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DEFINITE INTEGRALS OF GENERALIZED CERTAIN CLASS OF INCOMPLETE ELLIPTIC INTEGRALS

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Abstract. Elliptic- integral have their importance and potential in certain problems in radiation physics and nuclear technology, studies of crystallographic minimal surfaces, the theory of scattering of acoustic or electromagnetic waves by means of an elliptic disk, studies of elliptical crack problems in fracture mechanics. A number of earlier works on the subject contains remarkably large number of general families of elliptic- integrals and indeed also many definite integrals of such families with respect to their modulus (or complementary modulus) are known to arise naturally. Motivated essentially by these and many other potential avenues of their applications, our aim here is to give a systematic account of the theory of a certain family of generalized incomplete elliptic integrals in a unique and generalized manner. The results established in this paper are of manifold generality and basic in nature. By making use of the familiar Riemann-Liouville fractional differ integral operators, we establish many explicit hypergeometric representations and apply these representation in deriving several definite integrals pertaining to their, not only with respect to the modulus (or complementary modulus), but also with respect to the amplitude of generalized incomplete elliptic integrals involved therein.

1. Introduction and Preliminaries

Let

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \tag{1}$$

be the equation of ellipse in the (x,y)-coordinate plane with eccentri k given by

$$k=\sqrt{1-\frac{a^2}{b^2}}, \qquad (0<a<b).$$

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Now parameters of (1) as follows:

$$x = a\cos\theta \qquad \text{and} \qquad y = b\sin\theta, \qquad (0 \le \theta \le 2\pi)$$

the arc-length of the ellipse (1) from $\theta = 0$ to $\theta = \varphi$ is given by

$$L_{0,\phi} = \int_0^{\phi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = b\int_0^{\phi} \sqrt{1-k^2\sin^2\theta}\, d\theta. \tag{2}$$

The integrals which are defined in (2) belong to a family of integrals known as incomplete elliptic integrals. In fact, in Legendre's normal form, the incomplete elliptic integrals of the first and second kind (with modulus |k|and amplitude φ) are defined by (cf., e.g., [5] and [12]; see also [2, Chapter 17], [11, Chapter 9] and [10, Vol. II, Chapter 13]):

$$F(\phi, k) = \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^{\sin\phi} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \qquad \left(\left|k^2\right| < 1; 0 \le \phi \le \frac{\pi}{2}\right) \tag{3}$$

and [cf. Equation (2)]

$$E(\phi, k) = \int_0^{\phi} \sqrt{1-k^2\sin^2\theta}\, d\theta = \int_0^{\sin\phi} \frac{\sqrt{(1-k^2t^2)}}{\sqrt{(1-t^2)}}\, dt, \qquad \left(\left|k^2\right| < 1; 0 \le \phi \le \frac{\pi}{2}\right). \tag{4}$$

In our present investigation we take necessary constraint $\left|k^2\right| < 1$ on the modulus $|k|$ has been retained (instead of the $0 \le k < 1$) and the amplitude φ may take complex values. In particular, when $\phi = \frac{\pi}{2}$, the definitions (3) and (4) reduce into the corresponding complete elliptic integrals $K(k)$ and $E(k)$ of the first and second kind, which are defined by

$$K(k) = F\left(\frac{\pi}{2}, k\right) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^{1} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \qquad \left(\left|k^2\right| < 1\right) \tag{5}$$

$$E(k) = E\left(\frac{\pi}{2}, k\right) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\, d\theta = \int_0^{1} \frac{\sqrt{(1-k^2t^2)}}{\sqrt{(1-t^2)}}\, dt, \qquad \left(\left|k^2\right| < 1\right) \tag{6}$$

respectively.

In the study of vast literature on the complete elliptic integrals, one can find several extensively investigated unifications of the complete elliptic integrals $K(k)$ and $E(k)$ of the first and second kind. For example, in his systematic study of some problems involving high energies in quantum mechanics, by evaluating the first term in a certain Born series in two different ways and comparing the resulting expressions, Barton [3] found an interesting integral formula involving $K(\sqrt{1-k^2})$, which Bushell [4] not only proved directly, but also derived a number of additional results analogous to Barton's integral, thereby extending several known integral formulas recorded by, for example, Byrd and Friedman [5, p.274] (see also the works by Kaplan [3], Müler [20] and Prudnikov et al. [23, p.179–194, Sections 2.15 and 2.16]). Furthermore, Bushell [4] gave a generalization of Barton's integral in terms of the case when $\gamma \geq 0$ of the following unification of the complete elliptic integrals K(k) and E(k) of the first and second kind [4, p.2, Equation (2.2)]:

$$H(k,\gamma) = \int_0^1 \frac{\left(1-k^2 t^2\right)^{\gamma-\frac{1}{2}}}{\sqrt{\left(1-t^2\right)}}\, dt, \qquad \left(\left|k^2\right| < 1; \gamma \in C\right) \tag{7}$$

so that,

$$H(k;0) = K(k) \text{ and } \quad H(k;1) = E(k), \quad \left(\left|k^2\right| < 1\right) \tag{8}$$

where necessary constraint $\gamma \geq 0$ in Bushell's version of the definition (7) has naturally been waived.

Here we define a new elliptic function (generalized elliptic function of third kind)

$$R(\varphi, k, \xi; \alpha, \gamma) = \int_0^{\varphi} \frac{1}{\left(1+\xi \sin^2\theta\right)^{\alpha}\left(1-k^2\sin^2\theta\right)^{1/2-\gamma}}\, d\theta, \tag{9}$$

$$R(\varphi, k, \xi; \alpha, \gamma) = \int_0^{\sin\varphi} \frac{1}{\left(1+\xi v^2\right)^{\alpha}\sqrt{\left(1-v^2\right)}\left(1-k^2 v^2\right)^{1/2-\gamma}}\, dv, \tag{10}$$

$$\left(\left|k^2\right| < 1; 0 \leq \varphi \leq \frac{\pi}{2}; \gamma \in C, \alpha \geq 0\right)$$

where ξ is elliptic characteristic and $\xi > -1$.

Also here we define a new elliptic function

$$I(\varphi, k, \xi; \gamma) = \int_0^{\varphi} \frac{1}{\left(1+\xi\sin^2\theta\right)\left(1-k^2\sin^2\theta\right)^{1/2-\gamma}}\, d\theta, \tag{11}$$

$$I(\varphi, k, \xi; \gamma) = \int_0^{\sin\varphi} \frac{1}{\left(1+\xi v^2\right)\sqrt{\left(1-v^2\right)}\left(1-k^2 v^2\right)^{1/2-\gamma}}\, dv, \quad \left(\left|k^2\right| < 1; 0 \leq \varphi \leq \frac{\pi}{2}; \gamma \geq 0\right) \tag{12}$$

we can reduce the result of (11) and (12) by putting $\alpha = 1$ in (9) and (10).

If we put $\gamma = 0$ and $\alpha = 1$ then our function defined in (9) reduces into elliptic integral of the third kind or Legendre's incomplete elliptic integral

$$\Pi(\varphi, k, \xi) = \int_0^{\varphi} \frac{1}{\left(1+\xi\sin^2\theta\right)\left(1-k^2\sin^2\theta\right)^{1/2}}\, d\theta, \quad \left(\left|k^2\right| < 1; 0 \leq \varphi \leq \frac{\pi}{2}\right). \tag{13}$$

If $\xi = 0$, then our function of (9) reduces into elliptic integral of third kind

$$H(\varphi, k; \gamma) = \int_0^{\varphi} \frac{1}{\left(1-k^2\sin^2\theta\right)^{1/2-\gamma}}\, d\theta, \qquad \left(\left|k^2\right| < 1; 0 \leq \varphi \leq \frac{\pi}{2}\right) \tag{14}$$

and if $\gamma = 0$, $\xi = 0$, then our function (9) reduces to $F(\phi, k)$

$$F\left(\varphi, k\right) = \int_0^{\varphi} \frac{1}{\left(1 - k^2 \sin^2\theta\right)^{1/2}}\, d\theta, \qquad \left(\left|k^2\right| < 1; 0 \le \varphi \le \frac{\pi}{2}\right). \tag{15}$$

If $\gamma = 1$ and $\xi = 0$, then our function (9) reduces to $E(\phi, k)$

$$E\left(\varphi, k\right) = \int_0^{\varphi} \sqrt{\left(1 - k^2 \sin^2\theta\right)}\, d\theta, \qquad \left(\left|k^2\right| < 1; 0 \le \varphi \le \frac{\pi}{2}\right). \tag{16}$$

Same the manner if we put $\gamma = 1$, $\xi = 0$ and $\phi = \pi/2$, then our function of (9) reduces to $K(k) = F(\pi/2, k)$

$$K(k) = \int_0^{\pi/2} \frac{1}{\sqrt{\left(1 - k^2 \sin^2\theta\right)}}\, d\theta, \qquad \left(\left|k^2\right| < 1\right) \tag{17}$$

and similarly,

$$E(k) = \int_0^{\pi/2} \sqrt{\left(1 - k^2 \sin^2\theta\right)}\, d\theta, \qquad \left(\left|k^2\right| < 1\right) \tag{18}$$

for their details, see [8], [9], [32, Section 3], [25], [26] and [31, p.304].

Theorem 1. *If* $\overset{\omega}{{}_2R_1}(a, b; c; z)$ *is a the* ω*-Gauss hypergeometric function [32, 33] whose series representation is given by*

$$\overset{\omega}{{}_2R_1}(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n\, \Gamma(b + \omega n)}{\Gamma(c + \omega n)} . \frac{z^n}{n!}, \qquad |z| < 1$$

also consider that $\{\tau, \eta, \lambda, \mu\} \ge 0, \qquad (\tau + \eta > 0; \lambda + \mu > 0)$

and

$$\sum_{n=1}^{\infty} \left|\frac{a_n}{n^{1/2(1+\rho)}}\right| < \infty, \qquad \left[\tau = 0; R\left(\rho\right) > -1\right]$$

$$\sum_{n=1}^{\infty} \left|\frac{a_n}{n^{1+\sigma/2}}\right| < \infty, \qquad \left[\eta = 0; R\left(\sigma\right) > -2\right]$$

then

$$\int_0^1 k^{\rho} \left(\sqrt{\left(1 - k^2\right)}\right)^{\sigma} \overset{\omega}{{}_2R_1}\left(a, b; c; z k^{\tau} \left(\sqrt{\left(1 - k^2\right)}\right)^{\eta}\right) R\left(\varphi, \zeta k^{\lambda} \left(\sqrt{\left(1 - k^2\right)}\right)^{\mu}, \xi; \alpha, \gamma\right) dk$$
$$= \frac{\sin\varphi}{2} B\left(\frac{\rho + 1}{2}, \frac{\sigma + 2}{2}\right) . F^{3:2;1;1;1}_{2:1;0;0;0}$$
$$.\left[\begin{array}{l} \left(\tfrac{1}{2} : 0, 1, 1, 1\right), \left(\tfrac{\rho+1}{2} : \tfrac{\tau}{2}, \lambda, 0, 0\right), \left(\tfrac{\sigma+2}{2} : \tfrac{\eta}{2}, \mu, 0, 0\right) : (a, 1), (b, \omega); \left(\tfrac{1}{2} - \gamma, 1\right); \left(\tfrac{1}{2}, 1\right); (\alpha, 1); \\ \qquad \left(\tfrac{3}{2} : 0, 1, 1, 1\right), \left(\tfrac{\rho+\sigma+3}{2} : \tfrac{\eta+\tau}{2}, \lambda + \mu, 0, 0\right) : (c, \omega); -; -; \end{array}\right.$$
$$\left. z, \zeta^2 \sin^2\varphi, \sin^2\varphi, -\xi \sin^2\varphi \right]$$

where $R\left(\rho\right) > -1$ *and* $|\zeta| < 1$ [*or* $|\zeta| = 1$ *and* $R\left(\rho + 2\lambda\right) > -1$].

To prove the above Theorem we need the following formulas and relations which we have defining below:

A general family of integral formulas:

Srivastava & Daoust multivariable hypergeometric function defined by [27, 28]; see also [13]; [29, p.37 et seq.] and [30, p.64 et seq.]

$$F^{p:q_1;\dots;q_r}_{l:m_1;\dots;m_r}\left[\begin{matrix}\left(a_j;\alpha'_j,\dots\alpha_j^{(r)}\right)_{1,p}:\left(c'_j,\gamma'_j\right)_{1,q_1};\dots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,q_r};\\ \left(b_j;\beta'_j,\dots\beta_j^{(r)}\right)_{1,l}:\left(d'_j,\delta'_j\right)_{1,m_1};\dots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,m_r};\end{matrix} z_1,\dots,z_r\right]$$

$$=\sum_{n_1,\dots,n_r=0}^{\infty}\frac{\prod\limits_{j=1}^{p}\left(a_j\right)_{n_1\alpha'_j+\dots+n_r\alpha_j^{(r)}}\prod\limits_{j=1}^{q_1}\left(c'_j\right)_{n_1\gamma'_j}\cdots\prod\limits_{j=1}^{q_r}\left(c_j^{(r)}\right)_{n_r\gamma_j^{(r)}}}{\prod\limits_{j=1}^{l}\left(b_j\right)_{n_1\beta'_j+\dots+n_r\beta_j^{(r)}}\prod\limits_{j=1}^{m_1}\left(d'_j\right)_{n_1\delta'_j}\cdots\prod\limits_{j=1}^{m_r}\left(d_j^{(r)}\right)_{n_r\gamma_j^{(r)}}}\frac{z_1^{n_1}}{n_1!}\cdots\frac{z_r^{n_r}}{n_r!},\qquad(19)$$

the multiple hypergeometric series converges absolutely under the parametric and variable constraints, and $(\lambda)_v$ denotes the Pochhammer symbol which is defined below:

$$(\lambda)_v=\frac{\Gamma(\lambda+v)}{\Gamma(\lambda)}=\left\{\begin{matrix}1 & (v=0:\lambda\in C\setminus\{0\})\\ \lambda(\lambda+1)\dots(\lambda+n-1) & (v=n\in N=N_0\setminus\{0\}:\lambda\in C)\end{matrix}\right\},\qquad(20)$$

here λ, $\nu\in$ C.

We have by the definition of well known Riemann-Liouville operator $D_z^{\mu}f(z)$ of fractional calculus [18, p.54, eqn. (3.13)]:

$$D_z^{\lambda-\mu}\left\{z^{\lambda-1}\prod_{j=1}^{r}\left\{\left(1-a_j z^{\mu_j}\right)^{-\alpha_j}\right\}\right\}=\frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}F^{1:1;\dots;1}_{1:0;\dots;0}$$

$$\left[\begin{matrix}(\lambda;\mu_1,\dots\mu_r):(\alpha_1,1);\dots;(\alpha_r,1);\\ (\mu;\mu_1,\dots\mu_r):-------;\dots;--------;\end{matrix}a_1z^{\mu_1},\dots,a_rz^{\mu_r}\right],$$

$$\left[R(\lambda)>0;\mu_j>0\ \left(j=1,\dots,r\right);\max\left\{\left|a_1z^{\mu_1}\right|,\dots,\left|a_rz^{\mu_r}\right|\right\}<1\right],\qquad(21)$$

where the (Riemann-Liouville) fractional differintegral operator D_z^{v} is defined by [10, Chapter 13], [15, 18, 20, 21 and 23]

$$D_z^{v}\{f(z)\}=\left\{\begin{matrix}\dfrac{1}{\Gamma(-v)}\int_0^z(z-\zeta)^{-v-1}f(\zeta)\,d\zeta, & [R(v)<0]\\ \dfrac{d^n}{dz^n}D_z^{v-n}\{f(z)\}, & [0\le R(v)<n:n\in N_0]\end{matrix}\right.\qquad(22)$$

which shows the defining integral in (22) exists.

When we apply $r=3$ of the fractional differintegral formula (21) to the second integral in the definition (9) with, of course, $\lambda=\mu-1=1$ and $z=\sin\varphi$, we find that

$$R(\varphi,k,\xi;\alpha,\gamma)=\sin\varphi\,F^{1:1;1;1}_{1:0;0;0}\left[\begin{matrix}(1:2,2,2):(1/2-\gamma,1);(1/2,1);(\alpha,1);\\ (2:2,2,2):\quad;\quad;\quad;\end{matrix}k^2\sin^2\varphi,\sin^2\varphi,-\xi\sin^2\varphi\right],\qquad(23)$$

conditions are already defined in the (9) and (10).

Same the manner

$$I\left(\varphi, k, \xi; \gamma\right) = \sin\varphi\, F^{1:1;1;1}_{1:0;0;0}\left[\begin{matrix}(1;2,2,2):(1/2-\gamma,1);(1/2,1);(1,1);\\(2;2,2,2):\quad ;\quad ;\quad ;\end{matrix}\, k^2\sin^2\varphi, \sin^2\varphi, -\xi\sin^2\varphi\right], \tag{24}$$

conditions are already defined in the (11) and (12).

With the help of definition of Pochhammer symbol

$$\frac{(1)_{2l+2m+2n}}{(2)_{2l+2m+2n}} = \frac{\Gamma(2l+2m+2n+1)}{\Gamma(2l+2m+2n+2)} = \frac{\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{3}{2}\right)}\frac{\left(\frac{1}{2}\right)_{l+m+n}}{\left(\frac{3}{2}\right)_{l+m+n}} = \frac{\left(\frac{1}{2}\right)_{l+m+n}}{\left(\frac{3}{2}\right)_{l+m+n}}, \tag{25}$$

where we have used the Legendre duplication formula for the Gamma function:

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right), \tag{26}$$

with the help of above relation defined by equation (23), we can be write the relation such as

$$R\left(\varphi, k, \xi; \alpha, \gamma\right) = \sin\varphi\,.\,F_1\left[\frac{1}{2} : \frac{1}{2}-\gamma, \frac{1}{2}, \alpha; \frac{3}{2}; k^2\sin^2\varphi, \sin^2\varphi, -\xi\sin^2\varphi\right], \tag{27}$$

$$\left(\left|k^2\right| < 1; 0 \le \varphi \le \frac{\pi}{2}; \gamma \in C, \alpha \ge 0\right),$$

and similarly $I\left(\varphi, k, \xi; \gamma\right)$ can be defined in the similar manner, where F1 denotes the particular case of the Srivastava-Daoust multivariable hypergeometric function for three variables defined in above equation (19).

Here, to prove the Theorem, we also need the following binomial expansion:

$$(1-z)^{-\lambda} = \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}z^n, \qquad (|z| < 1; \lambda \in C) \tag{28}$$

and

$$(1+z)^{-\lambda} = \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}(-z)^n, \qquad (|z| < 1; \lambda \in C). \tag{29}$$

By the help of the binomial expansion, we can say that

$$\left(1-k^2\sin^2\theta\right)^{\gamma-\frac{1}{2}} = \sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}-\gamma\right)_n}{n!}k^{2n}\sin^{2n}\theta, \tag{30}$$

and

$$\left(1+\xi\sin^2\theta\right)^{-\alpha} = \sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}\left(-\xi\sin^2\theta\right)^n. \tag{31}$$

We also know that trigonometric form of the Eulerian integral for the Beta function $\mathrm{B}(\alpha,\beta)$:

$$\int_0^{\pi/2}\sin^{2\alpha-1}\theta\cos^{2\beta-1}\theta\,d\theta = \frac{1}{2}B\left(\alpha,\beta\right), \qquad \left(\min\left\{R(\alpha)\,.\,R\left(\beta\right)\right\} > 0 : B\left(\alpha,\beta\right) = \frac{\Gamma(\alpha)\,\Gamma\left(\beta\right)}{\Gamma\left(\alpha+\beta\right)}\right) \tag{32}$$

with help of these formulas and relations, we can easily establish Theorem 1.

Corollary 1. *With the help of definition new elliptic function defined in (11), we can establish the following result*

$$\int_0^1 k^\rho \left(\sqrt{(1-k^2)}\right)^\sigma {}_2\overset{\omega}{R}_1 \left(a, b; c; zk^\tau \left(\sqrt{(1-k^2)}\right)^\eta\right) I \left(\phi, \zeta\, k^\lambda \left(\sqrt{(1-k^2)}\right)^\mu, \xi; \gamma\right) d\,k$$
$$= \frac{\sin\phi}{2} B\left(\frac{\rho+1}{2}, \frac{\sigma+2}{2}\right) . F^{3:2;1;1;1}_{2:1;0;0;0}$$
$$\cdot \left[\begin{matrix} \left(\frac{1}{2}:0,1,1,1\right), \left(\frac{\rho+1}{2}:\frac{\tau}{2},\lambda,0,0\right), \left(\frac{\sigma+2}{2}:\frac{\eta}{2},\mu,0,0\right):(a,1),(b,\omega);\left(\frac{1}{2}-\gamma,1\right);\left(\frac{1}{2},1\right);(1,1); \\ \left(\frac{3}{2}:0,1,1,1\right), \left(\frac{\rho+\sigma+3}{2}:\frac{\eta+\tau}{2},\lambda+\mu,0,0\right):(c,\omega);-;-; \end{matrix}\right.$$
$$\left. z, \zeta^2\sin^2\phi, \sin^2\phi, -\xi\sin^2\phi \right], \tag{33}$$

with help of Theorem 1 we can determine the above Corollary 1*, by putting* $\alpha = 1$.

Corollary 2. *With the help of definition of elliptical function defined in (13), we can establish the following result*

$$\int_0^1 k^\rho \left(\sqrt{(1-k^2)}\right)^\sigma {}_2\overset{\omega}{R}_1 \left(a, b; c; zk^\tau \left(\sqrt{(1-k^2)}\right)^\eta\right) \Pi \left(\phi, \zeta\, k^\lambda \left(\sqrt{(1-k^2)}\right)^\mu, \xi\right) d\,k$$
$$= \frac{\sin\phi}{2} B\left(\frac{\rho+1}{2}, \frac{\sigma+2}{2}\right) . F^{3:2;1;1;1}_{2:1;0;0;0}$$
$$\cdot \left[\begin{matrix} \left(\frac{1}{2}:0,1,1,1\right), \left(\frac{\rho+1}{2}:\frac{\tau}{2},\lambda,0,0\right), \left(\frac{\sigma+2}{2}:\frac{\eta}{2},\mu,0,0\right):(a,1),(b,\omega);\left(\frac{1}{2},1\right);\left(\frac{1}{2},1\right);(1,1); \\ \left(\frac{3}{2}:0,1,1,1\right), \left(\frac{\rho+\sigma+3}{2}:\frac{\eta+\tau}{2},\lambda+\mu,0,0\right):(c,\omega);-;-; \end{matrix}\right.$$
$$\left. z, \zeta^2\sin^2\phi, \sin^2\phi, -\xi\sin^2\phi \right], \tag{34}$$

with help of Theorem 1 we can determine the above Corollary 2*, by putting* $\alpha = 1$ *and* $\gamma = 0$.

Theorem 2. *Here we define a bounded sequence* $\{a_n\}_{n=0}^{\infty}$ *of complex numbers, let*

$$\phi(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad (|z| < 1) \tag{35}$$

here we take that

$$\min\{\tau, \eta, \lambda, \mu, \delta, \beta\} \geq 0, \qquad (\tau+\eta > 0; \lambda+\mu > 0; \delta+\beta > 0) \tag{36}$$

and

$$\sum_{n=1}^{\infty} \left|\frac{a_n}{n^{1/2(1+\rho)}}\right| < \infty, \qquad [\tau = 0; R(\rho) > -1] \tag{37}$$

$$\sum_{n=1}^{\infty} \left|\frac{a_n}{n^{1+\sigma/2}}\right| < \infty, \qquad [\eta = 0; R(\sigma) > -2] \tag{38}$$

$$\int_0^1 k^\rho \left(\sqrt{(1-k^2)}\right)^\sigma \phi\left(zk^\tau\left(\sqrt{(1-k^2)}\right)^\eta\right) R\left(\phi,\zeta k^\lambda\left(\sqrt{(1-k^2)}\right)^\mu,\xi;\alpha,\gamma\right)dk$$
$$= \frac{1}{2}\sin\phi\sum_{n=0}^{\infty} a_n B\left(\frac{\rho+n\tau+1}{2},\frac{\sigma+n\eta}{2}+1\right)z^n$$
$$\cdot F^{1:3;1;1}_{1:1;0;0}\left[\begin{matrix}\left(\frac{1}{2}:1,1,1\right):\left(\frac{\rho+n\tau+1}{2},\lambda\right),\left(\frac{\sigma+n\eta}{2}+1,\mu\right),\left(\frac{1}{2}-\gamma,1\right);(\alpha,1);\left(\frac{1}{2},1\right);\\ \left(\frac{3}{2}:1,1,1\right):\left(\frac{\rho+n\tau+\sigma+n\eta+3}{2},\lambda+\mu\right);\quad;\quad;\end{matrix}\zeta^2\sin^2\phi,\sin^2\phi,-\xi\sin^2\phi\right]. \tag{39}$$

Proof. With the help of defined relation which defined below and using Theorem 1 and defined definition and relation we can easily prove Theorem 2

$$\int_0^1 k^\rho \left(\sqrt{(1-k^2)}\right)^\sigma R\left(\phi,\zeta k^\lambda\left(\sqrt{(1-k^2)}\right)^\mu,\xi;\alpha,\gamma\right)dk$$
$$= \frac{1}{2}\sin\phi . B\left(\frac{\rho+1}{2},\frac{\sigma}{2}+1\right)$$
$$\cdot F^{1:3;1;1}_{1:1;0;0}\left[\begin{matrix}\left(\frac{1}{2}:1,1,1\right):\left(\frac{\rho+1}{2},\lambda\right),\left(\frac{\sigma}{2}+1,\mu\right),\left(\frac{1}{2}-\gamma,1\right);(\alpha,1);\left(\frac{1}{2},1\right);\\ \left(\frac{3}{2}:1,1,1\right):\left(\frac{\rho+\sigma+3}{2},\lambda+\mu\right);\quad;\quad;\end{matrix}\zeta^2\sin^2\phi,\sin^2\phi,-\xi\sin^2\phi\right]. \tag{40}$$

which holds true under the relevant parts of the above-d constraints, whenever each member exists.

Now if we substitute for ϕ (zk τ κ η) from (35) into the integrand of (39), and make use of term-by-term integration by means of the integral formula (40), we shall be led formally to the right-hand side of the assertion (39). The formal process of term-by-term integration can be justified by the Theorem on Dominated Convergence (cf. [34, p.20 et seq.]) under the various conditions d above, since we have either

$$u_n \sim \frac{a_n}{n^{s+\frac{1}{2}(1+\rho)}}, \qquad \left[n\to\infty; s\in N_0=\{0,1,2,\ldots\};\tau=0: R(\rho)>-1\right] \tag{41}$$

or

$$u_n \sim \frac{a_n}{n^{1+s+\frac{1}{2}\rho}}, \qquad \left[n\to\infty; s\in N_0;\eta=0; R(\sigma)>-2\right] \tag{42}$$

where un denotes the coefficient of zn in the series on the right-hand side of (39). This evidently completes the proof of Theorem 2. □

Corollary 1. *With the help of definition new elliptical function defined in* (11)*, we can establish the following result.*

$$\int_0^1 k^\rho \left(\sqrt{(1-k^2)}\right)^\sigma \phi\left(zk^\tau\left(\sqrt{(1-k^2)}\right)^\eta\right) I\left(\phi,\zeta k^\lambda\left(\sqrt{(1-k^2)}\right)^\mu,\xi;\gamma\right)dk$$
$$= \frac{1}{2}\sin\phi\sum_{n=0}^{\infty} a_n B\left(\frac{\rho+n\tau+1}{2},\frac{\sigma+n\eta}{2}+1\right)z^n . F^{1:3;1;1}_{1:1;0;0}$$
$$\ldots\left[\begin{matrix}\left(\frac{1}{2}:1,1,1\right):\left(\frac{\rho+n\tau+1}{2},\lambda\right),\left(\frac{\sigma+n\eta}{2}+1,\mu\right),\left(\frac{1}{2}-\gamma,1\right);\left(\frac{1}{2},1\right);(1,1);\\ \left(\frac{3}{2}:1,1,1\right):\left(\frac{\rho+n\tau+\sigma+n\eta+3}{2},\lambda+\mu\right);\quad;\quad;\end{matrix}\zeta^2\sin^2\phi,\sin^2\phi,-\xi\sin^2\phi\right]. \tag{43}$$

This corollary can be find with the help of Theorem 2 by putting $\alpha = 1$.

Corollary 2. *With the help of definition new elliptical function defined in* (13)*, we can establish the following result*

$$\int_0^1 k^\rho \left(\sqrt{(1-k^2)}\right)^\sigma \phi\left(zk^\tau\left(\sqrt{(1-k^2)}\right)^\eta\right).\Pi\left(\phi,\zeta k^\lambda\left(\sqrt{(1-k^2)}\right)^\mu,\xi\right)dk$$
$$=\frac{1}{2}\sin\phi\sum_{n=0}^{\infty}a_n B\left(\frac{\rho+n\tau+1}{2},\frac{\sigma+n\eta}{2}+1\right)z^n.F^{1:3;1;1}_{1:1;0;0}$$
$$\cdot\left[\begin{matrix}\left(\frac{1}{2}:1,1,1\right):\left(\frac{\rho+n\tau+1}{2},\lambda\right),\left(\frac{\sigma+n\eta}{2}+1,\mu\right),\left(\frac{1}{2},1\right);\left(\frac{1}{2},1\right);(1,1);\\ \left(\frac{3}{2}:1,1,1\right):\left(\frac{\rho+n\tau+\sigma+n\eta+3}{2},\lambda+\mu\right);\qquad ;\quad ;\end{matrix}\zeta^2\sin^2\phi,\sin^2\phi,-\xi\sin^2\phi\right].\tag{44}$$

This corollary can be find with the help of Theorem 2 by putting $\alpha = 1$ and $\gamma = 0$.

Corollary 3. *With the help of definition new elliptical function defined in* (9)*, we can establish the following result*

$$\int_0^1 k^\rho\left(\sqrt{(1-k^2)}\right)^\sigma\phi(zk).R(\phi,k,\xi;\alpha,\gamma)\,dk$$
$$=\frac{1}{2}\sin\phi\sum_{n=0}^{\infty}a_n z^n B\left(\frac{\rho+n+1}{2},\frac{\sigma}{2}+1\right).F^{1:3;1;1}_{1:1;0;0}$$
$$\cdot\left[\begin{matrix}\left(\frac{1}{2}:1,1,1\right):\left(\frac{\rho+n+1}{2},1\right),\left(\frac{\sigma}{2}+1,0\right),\left(\frac{1}{2}-\gamma,1\right);\left(\frac{1}{2},1\right);(\alpha,1);\\ \left(\frac{3}{2}:1,1,1\right):\left(\frac{\rho+n\tau+\sigma+n\eta+3}{2},\lambda+\mu\right);\qquad ;\quad ;\end{matrix}\sin^2\phi,\sin^2\phi,-\xi\sin^2\phi\right].\tag{45}$$

This corollary can be find with the help of Theorem 2 by putting $\tau = 1$, $\eta = 0$, $\zeta = 1$, $\lambda = 1$, and $\mu = 0$.

Remark 1. The integral formulas which is defined in (39) and (40) having integral with respect to k, can easily rewritten as the corresponding integrals with respect to κ by means of the following simple change of variables

$$k\mapsto\sqrt{v^2-\kappa^2}\qquad\text{and}\qquad dk\mapsto-\frac{\kappa}{\sqrt{v^2-\kappa^2}}d\kappa,\qquad\left(\kappa=\sqrt{1-k^2}\right)\qquad\text{with }\kappa\in(0,v).\tag{46}$$

Alternatively, we can also rewrite the integral formulas (39) and (40) by applying the following change of variables which is defined below as

$$\kappa\mapsto\sqrt{v^2-k^2}\qquad\text{and}\qquad d\kappa\mapsto-\frac{k}{\sqrt{v^2-k^2}}dk,\qquad\text{with }k\in(0,v).\tag{47}$$

Theorem 3. *The following families of integrals hold true*

$$\int_0^{\pi/2}\sin^{2(a-1)}\varphi\cos^{2b-1}\varphi R(\varphi,k,\xi;\alpha,\gamma)\,d\varphi$$

$$= \frac{1}{2} B\,(a,b)\, F^{2:1;1;1}_{2:0;0;0} \left[\begin{matrix} \left(\frac{1}{2}:1,1,1\right), (a:1,1,1): & \left(\frac{1}{2}-\gamma,1\right); \left(\frac{1}{2},1\right); (\alpha,1); \\ \left(\frac{3}{2}:1,1,1\right), (a+b:1,1,1): & -;-\_;-; \end{matrix} k^2, 1, -\xi \right], \tag{48}$$

$$\left[\left|k^2\right| < 1 : \min\{R\,(a)\,, R\,(b)\} > 0; \gamma \in C\right].$$

Put $\alpha = 1$ *and* $n = 0$ *it reduces into the previous paper, and*

$$\int_0^{\pi/2} \sin^{2(a-1)} \phi \cos^{2b-1} \phi\, I\,(\phi, k, \xi; \gamma)\, d\phi$$
$$= \frac{1}{2} B\,(a,\, b)\, F^{2:1;1;1}_{2:0;0;0} \left[\begin{matrix} \left(\frac{1}{2}:1,1,1\right), (a:1,1,1): & \left(\frac{1}{2}-\gamma,1\right); \left(\frac{1}{2},1\right); (1,1); \\ \left(\frac{3}{2}:1,1,1\right), (a+b:1,1,1): & -;-;-, \end{matrix} k^2, 1, -\xi \right]. \tag{49}$$

provided that the second member of each of the integral formulas (48) *and* (49) *exists.*

Proof. Upon substituting the explicit hypergeometric representation for $R\,(\varphi, k, \xi; \alpha, \lambda)$ from (23) into the integral on the left-hand side of the assertion (48) of Theorem 3, if we apply the trigonometric integral (32) term-by-term, we are led easily to the integral formula (48) just as d above. The derivation of the assertion (49) of Theorem 3 can be based similarly upon the fractional differintegral formula (21). The details involved are being omitted here. □

Corollary 1. *With the help of definition new elliptical function defined in* (11)*, we can establish the following result*

$$\int_0^{\pi/2} \sin^{2(a-1)} \varphi \cos^{2b-1} \varphi\, I\,(\varphi, k, \xi; \gamma)\, d\varphi$$
$$= \frac{1}{2} B\,(a,b)\, F^{2:1;1;1}_{2:0;0;0} \left[\begin{matrix} \left(\frac{1}{2}:1,1,1\right), (a:1,1,1): & \left(\frac{1}{2}-\gamma,1\right); \left(\frac{1}{2},1\right); (1,1); \\ \left(\frac{3}{2}:1,1,1\right), (a+b:1,1,1): & -;-\_;-; \end{matrix} k^2, 1, -\xi \right]. \tag{50}$$

This Corollary can be find with the help of Theorem 3 by putting $\alpha = 1$.

Corollary 2. *With the help of the definition of new elliptical integral of third kind defined in* (13)*, we can establish the following result*

$$\int_0^{\pi/2} \sin^{2(a-1)} j \cos^{2b-1} j\, \Pi\,(j, k, \xi)\, dj$$
$$= \frac{1}{2} B\,(a,\, b)\, F^{2:1;1;1}_{2:0;0;0} \left[\begin{matrix} \left(\frac{1}{2}:1,1,1\right), (a:1,1,1): & \left(\frac{1}{2},1\right); \left(\frac{1}{2},1\right); (1,1); \\ \left(\frac{3}{2}:1,1,1\right), (a+b:1,1,1): & -;-;-; \end{matrix} k^2, 1, -\xi \right] \tag{51}$$

This Corollary can be find with the help of Theorem 3 by putting $\alpha = 1$ and $\gamma = 0$.

Remark 2. By means of the following change of variables

$$\phi = arc\sin x \qquad and \qquad d\phi = \frac{dx}{\sqrt{1-x^2}} \qquad with\, x \in (0,1) \tag{52}$$

the integral formula (48) can easily be written in the form

$$\int_0^1 x^{2(a-1)} \left(1-x^2\right)^{b-1} R\,(arc\sin x, k, \xi; \alpha, \gamma)\, dx$$

$$= \frac{1}{2} B\,(a,b)\, F^{2:1;1;1}_{2:0;0;0} \left[\begin{matrix} \left(\frac{1}{2}:1,1,1\right), (a:1,1,1): & \left(\frac{1}{2}-\gamma,1\right); \left(\frac{1}{2},1\right); (\alpha,1); \\ \left(\frac{3}{2}:1,1,1\right), (a+b:1,1,1): & -; -\_; -; \end{matrix} \; k^2, 1, -\xi \right], \tag{53}$$

$$\left[\left|k^2\right| < 1 : \min\left\{R\,(\alpha)\,,\, R\left(\beta\right)\right\} > 0; \gamma \in C\right]$$

which may be compared with the integral formula (49) asserted by Theorem 3.

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