac{p\left|a_{-p}\right|}{r^{p+1}} + \sum_{k=1}^{\infty} k a_k r^{k-1} \leq \frac{p\left|a_{-p}\right|}{r^{p+1}} + \sum_{k=1}^{\infty} k a_k \\ &\leq \frac{p\left|a_{-p}\right|}{r^{p+1}} + \frac{(B-A)(p-\alpha)\left|a_{-p}\right|}{\left[\dfrac{\ell}{\ell+\lambda(1+p)}\right]^{m+1}\left[(1+p)(1+B)+(A-B)(p-\alpha)\right]},\end{aligned}$$

by (3.5). Also

$$\left|f'(z)\right| \geq \frac{p\left|a_{-p}\right|}{r^{p+1}} - \sum_{k=1}^{\infty} k a_k r^{k-1} \geq \frac{p\left|a_{-p}\right|}{r^{p+1}} - \sum_{k=1}^{\infty} k a_k$$
$$\geq \frac{p\left|a_{-p}\right|}{r^{p+1}} - \frac{(B-A)(p-\alpha)\left|a_{-p}\right|}{\left[\dfrac{\ell}{\ell+\lambda(1+p)}\right]^{m+1}\left[(1+p)(1+B)+(A-B)(p-\alpha)\right]}. \tag{3.6}$$

This completes the proof of Theorem 4.

4. Closure theorems

Let the functions $f_j(z)$ be defined for $j = 1,2,\dots,m$, by

$$f_j(z) = \frac{a_{-p,j}}{z^p} + \sum_{k=1}^{\infty}(-1)^{k-1}a_{k,j}z^k \ \ (a_{-p,j} > 0;\ a_{k,j} \geq 0;\ p \in \mathbb{N}) \text{ for } z \in U^*. \tag{4.1}$$

Theorem 5. *Let the function $f_j(z)$ be defined by (4.1) be in the class $\sum_a^*(p,m,\alpha,\lambda,\ell,A,B)$ for every $j = 1,2,\dots,m$. Then the function $F(z)$ defined by*

$$F(z) = \frac{b_{-p}}{z^p} + \sum_{k=1}^{\infty}(-1)^{k-1}b_k z^k \ \ (b_{-p} > 0;\ b_k \geq 0;\ p \in \mathbb{N}), \tag{4.2}$$

is a member of the class $\sum_a^*(p,m,\alpha,\lambda,\ell,A,B)$, where

$$b_{-p} = \frac{1}{m}\sum_{j=1}^{m} a_{-p,j} \quad \text{and} \quad b_k = \frac{1}{m}\sum_{j=1}^{m} a_{k,j}\ (k \in N). \tag{4.3}$$

Proof. Since $f_j(z) \in \sum_a^*(p,m,\alpha,\lambda,\ell,A,B)$, it follows from Theorem 2, that

$$\begin{aligned}\sum_{k=1}^{\infty}\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}&\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]a_{k,j}\\ &\leq (B-A)(p-\alpha)\left|a_{-p,j}\right|,\end{aligned} \tag{4.4}$$

for every $j = 1,2,\dots,m$. Hence,

$$\begin{aligned}&\sum_{k=1}^{\infty}\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]b_k\\ &= \sum_{k=1}^{\infty}\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]\left\{\frac{1}{m}\sum_{j=1}^{\infty}a_{k,j}\right\}\\ &= \frac{1}{m}\sum_{j=1}^{m}\left(\sum_{k=1}^{\infty}\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]a_{k,j}\right)\\ &\leq (B-A)(p-\alpha)\left(\frac{1}{m}\sum_{j=1}^{\infty}a_{-p,j}\right) = (B-A)(p-\alpha)b_{-p},\end{aligned}$$

which (in view of Theorem 2) implies that $F(z) \in \sum_a^*(p,m,\alpha,\lambda,\ell,A,B)$.

Theorem 6. *The class $\sum_a^*(p,m,\alpha,\lambda,\ell,A,B)$ is closed under convex linear combination.*

Proof. Let the functions $f_j(z)$ $(j=1,2)$ defined by (4.1) be in the class $\sum_a^*(p,m,\alpha,\lambda,\ell,A,B)$, it is sufficient to prove that the function

$$H(z) = tf_1(z)+(1-t)f_2(z) \qquad (0\le t\le 1), \tag{4.5}$$

is also in the class $\sum_a^*(p,m,\alpha,\lambda,\ell,A,B)$. Since for $0\le t\le 1$,

$$H(z)=\frac{ta_{-p,1}+(1-t)a_{-p,2}}{z^p}+\sum_{k=1}^{\infty}(-1)^{k-1}\{ta_{k,1}+(1-t)a_{k,2}\}z^k, \tag{4.6}$$

we observe that

$$\begin{aligned}
&\sum_{k=1}^{\infty}\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]\left\{ta_{k,1}+(1-t)a_{k,2}\right\}\\
&=t\sum_{k=1}^{\infty}\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]a_{k,1}+\\
&(1-t)\sum_{k=1}^{\infty}\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]a_{k,2}\\
&\le (B-A)(p-\alpha)\left\{ta_{-p,1}+(1-t)a_{-p,2}\right\},
\end{aligned} \tag{4.7}$$

with the aid of Theorem 2, hence $H(z)\in\sum_a^*(p,m,\alpha,\lambda,\ell,A,B)$. This completes the proof of Theorem 6.

Theorem 7. *Let*

$$f_0(z)=\frac{a_{-p}}{z^p} \tag{4.8}$$

and

$$f_k(z)=\frac{a_{-p}}{z^p}+(-1)^{k-1}\frac{(B-A)(p-\alpha)\left|a_{-p}\right|}{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left\{(k+p)(1+B)+(A-B)(p-\alpha)\right\}}z^k \quad (k\in N). \tag{4.9}$$

Then $f(z)\in\sum_a^*(p,m,\alpha,\lambda,\ell,A,B)$ if and only if it can be expressed in the form

$$f(z)=\sum_{k=0}^{\infty}\mu_k f_k(z),\ \textit{where}\ \mu_k\ge 0\ \ (k\ge 0)\ \textit{and}\ \sum_{k=0}^{\infty}\mu_k=1. \tag{4.10}$$

Proof. Suppose that

$$f(z)=\sum_{k=0}^{\infty}\mu_k f_k(z),\ \text{where}\ \mu_k\ge 0\ \ (k\ge 0)\ \text{and}\ \sum_{k=0}^{\infty}\mu_k=1.$$

Then

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} \mu_k f_k(z) \\ &= \mu_0 f_0(z) + \sum_{k=1}^{\infty} \mu_k f_k(z) \\ &= \frac{a_{-p}}{z^p} + \sum_{k=1}^{\infty} \mu_k (-1)^{k-1} \frac{(B-A)(p-\alpha)\left|a_{-p}\right|}{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]} z^k \ (k \in N). \end{aligned} \tag{4.11}$$

Since

$$\begin{aligned} &\sum_{k=1}^{\infty} \left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1} \left\{(k+p)(1+B)+(A-B)(p-\alpha)\right\}. \\ &\frac{(B-A)(p-\alpha)\left|a_{-p}\right|}{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left\{(k+p)(1+B)+(A-B)(p-\alpha)\right\}} \mu_k \\ &\quad = (B-A)(p-\alpha)\left|a_{-p}\right| \sum_{k=1}^{\infty} \mu_k \\ &\quad = (B-A)(p-\alpha)\left|a_{-p}\right|(1-\mu_0) \\ &\quad \le (B-A)(p-\alpha)\left|a_{-p}\right|, \end{aligned} \tag{4.12}$$

we have $f(z) \in \sum_a^* (p,m,\alpha,\lambda,\ell,A,B)$, by Theorem 2.

Conversely, suppose that the function $f(z)$ defined by (1.9) belongs to the class $\sum_a^* (p,m,\alpha,\lambda,\ell,A,B)$. Since

$$a_k \le \frac{(B-A)(p-\alpha)\left|a_{-p}\right|}{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left\{(k+p)(1+B)+(A-B)(p-\alpha)\right\}} \quad (k \in N), \tag{4.13}$$

by Corollary 1, setting

$$\mu_k = \frac{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left\{(k+p)(1+B)+(A-B)(p-\alpha)\right\}}{1/\ell(A-B)(p-\alpha)\left|a_{-p}\right|} a_k \quad (k \in N), \tag{4.14}$$

and

$$\mu_0 = 1 - \sum_{k=1}^{\infty} \mu_k f_k(z), \tag{4.15}$$

it follows that

$$f(z) = \sum_{k=0}^{\infty} \mu_k f_k(z).$$

This completes the proof of Theorem 7.

5. Integral operators

Theorem 8. *If* $f(z) = \dfrac{a_{-p}}{z^p} + \sum_{k=1}^{\infty}(-1)^{k-1}a_k z^k$ $(a_{-p} \neq 0;\ a_k \geq 0,\ p \in \mathbb{N})$ *is in the class* $\sum_a^*(p,m,\alpha,\lambda,\ell,A,B)$, *then*

$$F(z) = (c-p+1)z^{-c-1}\int_0^z t^c f(t)\,dt$$
$$= \frac{a_{-p}}{z^p} + \sum_{k=1}^{\infty}(-1)^{k-1}\left(\frac{c-p+1}{c+k+1}\right)a_k z^k, \tag{5.1}$$

$c > p-1$, *belongs to the class* $\sum_a^*(p,m,\gamma(p,\alpha,c,A,B),\lambda,\ell,A,B)$, *where*

$$\gamma(p,\alpha,c,A,B) = p - \frac{(c-p+1)(1+p)(1+B)(p-\alpha)}{(c+p+1)[(1+p)(1+B)+(A-B)(p-\alpha)]-(c-p+1)(A-B)(p-\alpha)}. \tag{5.2}$$

The result is sharp for the function $f(z)$ *given by*

$$f(z) = \frac{a_{-p}}{z^p} + \frac{(B-A)(p-\alpha)\left|a_{-p}\right|}{\left[\dfrac{\ell}{\ell+\lambda(1+p)}\right]^{m+1}\left[(1+p)(1+B)+(A-B)(p-\alpha)\right]}z. \tag{5.3}$$

Proof. Suppose $f(z) = \dfrac{a_{-p}}{z^p} + \sum_{k=1}^{\infty}(-1)^{k-1}a_k z^k \in \sum\limits_a^*(p,m,\alpha,\lambda,\ell,A,B)$, in view of Theorem 2, we shall find the largest value of γ for which

$$\sum_{k=1}^{\infty}\frac{\left[\dfrac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\gamma)\right]}{(B-A)(p-\alpha)\left|a_{-p}\right|}\left(\frac{c-p+1}{c+k+1}\right)a_k \leq 1.$$

It is sufficient to fined the range values of γ for which

$$\frac{(c-p+1)\left[(k+p)(1+B)+(A-B)(p-\gamma)\right]}{(c+k+1)(p-\gamma)} \leq \frac{\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]}{(p-\alpha)}$$

for each k.

Solving the above inequality for γ, we obtain

$$\gamma \leq p - \frac{(c-p+1)(k+p)(1+B)(p-\alpha)}{(c+k+1)\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]-(c-p+1)(A-B)(p-\alpha)}.$$

For each $p,\alpha,\lambda,\ell,A,B$ and c fixed let

$$F(k) = p - \frac{(c-p+1)(k+p)(1+B)(p-\alpha)}{(c+k+1)\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]-(c-p+1)(A-B)(p-\alpha)}.$$

Then $F(k+1)-F(k) = \dfrac{D}{G} > 0$ for each k, where

$$D = (c-p+1)(1+B^2)(p-\alpha)(k+p)(k+p+1)$$

and

$$G=\{(c+k+2)[(k+p+1)(1+B)+(A-B)(p-\alpha)]-(c-p+1)(A-B)(p-\alpha)\}$$
$$\times\{(c+k+1)[(k+p)(1+B)+(A-B)(p-\alpha)]-(c-p+1)(A-B)(p-\alpha)\}.$$

Hence $F(k)$ is an increasing function of k. Since

$$F(1)=p-\frac{(c-p+1)(1+p)(1+B)(p-\alpha)}{(c+2)\left[(1+p)(1+B)+(A-B)(p-\alpha)\right]-(c-p+1)(A-B)(p-\alpha)},$$

the result follows.

Remark.

(i) Putting $a_{-p}=1,\ p=1,\ m=0,\ A=-1$ and $B=1$, in the above results, we have the results obtained by Uralegaddi and Ganigi [11];

(ii) Putting $a_{-p}=1,\ p=1,\ A=-1$ and $B=1$, in the above results, we have the results obtained by Aouf and Darwish [1];

(iii) Putting $a_{-p}=1$ and $p=1$, in the above results, we have the reults obtained by Aouf et al. [2].

6. Convolution properties

Theorem 9. *If $f(z)$ and $g(z)$ belong to the class $B_m(p,\alpha,\lambda,\ell,A,B)$, then*

$$T(z)=\frac{a_{-p}b_{-p}}{z^p}+\sum_{k=1}^{\infty}(a_k^2+b_k^2)z^k, \tag{6.1}$$

is in the class $B_m(p,\alpha,\lambda,\ell,A_1,B_1)$ such that $A_1<-\mu^2+B_1(1-\mu^2)$, where

$$\mu=\frac{\sqrt{2(p-\alpha)(k+p)(\left|a_{-p}\right|\left|b_{-p}\right|)(B-A)}}{\sqrt{2\left|a_{-p}\right|\left|b_{-p}\right|(B-A)(p-\alpha)+\sqrt{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]}}}.$$

Proof. Since $f,g\in B_m(p,\alpha,\lambda,\ell,A,B)$. Theorem 2 yields

$$\sum_{k=1}^{\infty}\left(\frac{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]a_k}{(B-A)(p-\alpha)\left|a_{-p}\right|}\right)^2\le 1,$$

and

$$\sum_{k=1}^{\infty}\left(\frac{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]b_k}{(B-A)(p-\alpha)\left|b_{-p}\right|}\right)^2\le 1,$$

we obtain from the last two inequalities

$$\sum_{k=1}^{\infty}\frac{1}{2}\left(\frac{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]}{(B-A)(p-\alpha)\left|a_{-p}\right|\left|b_{-p}\right|}\right)^{2}(a_k^2+b_k^2)\le 1, \tag{6.2}$$

However, $T(z)\in B_m(p,\alpha,\lambda,\ell,A_1,B_1)$ if and only if

$$\sum_{k=1}^{\infty}\left(\frac{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B_1)+(A_1-B_1)(p-\alpha)\right]}{(B_1-A_1)(p-\alpha)\left|a_{-p}\right|\left|b_{-p}\right|}\right)(a_k^2+b_k^2)\le 1. \tag{6.3}$$

where $-1\le A_1<B_1\le 1$, $\lambda\ge 0$, $\ell>0$, but (6.2) implies (6.3) if

$$\frac{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B_1)+(A_1-B_1)(p-\alpha)\right]}{(B_1-A_1)(p-\alpha)\left|a_{-p}\right|\left|b_{-p}\right|} < \frac{1}{2}\left(\frac{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]}{(B-A)(p-\alpha)\left|a_{-p}\right|\left|b_{-p}\right|}\right)^{2}.$$

Hence, if

$$\frac{1+B_1}{B_1-A_1}<\frac{2(B-A)^2(p-\alpha)^2\left|a_{-p}\right|\left|b_{-p}\right|+\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]^2}{2(B-A)^2(p-\alpha)(k+p)\left|a_{-p}\right|\left|b_{-p}\right|}.$$

This is equivalent to

$$\frac{B_1-A_1}{1+B_1}>\frac{2(B-A)^2(p-\alpha)(k+p)\left|a_{-p}\right|\left|b_{-p}\right|}{2(B-A)^2(p-\alpha)^2\left|a_{-p}\right|\left|b_{-p}\right|+\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]^2}=\mu^2. \tag{6.4}$$

Hence we get $A_1<-\mu^2+B_1(1-\mu^2)$.

Theorem 10. *Let $f(z)$ and $g(z)$ belong to the class $B_m(p,\alpha,\lambda,\ell,A,B)$. Then the convolution (or Hadamard product) of two functions f and g belong to the class that is, $(f*g)(z)\in B_m(p,\alpha,\lambda,\ell,A_1,B_1)$, where $A_1<-v+B_1(1-v)$ and*

$$v=\frac{(B-A)^2(p-\alpha)\left|a_{-p}\right|\left|b_{-p}\right|}{(B-A)^2(p-\alpha)^2\left|a_{-p}\right|\left|b_{-p}\right|+\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]^2}.$$

Proof. Since $f, g \in B_m(p,\alpha,\lambda,\ell,A,B)$, by using the Cauchy-Schwarz inequality and Theorem 2, we obtain

$$\sum_{k=1}^{\infty} \frac{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]}{(B-A)(p-\alpha)\left|a_{-p}\right|\left|b_{-p}\right|}$$
$$\leq \left(\sum_{k=1}^{\infty} \frac{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]}{(B-A)(p-\alpha)\left|a_{-p}\right|} a_k\right)^{1/2}$$
$$\bullet \left(\sum_{k=1}^{\infty} \frac{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]}{(B-A)(p-\alpha)\left|b_{-p}\right|} b_k\right)^{1/2}$$
$$\leq 1, \tag{6.5}$$

we must fined the values of A_1, B_1 so that

$$\sum_{k=1}^{\infty} \frac{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B_1)+(A_1-B_1)(p-\alpha)\right]}{(B_1-A_1)(p-\alpha)\left|a_{-p}\right|\left|b_{-p}\right|} a_k b_k < 1. \tag{6.6}$$

Therefor, by (6.5), (6.6) holds true if

$$\sqrt{a_k b_k} \leq \frac{(B_1-A_1)\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]}{(B-A)\left[(k+p)(1+B_1)+(A_1-B_1)(p-\alpha)\right]}, \tag{6.7}$$

$k \geq m,\ m \geq p,\ a_k \neq 0,\ b_k \neq 0.$

By (6.5), we have

$$\sqrt{a_k b_k} < \frac{(B-A)(p-\alpha)\left|a_{-p}\right|}{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]},$$

therefor (6.7) holds true

if

$$\frac{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B_1)+(A_1-B_1)(p-\alpha)\right]}{(B_1-A_1)(p-\alpha)\left|a_{-p}\right|\left|b_{-p}\right|}$$
$$\leq \left[\frac{\left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]}{(B-A)(p-\alpha)\left|a_{-p}\right|\left|b_{-p}\right|}\right]^2,$$

which is equivalent to

$$\frac{1+B_1}{B_1-A_1} < \frac{(B-A)^2(p-\alpha)^2\left|a_{-p}\right|\left|b_{-p}\right| + \left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]^2}{(B-A)^2(p-\alpha)\left|a_{-p}\right|\left|b_{-p}\right|}.$$

Alternatively, we can write

$$\frac{B_1-A_1}{1+B_1} > \frac{(B-A)^2(p-\alpha)\left|a_{-p}\right|\left|b_{-p}\right|}{(B-A)^2(p-\alpha)^2\left|a_{-p}\right|\left|b_{-p}\right| + \left[\frac{\ell}{\ell+\lambda(k+p)}\right]^{m+1}\left[(k+p)(1+B)+(A-B)(p-\alpha)\right]^2} = v.$$

Hence we get $A_1 < -v + B_1(1-v)$.

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