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ON THE ARITHMETIC–GEOMETRIC MEAN INEQUALITY

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Abstract. We obtain some refinements of the Arithmetic–Geometric mean inequality. As an application, we find the maximum value of a multi-variable function.

1. Introduction

We assume that $a_1, a_2, \ldots, a_n$ are n positive real numbers, and as usual, we define their arithmetic and geometric means, respectively by

$$A = \frac{1}{n}\sum_{i=1}^{n} a_i, \qquad \text{and} \qquad G = \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}}.$$

We consider the functions $g(x) = \mathrm{e}^x - x^{\mathrm{e}}$ and $h(x) = x^{1/x}$ over $(0,\infty)$. The function h has an absolute maximum at $x = \mathrm{e}$. Thus, if $x > 0$ then $\mathrm{e}^{1/\mathrm{e}} \geqslant x^{1/x}$, or equivalently $g(x) \geqslant 0$, with equality if and only if $x = \mathrm{e}$. For $i = 1,2,\ldots,n$, we take $x = a_i\mathrm{e}/G$ in $\mathrm{e}^x \geqslant x^{\mathrm{e}}$, and then we multiply the resulting inequalities to get

$$\mathrm{e}^{\frac{\mathrm{e}}{G}nA} = \mathrm{e}^{\frac{\mathrm{e}}{G}\sum_{i=1}^{n} a_i} \geqslant \left(\prod_{i=1}^{n}\frac{a_i\mathrm{e}}{G}\right)^{\mathrm{e}} = \left(\frac{\mathrm{e}^n G^n}{G^n}\right)^{\mathrm{e}} = \mathrm{e}^{n\mathrm{e}},$$

from which we obtain $A \geqslant G$, with equality if and only if $a_i\mathrm{e}/G = \mathrm{e}$ for $i = 1,2,\ldots,n$, or equivalently for when $a_1 = a_2 = \cdots = a_n$.

The above argument for obtaining the Arithmetic–Geometric mean inequality is due to Schaumberger [1]. In this note we replace $g(x)$ by a smaller positive function to get some refinements of the this inequality. More precisely, we obtain the following result.

Theorem 1.1. *Assume that $a_1, a_2, \ldots, a_n$ are n positive real numbers with arithmetic and geometric means A and G, respectively. Then, we have*

$$A \geqslant G + \mathscr{R} \geqslant G,$$

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where

$$\mathscr{R} = \frac{G}{n\mathrm{e}} \log\left(1 + \frac{1}{\mathrm{e}^{n\mathrm{e}}} \prod_{i=1}^{n} \left(\frac{a_i}{G} - 1\right)^2\right) \geqslant 0,$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

2. Proof of Theorem 1.1

Lemma 2.1. *For* $x > 0$ *we define*

$$f(x) = \mathrm{e}^x - x^{\mathrm{e}} - \frac{1}{\mathrm{e}^2}(x - \mathrm{e})^2.$$

The inequality $f(x) \geqslant 0$ *is valid for* $x > 0$, *with equality if and only if* $x = \mathrm{e}$. *Moreover,* $\frac{1}{\mathrm{e}^2}$ *is the best possible constant for which the above inequality is valid.*

Proof. As Figure 1 shows, $f(x)$ takes its minimum value equal to 0 at $x = \mathrm{e}$. Also, we have $\lim_{x \to 0^+} f(x) = 0$, which proves optimal choose of the constant $\frac{1}{\mathrm{e}^2}$. This completes the proof. □

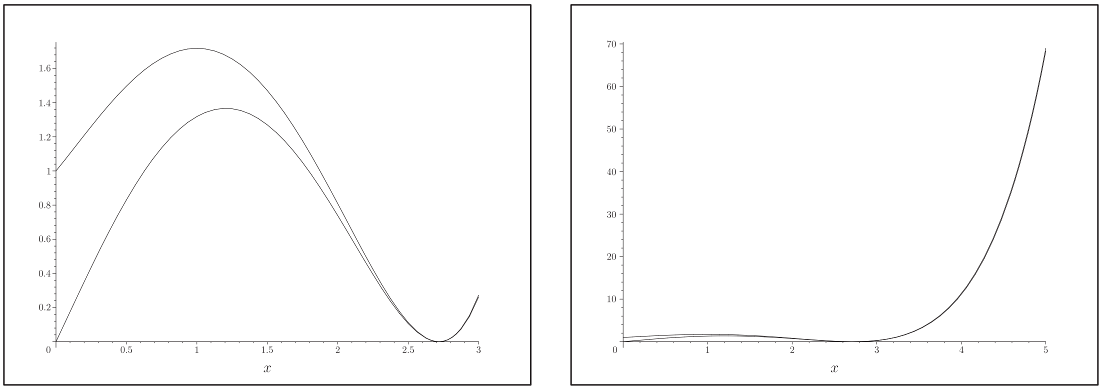

Figure 1: Graphs of the functions $f(x) = \mathrm{e}^x - x^{\mathrm{e}} - \frac{1}{\mathrm{e}^2}(x - \mathrm{e})^2$ and $g(x) = \mathrm{e}^x - x^{\mathrm{e}}$ over the intervals $(0, 3)$ and $(0, 5)$.

Proof of Theorem 1.1. We apply Lemma 2.1 by taking $x = a_i\mathrm{e}/G$ in $f(x)$, from which we obtain

$$\mathrm{e}^{\frac{\mathrm{e}}{G}a_i} \geqslant \left(\frac{a_i\mathrm{e}}{G}\right)^{\mathrm{e}} + \frac{1}{\mathrm{e}^2}\left(\frac{a_i\mathrm{e}}{G} - \mathrm{e}\right)^2 = \left(\frac{a_i\mathrm{e}}{G}\right)^{\mathrm{e}} + \left(\frac{a_i}{G} - 1\right)^2 \qquad (\text{for } i = 1, 2, \ldots, n).$$

We multiply these inequalities to get

$$\mathrm{e}^{\frac{\mathrm{e}}{G}nA} = \mathrm{e}^{\frac{\mathrm{e}}{G}\sum_{i=1}^{n} a_i} \geqslant \left(\prod_{i=1}^{n} \frac{a_i\mathrm{e}}{G}\right)^{\mathrm{e}} + \prod_{i=1}^{n}\left(\frac{a_i}{G} - 1\right)^2 = \mathrm{e}^{n\mathrm{e}} + \prod_{i=1}^{n}\left(\frac{a_i}{G} - 1\right)^2.$$

Thus, we have

$$\mathrm{e}^{\frac{\mathrm{e}}{G}nA} \geqslant \mathrm{e}^{n\mathrm{e}}\left(1 + \frac{1}{\mathrm{e}^{n\mathrm{e}}}\prod_{i=1}^{n}\left(\frac{a_i}{G} - 1\right)^2\right).$$

Finally, we take logarithm and we divide the resulting inequality by ne to obtain

$$\frac{A}{G} \geqslant 1 + \frac{1}{n\mathrm{e}} \log\left(1 + \frac{1}{\mathrm{e}^{n\mathrm{e}}} \prod_{i=1}^{n} \left(\frac{a_i}{G} - 1\right)^2\right),$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$. This completes the proof. □

3. Some applications

On may rewrite the Arithmetic–Geometric mean inequality in the forms

$$A - G \geqslant 0, \qquad \text{and} \qquad \frac{A}{G} - 1 \geqslant 0.$$

As the first application of Theorem 1.1, we obtain the following refinement of the above mentioned inequalities.

Theorem 3.1. *Assume that $a_1, a_2, \ldots, a_n$ are n positive real numbers which are not simultaneously equal, with arithmetic and geometric means A and G, respectively. Then, we have*

$$A - G \geqslant \frac{G\mathrm{e}^{n\mathrm{e}(\frac{A}{G}-1)}}{n\mathrm{e}(\mathrm{e}^{n\mathrm{e}\frac{A}{G}} - \mathrm{e}^{n\mathrm{e}})} \prod_{i=1}^{n} \left(\frac{a_i}{G} - 1\right)^2 \geqslant 0,$$

or equivalently

$$\frac{A}{G} - 1 \geqslant \frac{\mathrm{e}^{n\mathrm{e}(\frac{A}{G}-1)}}{n\mathrm{e}^{n\mathrm{e}+1}\left(\mathrm{e}^{n\mathrm{e}(\frac{A}{G}-1)} - 1\right)} \prod_{i=1}^{n} \left(\frac{a_i}{G} - 1\right)^2 \geqslant 0.$$

Proof. Assume that $a_1, a_2, \ldots, a_n$ are n positive real numbers which are not simultaneously equal, so that $A > G$. By using the result of Theorem 1.1, we have $\mathscr{R} \leq A - G$, which is equivalent to

$$\frac{1}{\mathrm{e}^{n\mathrm{e}}} \prod_{i=1}^{n} \left(\frac{a_i}{G} - 1\right)^2 \leqslant \mathrm{e}^{n\mathrm{e}(\frac{A}{G}-1)} - 1.$$

On the other hand, for $0 \leqslant x \leqslant \beta$, we have $\log(1 + x) \geqslant \frac{\log(1+\beta)}{\beta} x$ because $\frac{\log(1+x)}{x}$ is decreasing on $(0, \beta]$. We use this inequality by putting $x = \frac{1}{\mathrm{e}^{n\mathrm{e}}} \prod_{i=1}^{n} \left(\frac{a_i}{G} - 1\right)^2 \geqslant 0$ and $\beta = \mathrm{e}^{n\mathrm{e}(\frac{A}{G}-1)} - 1$ to get

$$\mathscr{R} = \frac{G}{n\mathrm{e}} \log\left(1 + \frac{1}{\mathrm{e}^{n\mathrm{e}}} \prod_{i=1}^{n} \left(\frac{a_i}{G} - 1\right)^2\right) \geqslant \frac{G}{n\mathrm{e}} \left(\frac{\mathrm{e}^{n\mathrm{e}(\frac{A}{G}-1)}}{\mathrm{e}^{n\mathrm{e}\frac{A}{G}} - \mathrm{e}^{n\mathrm{e}}} \prod_{i=1}^{n} \left(\frac{a_i}{G} - 1\right)^2\right).$$

This completes the proof. □

As the second application of Theorem 1.1, we observe that it allows us to find the maximum value of a multi-variable function, without using partial derivative tests.

Theorem 3.2. *We have*

$$\max_{a_i>0} \frac{G}{ne} \log\left(1+\frac{1}{e^{ne}}\prod_{i=1}^{n}\left(\frac{a_i}{G}-1\right)^2\right) = A-G.$$

Remark 3.3. We assume that $a_i > 0$, and then we replace a_i by $1/a_i$, from which the inequality $A \geqslant G$ implies validity of the well-known Geometric-Harmonic mean inequality, asserting that $G \geqslant H$, where H refers to the harmonic mean of the positive real numbers $a_1, a_2, \ldots, a_n$. We observe that the replacement $a_i \to 1/a_i$ gives the replacements $A \to 1/H$ and $G \to 1/G$. By applying this fact, one may rewrite all of the above results concerning the means A and G, to obtain similar results concerning the means G and H.

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