



INVERSE SPECTRAL-SCATTERING PROBLEM FOR THE STURM-LIOUVILLE OPERATOR ON A NONCOMPACT STAR-TYPE GRAPH

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Abstract. We study the Sturm-Liouville operator on a noncompact star-type graph consisting of a finite number of compact and noncompact edges under standard matching conditions in the internal vertex. We introduce and investigate the so-called spectral-scattering data, which generalize the classical spectral data for the Sturm-Liouville operator on the half-line and the scattering data on the line. Developing the idea of the method of spectral mappings we prove that the specification of the spectral-scattering data uniquely determines the Sturm-Liouville operator on the graph.

1. Introduction

We study an inverse problem for the Sturm-Liouville differential operator on an arbitrary noncompact star-type graph possessing a finite number of edges. This inverse problem consists in recovering the potential of the Sturm-Liouville operator on the graph along with the coefficients of boundary conditions in boundary vertices from appropriate spectral characteristics. Differential operators on graphs (spatial networks) often appear in mathematics, mechanics, physics, geophysics, physical chemistry, biology, electronics, nanoscale technology and other branches of natural sciences and engineering (see [1]–[14]). Recently there has increased interest in spectral theory of Sturm-Liouville or Schrödinger operators on graphs (for a good review of such publications see [15], [16]). Most of the works in this direction are devoted to the so-called *direct* problems of studying properties of the spectrum and the root functions. *Inverse* spectral problems on graphs are because of their nonlinearity more difficult and their intensive investigation started only several years ago.

Most complete results on inverse spectral problems for the Sturm-Liouville operator on graphs were obtained for *compact* graphs including trees and also graphs with cycles (see

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[17]–[23] and the references therein). Certain systems of spectra or Weyl functions were shown to be appropriate input data in the compact case. In [24] an inverse spectral problem was studied for the Sturm-Liouville operator on a noncompact tree with a single noncompact edge, where Weyl functions associated with finite boundary vertices were introduced and used as the spectral data. Inverse spectral problems for higher-order differential operators on compact trees were studied in [25] and on trees with a single noncompact edge in [26].

In the case of graphs with more than one noncompact edges it is natural to use the so-called scattering data, which generalize the scattering data for the Sturm-Liouville operators on the line [27]–[31]. For the first time the inverse scattering problem on a noncompact graph was discussed in [32], but only for a very special case of star-type graphs without compact edges. In [33] this case was reduced to the inverse scattering problem for the matrix Sturm-Liouville operator on the half-line with a special selfadjoint boundary condition in the origin. In [34]–[36] inverse scattering problems were studied for graphs consisting of a cycle and a finite number rays.

In presence of compact boundary edges the scattering data are not sufficient to determine the potential on all edges of the graph. For example, in [37], [38] some non-uniqueness results were obtained for inverse scattering problems on general noncompact graphs.

In this paper we provide a formulation and prove the uniqueness theorem for the inverse problem of recovering the Sturm-Liouville operator on a noncompact star-type graph possessing compact edges. In this situation it is natural to consider a mixture of inverse spectral and inverse scattering problems. The spectral-scattering data, which we use, can be subdivided into two parts. The "spectral" part consists of the so-called Weyl functions associated with the compact edges, which generalize the Weyl function of the Sturm-Liouville operator on the half-line (see, e.g., [31]) and on a noncompact tree with a single infinite edge (see [24]). The "scattering" part, in turn, includes a portion of negative eigenvalues together with the so-called reflection coefficients and norming constants associated with all but one noncompact edges. Such scattering data generalizes the left (or right) scattering data for the Sturm-Liouville operator on the line (see [27]–[31]). Since some eigenvalues are poles of the Weyl functions from the "spectral" part, in the "scattering" part it suffices to specify only the so-called invisible from compact edges eigenvalues. Developing the ideas of the method of spectral mappings [39], we prove that the specification of these spectral-scattering data uniquely determines the potential of the Sturm-Liouville operator on the graph along with the coefficients of the boundary conditions in the boundary vertices. We note that the case without compact edges was studied separately in [40] and also with general matching conditions in [41]. The presence of compact edges causes, inter alia, new qualitative difficulties in the investigation of inverse problems due to a more complicated behavior of the scattering solutions and reflection coefficients.

In the next section we introduce and investigate special solutions of the differential equation on the graph, which possess necessary asymptotical and analytical properties. In Section 3 properties of spectral characteristics are studied and the spectral-scattering data are introduced. In Section 4 we prove the uniqueness theorem of the inverse problem.

2. Special solutions

In a finite-dimensional Euclidean space we consider a noncompact star-type graph Γ with the vertices v_0, v_1, \dots, v_p and the edges $\varepsilon_1, \dots, \varepsilon_m$, where $\varepsilon_j = [v_0, v_j], j = \overline{1, p}$, are segments and $\varepsilon_j, j = \overline{p+1, m}$, are rays with the common vertex v_0 . Let for definiteness $p \geq 1$ and $m - p \geq 2$. Moreover, for simplicity we assume that the length of each compact edge is equal to 1 (it follows from the proofs that our method works also for arbitrary lengths). The vertices $v_j, j = \overline{1, p}$, are called *boundary* vertices. The vertex v_0 is called *internal* vertex and considered as the initial point of all edges, which, in turn, are parameterized by $x \geq 0$, and $x = 0$ corresponds always to the internal vertex. For the compact edges we have $x \in [0, 1]$ and $x = 1$ corresponds to the boundary vertex. Any function y on Γ can be represented as a vector $y = [y_j(x)]_{j=\overline{1, m}}$ (but not a vector-function), where the function $y_j(x)$ is defined on the edge ε_j . Let $q = [q_j(x)]_{j=\overline{1, m}}$ be a real-valued function on Γ , which we will call the *potential*. We assume that

$$q_j(x) \in L(0, 1), \quad j = \overline{1, p}; \quad q_j(x), xq_j(x) \in L(0, \infty), \quad j = \overline{p+1, m}.$$

Consider the Sturm-Liouville equation on Γ :

$$\ell_j y_j := -y_j'' + q_j(x)y_j = \lambda y_j, \tag{1}$$

where $x \in (0, 1)$ for $j = \overline{1, p}$; $x \in (0, \infty)$ for $j = \overline{p+1, m}$ and λ is the spectral parameter. Let the function $y = [y_j(x)]_{j=\overline{1, m}}$ satisfy the following m matching conditions in the vertex v_0 :

$$\left. \begin{aligned} y_1(0) = y_j(0), \quad j = \overline{2, m}, \quad (\text{continuity condition}) \\ \sum_{j=1}^m y_j'(0) = 0 \quad (\text{Kirchhoff's condition}). \end{aligned} \right\} \tag{2}$$

and the following boundary conditions in the boundary vertices:

$$U_j(y_j) := y_j'(1) + H_j y_j(1) = 0, \quad j = \overline{1, p}, \tag{3}$$

where $H_j \in \mathbb{R}, j = \overline{1, p}$. Denote by $L = L(q, H)$ the boundary value problem (1)–(3), where $H = [H_j]_{j=\overline{1, p}}$. We also consider the corresponding operator

$$\mathcal{L} : D(\mathcal{L}) \rightarrow L_2(\Gamma), \quad y = [y_j]_{j=\overline{1, m}} \mapsto \mathcal{L}y := [\ell_j y_j]_{j=\overline{1, m}},$$

where the domain of definition $D(\mathcal{L})$ consists of functions $y = [y_j(x)]_{j=\overline{1,m}}$ such that

$$y_j \in W_1^2[0, 1], \quad j = \overline{1, p}; \quad y_j \in W_{1,loc}^2[0, \infty), \quad y_j, \ell_j y_j \in L_2(0, \infty), \quad j = \overline{p+1, m},$$

and y satisfies (2), (3). The matching conditions (2) are called the standard matching conditions. In electrical circuits (2) express Kirchhoff’s law, in elastic string network they express the balance of tension, etc.

Let $\lambda = \rho^2$. Denote

$$\begin{aligned} \mathbb{R}^* &= \mathbb{R} \setminus \{0\}, \quad \Omega_{\pm} = \{\rho : \pm \operatorname{Im} \rho > 0\}, \\ Q_{0j}(x) &= \int_x^\infty |q_j(t)| dt, \quad Q_{1j}(x) = \int_x^\infty Q_{0j}(t) dt = \int_x^\infty (t-x)|q_j(t)| dt. \end{aligned}$$

Let $C_j(x, \lambda), S_j(x, \lambda)$ be the solutions of equation (1) on the edge ε_j satisfying the initial conditions

$$C_j(0, \lambda) = S'_j(0, \lambda) = 1, \quad C'_j(0, \lambda) = S_j(0, \lambda) = 0. \tag{4}$$

On the compact edges $\varepsilon_j, j = \overline{1, p}$, we introduce also the solutions $\psi_j(x, \lambda), \theta_j(x, \lambda)$ such that

$$\psi_j(1, \lambda) = -\theta'_j(1, \lambda) = 1, \quad U_j(\psi_j(x, \lambda)) = \theta_j(1, \lambda) = 0. \tag{5}$$

For $|\rho| \rightarrow \infty$ the following estimates hold (see, e.g., [31]):

$$\left. \begin{aligned} \psi_j(x, \lambda) &= \cos \rho(1-x) - \omega_j(x) \frac{\sin \rho(1-x)}{\rho} + o\left(\frac{1}{\rho} \exp(|\operatorname{Im} \rho|(1-x))\right), \\ \psi'_j(x, \lambda) &= \rho \sin \rho(1-x) + \omega_j(x) \cos \rho(1-x) + o(\exp(|\operatorname{Im} \rho|(1-x))), \end{aligned} \right\} \tag{6}$$

where

$$\omega_j(x) = -H_j - \frac{1}{2} \int_x^1 q_j(t) dt, \quad j = \overline{1, p}.$$

The following theorem introduces the Jost solution $e_j(x, \rho)$ on the noncompact edges $\varepsilon_j, j = \overline{p+1, m}$, with prescribed behaviour in ∞ (see [31]).

Theorem 1. *Let $\rho \in \overline{\Omega_+}$ and $j \in \{p+1, \dots, m\}$. Equation (1) has a unique solution $y_j = e_j(x, \rho)$ satisfying the integral equation*

$$e_j(x, \rho) = \exp(i\rho x) + \int_x^\infty \frac{\sin \rho(t-x)}{\rho} q_j(t) e_j(t, \rho) dt. \tag{7}$$

The function $e_j(x, \rho)$ has the following properties:

- (i) For each fixed x the functions $e_j^{(v)}(x, \rho), v = 0, 1$, are analytic in Ω_+ and continuous in $\overline{\Omega_+}$.

(ii) For $v = 0, 1$

$$e_j^{(v)}(x, \rho) = (i\rho)^v \exp(i\rho x)(1 + o(1)), \quad x \rightarrow +\infty, \tag{8}$$

uniformly in $\overline{\Omega_+}$. Moreover, for $\rho \in \overline{\Omega_+}$

$$\left. \begin{aligned} |e_j(x, \rho) \exp(-i\rho x)| &\leq \exp(Q_{1j}(x)), \\ |e_j(x, \rho) \exp(-i\rho x) - 1| &\leq Q_{1j}(x) \exp(Q_{1j}(x)), \\ |e'_j(x, \rho) \exp(-i\rho x) - i\rho| &\leq Q_{0j}(x) \exp(Q_{1j}(x)). \end{aligned} \right\} \tag{9}$$

(iii) For each fixed $\rho \in \Omega_+$ $e_j(x, \rho) \in L_2(0, \infty)$. Moreover, $e_j(x, \rho)$ is a unique solution of (1) (up to a multiplicative constant) having this property.

(iv) For $|\rho| \rightarrow \infty$, $\rho \in \overline{\Omega_+}$, $v = 0, 1$

$$e_j^{(v)}(x, \rho) = (i\rho)^v \exp(i\rho x) \left(1 + \frac{\omega_j(x)}{i\rho} + o\left(\frac{1}{\rho}\right) \right), \quad \omega_j(x) = -\frac{1}{2} \int_x^\infty q_j(t) dt \tag{10}$$

uniformly for $x \geq 0$.

(v) For $\rho \in \mathbb{R}^*$ the functions $e_j(x, \rho)$, $e_j(x, -\rho)$ form a fundamental system of solutions for (1), and

$$\langle e_j(x, \rho), e_j(x, -\rho) \rangle \equiv -2i\rho, \tag{11}$$

where $\langle y, z \rangle := yz' - y'z$.

The next lemma (see [31]) gives properties of the Jost solutions $e_{j,r}(x, \rho)$ related to the potentials $q_{j,r}(x)$ that approximate $q_j(x)$.

Lemma 1. *If*

$$\lim_{r \rightarrow \infty} \int_0^\infty (1+x)|q_{j,r}(x) - q_j(x)| dx = 0,$$

then

$$\lim_{r \rightarrow \infty} \sup_{\rho \in \overline{\Omega_+}} \sup_{x \geq 0} |(e_{j,r}^{(v)}(x, \rho) - e_j^{(v)}(x, \rho)) \exp(-i\rho x)| = 0, \quad v = 0, 1.$$

Here $e_{j,r}(x, \rho)$ are the Jost solutions for the potentials $q_{j,r}(x)$.

Let $\Psi_k(\lambda) = [\Psi_{kj}(x, \lambda)]_{j=\overline{1, m}}$, $k = \overline{1, p}$, be solutions of (1), (2) satisfying the following boundary conditions in the boundary vertices:

$$U_j(\Psi_{kj}(x, \lambda)) = -\delta_{jk}, \quad j = \overline{1, p}, \tag{12}$$

and having the following asymptotics on the noncompact edges:

$$\Psi_{kj}(x, \lambda) = O(\exp(i\rho x)), \quad x \rightarrow \infty, \quad \rho \in \overline{\Omega_+}, \quad j = \overline{p+1, m}. \tag{13}$$

The functions $\Psi_k(\lambda)$ and $M_k(\lambda) := \Psi_{kk}(1, \lambda)$ are called respectively the *Weyl solution* and the *Weyl function* associated with the boundary vertex $v_k, k = \overline{1, p}$.

We introduce also the solutions $f_k(\rho) = [f_{kj}(x, \rho)]_{j=\overline{1, m}}, k = \overline{p+1, m}$, of (1)–(3) having the following asymptotics on the noncompact edges:

$$f_{kk}(x, \rho) \sim \exp(-i\rho x), \quad f_{kj}(x, \rho) = O(\exp(i\rho x)), \quad x \rightarrow \infty, \rho \in \Omega_+, j = \overline{p+1, m} \setminus k. \quad (14)$$

The function $f_k(\rho)$ is called the *scattering solution* associated with the edge $\varepsilon_k, k = \overline{p+1, m}$.

In order to investigate the Weyl and scattering solutions we construct the auxiliary solutions $g_k(\rho) = [g_{kj}(x, \rho)]_{j=\overline{1, m}}, k = \overline{1, m}$, of (1), (2) in the following way:

$$g_{kj}(x, \rho) = e_j(x, \rho) \prod_{l \neq k, j} e_l(0, \rho), \quad j \neq k, \quad (15)$$

$$g_{kk}(x, \rho) = C_k(x, \lambda) \prod_{l \neq k} e_l(0, \rho) - S_k(x, \lambda) \sum_{j \neq k} e'_j(0, \rho) \prod_{l \neq k, j} e_l(0, \rho), \quad (16)$$

where here and in the sequel for brevity we use the notation

$$e_j(x, \rho) := \psi_j(x, \rho^2), \quad j = \overline{1, p}. \quad (17)$$

Obviously $g_{kj}(x, \rho)$ for each fixed $x \geq 0$ is analytic in Ω_+ and continuous in $\overline{\Omega_+}$. Moreover, we have

$$\left. \begin{aligned} U_j(g_{kj}(x, \rho)) &= 0, & j &= \overline{1, p} \setminus k, \\ g_{kj}(x, \rho) &= O(\exp(i\rho x)), \quad x \rightarrow \infty, \rho \in \overline{\Omega_+}, & j &= \overline{p+1, m} \setminus k. \end{aligned} \right\} \quad (18)$$

Lemma 2. *The following representation holds*

$$\Psi_k(\lambda) = \frac{1}{\Delta(\lambda)} g_k(\rho), \quad k = \overline{1, p}, \quad (19)$$

where

$$\Delta(\lambda) = \sum_{j=1}^m e'_j(0, \rho) \prod_{l \neq j} e_l(0, \rho). \quad (20)$$

Proof. According to (12), (13), (18) we have $\Psi_k(\lambda) = -(U_k(g_{kk}(x, \rho)))^{-1} g_k(\rho), k = \overline{1, p}$. Using (3)–(5) we get

$$U_k(C_k(x, \lambda)) = -\psi'_k(0, \lambda) = -e'_k(0, \rho), \quad U_k(S_k(x, \lambda)) = \psi_k(0, \lambda) = e_k(0, \rho), \quad k \in \overline{1, p},$$

which together with (16) gives $U_k(g_{kk}(x, \rho)) = -\Delta(\lambda)$. □

It is obvious that for $k = \overline{1, p}$ the following relations hold

$$\Psi_{kk}(x, \lambda) = \theta_k(x, \lambda) + M_k(\lambda) \psi_k(x, \lambda) = \frac{g_{kk}(x, \rho)}{\Delta(\lambda)}, \quad M_k(\lambda) = \frac{g_{kk}(1, \rho)}{\Delta(\lambda)}. \quad (21)$$

Thus, the Weyl solutions and the Weyl functions have a jump along the real semi-axis $[0, \infty)$ in the λ -plane.

Further, let $k = \overline{p+1, m}$. Then for $\rho \in \mathbb{R}^*$ we have

$$g_{kk}(x, \rho) = a(\rho)e_k(x, -\rho) + b_k(\rho)e_k(x, \rho), \quad k = \overline{p+1, m}, \tag{22}$$

where $a(\rho)$ does not depend on k . Indeed, using (11) we have

$$\begin{bmatrix} a(\rho) \\ b_k(\rho) \end{bmatrix} = \frac{1}{2i\rho} \begin{bmatrix} e'_k(0, \rho) & -e_k(0, \rho) \\ -e'_k(0, -\rho) & e_k(0, -\rho) \end{bmatrix} \begin{bmatrix} g_{kk}(0, \rho) \\ g'_{kk}(0, \rho) \end{bmatrix}.$$

Thus, using (15), (16), (20) we calculate

$$a(\rho) = \frac{1}{2i\rho} \langle g_{kk}(x, \rho), e_k(x, \rho) \rangle = \frac{1}{2i\rho} \Delta(\rho^2), \tag{23}$$

$$b_k(\rho) = -\frac{1}{2i\rho} \left(e'_k(0, -\rho) \prod_{l \neq k} e_l(0, \rho) + e_k(0, -\rho) \sum_{j \neq k} e'_j(0, \rho) \prod_{l \neq k, j} e_l(0, \rho) \right). \tag{24}$$

The second equality in (23) is valid for $k = \overline{1, m}$.

We note that (23) gives an analytic continuation of $a(\rho)$ in Ω_+ . Hence the function $a(\rho)$ is analytic in Ω_+ and $\rho a(\rho)$ is continuous in $\overline{\Omega_+}$. Moreover, using (20), (23) together with the asymptotics of the Jost solutions (10) and the entire solutions $e_j(x, \rho) = \psi_j(x, \rho^2)$, $j = \overline{1, p}$, (6) for each $\varepsilon > 0$ we get

$$a(\rho) = \frac{m}{2^{p+1}} \exp(-pi\rho) \left(1 + \frac{1}{i\rho} \sum_{j=1}^m \omega_j(0) + o\left(\frac{1}{\rho}\right) \right), \quad |\rho| \rightarrow \infty, \quad \rho \in \Omega_\varepsilon, \tag{25}$$

where $\Omega_\varepsilon = \{\rho : \arg \rho \in [\varepsilon, \pi - \varepsilon]\}$.

Lemma 3. For $\rho \in \mathbb{R}^*$, $k = \overline{p+1, m}$ the following relations hold:

$$\overline{a(\rho)} = a(-\rho), \quad \overline{b_k(\rho)} = b_k(-\rho), \tag{26}$$

$$|a(\rho)|^2 - |b_k(\rho)|^2 = \sum_{\substack{j \neq k \\ j=p+1}}^m \prod_{l \neq k, j} |e_l(0, \rho)|^2. \tag{27}$$

Proof. By virtue of (7), (17) we have $e_j(x, -\rho) = e_j(x, \rho) = \overline{e_j(x, \rho)}$ for $j = \overline{1, p}$ and $e_j(x, -\rho) = \overline{e_j(x, \rho)}$ for $j = \overline{p+1, m}$, $\rho \in \mathbb{R}$. Thus, (26) follows from (23), (24). Further, (11), (22) give

$$\langle g_{kk}(x, \rho), g_{kk}(x, -\rho) \rangle = 2i\rho(a(\rho)a(-\rho) - b_k(\rho)b_k(-\rho)), \quad k = \overline{p+1, m}. \tag{28}$$

On the other hand, according to (11), (16) we have

$$\langle g_{kk}(x, \rho), g_{kk}(x, -\rho) \rangle = 2i\rho \sum_{\substack{j \neq k \\ j=p+1}}^m \prod_{l \neq k, j} e_l(0, \rho)e_l(0, -\rho), \quad k = \overline{p+1, m},$$

which together with (26), (28) give (27). □

Corollary 1. *Let $\rho \in \mathbb{R}^*$. Since $e_j(x, \rho) \neq 0$, $x \geq 0$, $j = \overline{p+1, m}$, according to (20), (23), (27) we have $a(\rho) = 0$ only if $\psi_{l_\nu}(0, \rho^2) = 0$ for at least two distinct indices $l_\nu \in \{1, \dots, p\}$, $\nu = 1, 2$. Moreover, for all pairs of distinct indices $k, j \in \{p+1, \dots, m\}$ we have*

$$\left| \frac{1}{a(\rho)} \prod_{l \neq k, j} e_l(0, \rho) \right| \leq 1, \quad \rho \in \mathbb{R}_1^* := \{\rho : \rho \in \mathbb{R}^*, a(\rho) \neq 0\}. \quad (29)$$

Lemma 4. *For $\rho \in \Omega_+$ such that $a(\rho) \neq 0$ the following representation holds:*

$$f_k(\rho) = \frac{1}{a(\rho)} g_k(\rho), \quad k = \overline{p+1, m}. \quad (30)$$

Proof. According to (8), (14) we have

$$\langle e_k(x, \rho), f_{kk}(x, \rho) \rangle = -2i\rho. \quad (31)$$

Let us first show that if $a(\rho) \neq 0$ then

$$f_k(\rho) = D_k(\rho) g_k(\rho). \quad (32)$$

For $a(\rho) \neq 0$ and $j \neq k$ it is easy to show that $f_{kj}(x, \rho) = D_{kj}(\rho) g_{kj}(x, \rho)$, whence according to (2), (15) we get

$$(D_{kj_1}(\rho) - D_{kj_2}(\rho)) \prod_{j \neq k} e_j(0, \rho) = 0, \quad j_1, j_2 \neq k.$$

Thus, if

$$\prod_{j \neq k} e_j(0, \rho) \neq 0, \quad (33)$$

then all $D_{kj}(\rho)$, $j \neq k$, are equal and we put $D_k(\rho) := D_{kj}(\rho)$. Let (33) be false. Then there exists a unique $j_0 \neq k$ such that $e_{j_0}(0, \rho) = 0$ (otherwise (20), (23) infer $a(\rho) = 0$). Taking (15) into account we obtain $g_{kj}(x, \rho) \equiv 0$, for $j \neq k, j_0$ and put $D_k(\rho) = D_{kj_0}(\rho)$. Thus, we have $f_{kj}(x, \rho) = D_k(\rho) g_{kj}(x, \rho)$, $j \neq k$. Since both the functions $f_k(\rho)$, $g_k(\rho)$ satisfy the matching conditions (2) we get

$$f_{kk}(x, \rho) = D_k(\rho) g_{kk}(x, \rho) \quad (34)$$

and arrive at (32). Substituting (34) into (31) and using (23) we obtain

$$-2i\rho = D_k(\rho) \langle e_k(x, \rho), g_{kk}(x, \rho) \rangle = -2i\rho D_k(\rho) a(\rho),$$

which together with (32) gives (30). □

For $\rho \in \mathbb{R}_1^*$ we consider (30) as a definition of the function $f_k(\rho)$. By virtue of the continuity of the functions $e_k(x, \rho)$, $f_{kk}(x, \rho)$ with respect to ρ in their domains of definition, formula (31) remains valid also for $\rho \in \mathbb{R}_1^*$.

3. Spectral-scattering data

According to (22) and (30) we get

$$f_{kk}(x, \rho) = e_k(x, -\rho) + s_k(\rho)e_k(x, \rho), \quad \rho \in \mathbb{R}_1^*, \quad k = \overline{p+1, m}, \tag{35}$$

where the function

$$s_k(\rho) = \frac{b_k(\rho)}{a(\rho)} \tag{36}$$

is called the *reflection coefficient* associated with the edge ε_k , $k = \overline{p+1, m}$.

Let us point out some properties of $s_k(\rho)$. By virtue of (23), (24), (26), (36) the functions $s_k(\rho)$ are continuous for $\rho \in \mathbb{R}_1^*$, and

$$\overline{s_k(\rho)} = s_k(-\rho).$$

Moreover, (27) implies

$$|s_k(\rho)|^2 = 1 - \frac{1}{|a(\rho)|^2} \sum_{\substack{j \neq k \\ j=p+1}}^m \prod_{l \neq k, j} |e_l(0, \rho)|^2,$$

and consequently we have

$$|s_k(\rho)| < 1, \quad \rho \in \mathbb{R}_1^*.$$

Let us now study the properties of the discrete spectrum.

Definition 1. The values of the parameter λ , for which equation (1) has nonzero solutions $y = [y_j(x)]_{j=\overline{1, m}}$, satisfying the matching conditions (2), the boundary conditions (3) and $y_j(x) \in L_2(0, \infty)$, $j = \overline{p+1, m}$, are called *eigenvalues* of L , and the corresponding nonzero solutions are called *eigenfunctions*.

Denote by Λ the set of all eigenvalues of L and put

$$\begin{aligned} \Lambda_1 &:= \{\lambda = \rho^2 : \rho \in \Omega_+, a(\rho) = 0\}, \quad \Lambda_2 := \{\lambda = \rho^2 : \rho \in \mathbb{R}^*, a(\rho) = 0\}, \\ \Delta_0(\lambda) &:= \sum_{j=1}^p \psi'_j(0, \lambda) \prod_{\substack{l \neq j \\ l=1}}^p \psi_l(0, \lambda), \\ \Delta_1(\lambda) &:= \prod_{j=1}^p \psi_j(0, \lambda), \\ \Lambda_3 &:= \{\lambda : \lambda \geq 0, \Delta_0(\lambda) = \Delta_1(\lambda) = 0\}. \end{aligned} \tag{37}$$

According to (20), (23) we have

$$a(\rho) = \Delta_0(\rho^2)a_1(\rho) + \Delta_1(\rho^2)a_0(\rho), \tag{38}$$

where

$$a_0(\rho) := \frac{1}{2i\rho} \sum_{j=p+1}^m e'_j(0, \rho) \prod_{\substack{l \neq j \\ l=p+1}}^m e_l(0, \rho), \quad a_1(\rho) := \frac{1}{2i\rho} \prod_{j=p+1}^m e_j(0, \rho). \quad (39)$$

Consider the boundary value problem L_0 for equation (1) on the graph Γ_0 that is obtained from Γ by removing the noncompact edges ε_j , $j = \overline{p+1, m}$, with the new matching conditions

$$y_1(0) = \dots = y_p(0), \quad y'_1(0) + \dots + y'_p(0) = 0$$

and the boundary conditions (3). It is obvious that the eigenvalues of L_0 coincide with the zeros of the entire function $\Delta_0(\lambda)$.

Lemma 5. $\Lambda_2 = \Lambda_3 \setminus \{0\}$.

Proof. According to (38) it is sufficient to prove the inclusion $\Lambda_2 \subset \Lambda_3$. Let $\lambda = \rho^2 \in \Lambda_2$, then (26), (38) give

$$\begin{cases} \Delta_0(\lambda) a_1(\rho) + \Delta_1(\lambda) a_0(\rho) = 0, \\ \Delta_0(\lambda) a_1(-\rho) + \Delta_1(\lambda) a_0(-\rho) = 0. \end{cases} \quad (40)$$

By virtue of (11), (39) the determinant of this linear algebraic system does not vanish:

$$a_1(\rho) a_0(-\rho) - a_1(-\rho) a_0(\rho) = \frac{1}{2i\rho} \sum_{j=p+1}^m \prod_{\substack{l \neq j \\ l=p+1}}^m |e_l(0, \rho)|^2 \neq 0.$$

Thus, from (40) it follows that $\lambda \in \Lambda_3$. □

Theorem 2. *The set $\Lambda = \{\lambda_n\}$ is at most countable. All eigenvalues λ_n are real. Eigenfunctions related to different eigenvalues are orthogonal in $L_2(\Gamma)$. For each eigenvalue there exist up to $m-1$ linearly independent eigenfunctions. Moreover,*

- (i) $\Lambda = \Lambda_1 \cup \Lambda_3$;
- (ii) $y_j(x) \equiv 0$, $j = \overline{p+1, m}$, for each eigenfunction $y = [y_j(x)]_{j=\overline{1, m}}$ related to $\lambda \in \Lambda_3$.
- (iii) for all $\lambda_n = \rho_n^2 \in \Lambda$

$$g_{kk}(x, \rho_n) = d_{kn} e_k(x, \rho_n), \quad k = \overline{1, m}. \quad (41)$$

- (iv) $g_k(\rho_n)$ is an eigenfunction related to λ_n if and only if $d_{kn} \neq 0$.

Proof. Since \mathcal{L} is a selfadjoint operator in $L_2(\Gamma)$, it is known [42] that its eigenvalues are real and eigenfunctions related to different eigenvalues are orthogonal in $L_2(\Gamma)$.

The components $y_j(x)$ of each eigenfunction $y = [y_j(x)]_{j=\overline{1, m}}$ corresponding to an eigenvalue $\lambda = \rho^2$, $\rho \in \Omega_+$, have the form $y_j(x) = A_j e_j(x, \rho)$, where A_j are constants, which do not

vanish simultaneously. Substituting this into the matching conditions (2) we arrive at a homogeneous system of linear algebraic equations with respect to A_j , whose determinant equals to $\Delta(\rho^2)$. Thus, the set of eigenvalues $\lambda = \rho^2, \rho \in \Omega_+$, coincides with Λ_1 . Moreover, since the rank of the determinant $\Delta(\lambda)$ exceeds 0, the number of linearly independent eigenfunctions related to an eigenvalue from Λ_1 is not greater than $m - 1$.

Further, let $\lambda \geq 0$ be an eigenvalue and let $y = [y_j(x)]_{j=\overline{1,m}}$ be a corresponding eigenfunction. Using the same arguments as in [24] one can show that $y_j(x) \equiv 0, j = \overline{p+1,m}$. Thus, $y^0 := [y_j(x)]_{j=\overline{1,p}}$ is an eigenfunction of the boundary value problem L_0 , and hence $\lambda \in \Lambda_3$. Conversely, each $\lambda \in \Lambda_3$ is an eigenvalue of L_0 and the corresponding eigenfunctions vanish in v_0 , hence it is an eigenvalue also of L . Thus, the set of real nonnegative eigenvalues coincides with Λ_3 and (i), (ii) are proved. Moreover, it is obvious that the number of linearly independent eigenfunctions related to each $\lambda \in \Lambda_3$ does not exceed $p - 1$.

For $\lambda_n \in \Lambda$ according to (23), (38), (39) and the first assertion (i) of the theorem we have $\Delta(\lambda_n) = 0$. Using the second equality in (23) for all $k = \overline{1,m}$ we obtain (41). Further, let $d_{kn} \neq 0$. Then for $\lambda_n \in \Lambda_1$ it is obvious that $g_k(\rho_n)$ is an eigenfunction. If $\lambda_n = \rho_n^2 \in \Lambda_3$ we have $e_j(x, \rho_n) \notin L_2(0, \infty), j = \overline{p+1,m}$. However, for $k = \overline{1,p}$ the function $g_k(\rho_n)$ is an eigenfunction anyway because in this case according to (15), (37) we have $g_{kj}(x, \rho) \equiv 0$ for $j = \overline{p+1,m}$. Note that according to (16) for $k = \overline{p+1,m}$ and $\lambda_n \in \Lambda_3$ we have $d_{kn} = 0$. Conversely, let $d_{kn} = 0$. Then $g_{kk}(x, \rho_n) \equiv 0$ and according to (16) there are two distinct indices $j_1, j_2 \neq k$ such that $e_{j_\nu}(0, \rho_n) = 0, \nu = 1, 2$. By virtue of (15) we get $g_k(\rho_n) = 0$, and hence $g_k(\rho_n)$ is not an eigenfunction. □

Theorem 3. *The set of negative eigenvalues Λ_1 is finite.*

Proof. According to (25) $\Lambda_1 \subset (-M, 0), M < \infty$. Hence, the set of negative eigenvalues $\lambda_n = \rho_n^2, \text{Im} \rho_n > 0$, has a uniquely possible accumulation point in the origin. Let us assume to the contrary that Λ_1 is infinite, then $\tau_n := -i\rho_n \rightarrow 0, n \rightarrow \infty$. Since $e_j(0, \rho_n) \neq 0, j = \overline{1,m}$, for sufficiently large n (for $j = \overline{p+1,m}$ see, e.g., [27]), we have $g_k(\rho_n) \neq 0, k = \overline{1,m}$, for large n , and hence $g_k(\rho_n)$ is an eigenfunction. According to (7), (16), (41) we get $d_{kn} \in \mathbb{R}$. One can choose such increasing sequences of natural numbers $\{n_\nu\}, \{n'_\nu\}$ that for all $\nu \in \mathbb{N}$

$$n_\nu \neq n'_\nu, \quad d_{1n_\nu} d_{1n'_\nu} > 0, \quad e_j(0, \rho_{n_\nu}) e_j(0, \rho_{n'_\nu}) > 0, \quad j = \overline{1,m}. \tag{42}$$

By virtue of (9), there exists $A > 0$ such that

$$\int_A^\infty e_j(x, i\tau') e_j(x, i\tau'') dx \geq \frac{\exp(-2AM)}{8M}, \quad \tau', \tau'' \in (0, M], \quad j = \overline{p+1,m}.$$

Hence for sufficiently large ν we obtain

$$\int_0^\infty e_j(x, \rho_{n_\nu}) e_j(x, \rho_{n'_\nu}) dx = \int_A^\infty e_j(x, \rho_{n_\nu}) e_j(x, \rho_{n'_\nu}) dx$$

$$+ \int_0^A e_j(x, \rho_{n_v})(e_j(x, \rho_{n'_v}) - e_j(x, \rho_{n_v})) dx + \int_0^A e_j^2(x, \rho_{n_v}) dx > 0, \quad j = \overline{p+1, m}, \quad (43)$$

$$\begin{aligned} \int_0^1 e_j(x, \rho_{n_v}) e_j(x, \rho_{n'_v}) dx &= \int_0^1 \psi_j(x, \lambda_{n_v}) \psi_j(x, \lambda_{n'_v}) dx \\ &= \int_0^1 \psi_j(x, \lambda_{n_v})(\psi_j(x, \lambda_{n'_v}) - \psi_j(x, \lambda_{n_v})) dx + \int_0^1 \psi_j^2(x, \lambda_{n_v}) dx > 0, \quad j = \overline{1, p}. \end{aligned} \quad (44)$$

Take, for example, $g_{p+1}(\rho)$. According to (15), (41)–(44) we have

$$\begin{aligned} &\sum_{j=1}^m \int_0^\infty g_{p+1,j}(x, \rho_{n_v}) g_{p+1,j}(x, \rho_{n'_v}) dx \\ &= d_{p+1, n_v} d_{p+1, n'_v} \int_0^\infty e_{p+1}(x, \rho_{n_v}) e_{p+1}(x, \rho_{n'_v}) dx \\ &\quad + \sum_{\substack{j \neq p+1 \\ j=1}}^m \prod_{l \neq p+1, j} e_l(0, \rho_{n_v}) e_l(0, \rho_{n'_v}) \int_0^\infty e_j(x, \rho_{n_v}) e_j(x, \rho_{n'_v}) dx > 0, \end{aligned} \quad (45)$$

where $g_{p+1,j}(x, \rho) = 0$ and $e_j(x, \rho) = 0$ for $x > 1, j = \overline{1, p}$. Inequality (45) contradicts the orthogonality of the eigenfunctions $g_{p+1}(\rho_{n_v}), g_{p+1}(\rho_{n'_v})$ related to the different eigenvalues $\lambda_{n_v}, \lambda_{n'_v}$. □

Thus, the set of negative eigenvalues has the form

$$\Lambda_1 = \{\lambda_n\}_{n=\overline{1, N}}, \quad \lambda_n = \rho_n^2, \quad \rho_n = i\tau_n, \quad 0 < \tau_N < \dots < \tau_1.$$

For briefness denote

$$f_k(\rho) := \Psi_k(\rho^2), \quad f_{kj}(x, \rho) := \psi_{kj}(x, \rho^2), \quad j = \overline{1, m}, \quad k = \overline{1, p}. \quad (46)$$

Theorem 4. *The functions $f_k(\rho), k = \overline{1, m}$, have at most simple poles in $\rho_n, n = \overline{1, N}$. The corresponding nonzero residues are eigenfunctions related to the eigenvalues $\lambda_n = \rho_n^2$.*

Proof. Let ρ_n be a pole of order $s \geq 1$ of the function $f_k(\rho)$. According to (3), (5), (8), (12), (15), (17), (19), (30), (31), (46) we have

$$\langle f_{kj}(x, \rho), e_j(x, \rho) \rangle = \delta_{kj} \begin{cases} 1, & k = \overline{1, p}, \\ 2i\rho, & k = \overline{p+1, m}, \end{cases} \quad j = \overline{1, m}, \quad (47)$$

where δ_{kj} is the Kronecker delta. Consider the function $u = [u_j(x)]_{j=\overline{1, m}}$, determined by the formula

$$u = \lim_{\rho \rightarrow \rho_n} (\rho - \rho_n)^s f_k(\rho). \quad (48)$$

Formulae (47), (48) give

$$\langle u_j(x), e_j(x, \rho_n) \rangle = 0, \quad j = \overline{1, m}, \quad (49)$$

and consequently

$$-u_j'' + q_j(x)u_j = \lambda_n u_j, \quad x \in \varepsilon_j, \quad j = \overline{1, m}; \quad U_j(u_j) = 0, \quad j = \overline{1, p}. \tag{50}$$

Since the function $f_k(\rho)$ satisfies the matching conditions (2), the function u satisfies them too. Thus, u is an eigenfunction corresponding to the eigenvalue λ_n .

Let us assume to the contrary that $s > 1$. Determine the function $v = [v_j(x)]_{j=\overline{1, m}}$ by the formula

$$v = \lim_{\rho \rightarrow \rho_n} \frac{d}{d\rho} ((\rho - \rho_n)^s f_k(\rho)).$$

By virtue of (47), (48) we have

$$\langle v_j(x), e_j(x, \rho_n) \rangle + \langle u_j(x), \dot{e}_j(x, \rho_n) \rangle = 0, \quad j = \overline{1, m}, \tag{51}$$

where $\dot{e}_j(x, \rho) = \frac{d}{d\rho} e_j(x, \rho)$. Differentiating (51) with respect to x and then substituting

$$e_j''(x, \rho_n) = (q_j(x) - \rho_n^2)e_j(x, \rho_n), \quad \dot{e}_j''(x, \rho_n) = (q_j(x) - \rho_n^2)\dot{e}_j(x, \rho_n) - 2\rho_n \dot{e}_j(x, \rho_n)$$

and (50) into the relation obtained, we arrive at

$$-v_j''(x) + q_j(x)v_j(x) = \rho_n^2 v_j(x) + 2\rho_n u_j(x), \quad x \in \varepsilon_j, \quad j = \overline{1, m}. \tag{52}$$

Using (50), (52) we obtain

$$\sum_{j=1}^p \langle v_j(x), u_j(x) \rangle \Big|_{x=0}^1 + \sum_{j=p+1}^m \langle v_j(x), u_j(x) \rangle \Big|_{x=0}^\infty = 2\rho_n \left(\sum_{j=1}^p \int_0^1 u_j^2(x) dx + \sum_{j=p+1}^m \int_0^\infty u_j^2(x) dx \right). \tag{53}$$

Since the functions u, v satisfy the matching conditions (2), we get

$$\sum_{j=1}^m \langle v_j(x), u_j(x) \rangle \Big|_{x=0} = 0. \tag{54}$$

From (49) we get $u_j(x) = C_j e_j(x, \rho_n)$, $j = \overline{1, m}$, which together with (3), (5), (17) gives

$$\langle v_j(x), u_j(x) \rangle \Big|_{x=1} = -C_j U_j(v_j(x)) = 0, \quad j = \overline{1, p}. \tag{55}$$

Moreover, for $j = \overline{p+1, m}$ we have $\dot{e}_j^{(v)}(x, \rho_n) = O(1)$, $x \rightarrow \infty$, $v = 0, 1$, (see [31]). By virtue of (51) we have

$$\begin{aligned} \lim_{x \rightarrow \infty} \langle v_j(x), u_j(x) \rangle &= C_j \lim_{x \rightarrow \infty} \langle v_j(x), e_j(x, \rho_n) \rangle = \\ &= C_j \lim_{x \rightarrow \infty} \langle \dot{e}_j(x, \rho_n), u_j(x) \rangle = 0, \quad j = \overline{p+1, m}, \end{aligned} \tag{56}$$

because $u_j^{(v)}(x) = o(1)$, $x \rightarrow \infty$, $v = 0, 1$. According to (7), (15), (16), (23), (30) the functions $f_k(i\tau)$, $\tau \geq 0$, are real-valued. Thus, by virtue of (48) the function $u(x)$ is real or imaginary depending on the fact if s is even or odd, respectively. Taking into account (53)–(56) we get $u(x) \equiv 0$, which is impossible. □

Corollary 2. For the eigenvalue $\lambda_{n_0} = \rho_{n_0}^2 \in \Lambda_1$ to be multiple, i.e. $a(\rho_{n_0}) = \dot{a}(\rho_{n_0}) = 0$, it is necessary and sufficient that there are at least three distinct numbers $j_1, j_2, j_3 \in \{1, \dots, m\}$, such that $e_{j_\nu}^{(\nu)}(0, \rho_{n_0}) = 0, \nu = \overline{1, 3}$.

Proof. The sufficiency follows from (20), (23). Let $a(\rho_{n_0}) = \dot{a}(\rho_{n_0}) = 0$, then according to Theorem 4 and formula (30) we have $g_{kj}(x, \rho_{n_0}) \equiv 0$ for all $k, j \in \{1, \dots, m\}$. Hence, for all k, j in the right-hand part of (15) for $\rho = \rho_{n_0}$ there should be at least one zero-multiplier $e_l(0, \rho_{n_0}) = 0, l \neq k, j$, which proves the necessity. \square

Lemma 6. For all $k, j \in \{p+1, \dots, m\}$ and μ such that $\mu^2 \in \Lambda_3 \cup \{0\}$ the following estimate holds:

$$a_{kj}(\rho) := \frac{1}{a(\rho)} \prod_{l \neq k, j} e_l(0, \rho) = O(1), \quad \rho \rightarrow \mu, \quad \rho \in \overline{\Omega_+}. \tag{57}$$

Proof. Fix $k, j \in \{p+1, \dots, m\}$ and μ such that $\mu^2 \in \Lambda_3 \cup \{0\}$. Suppose that the functions $e_l^{(\nu)}(0, \rho), l = \overline{p+1, m}, \nu = 0, 1$, are analytic in a vicinity of μ . Then, using (20), (23), (29) we deduce that the function $a_{kj}(\rho)$ has a removable singularity in μ , and hence (57) holds.

In the general case we cannot use these arguments. Therefore, we introduce the potential $q_{(r)} = [q_{j,r}(x)]_{j=\overline{1, m}}$, where

$$q_{j,r}(x) = q_j(x), \quad j = \overline{1, p}, \quad q_{j,r}(x) = \begin{cases} q_j(x), & 0 < x < r, \\ 0, & x > r, \end{cases} \quad j = \overline{p+1, m},$$

and consider the corresponding Jost solutions $e_{j,r}(x, \rho), j = \overline{p+1, m}$. Clearly, for each fixed x the functions $e_{j,r}^{(\nu)}(x, \rho), \nu = 0, 1$, are entire in ρ . Put

$$a_r(\rho) = \frac{1}{2i\rho} \sum_{j=1}^m e'_{j,r}(0, \rho) \prod_{l \neq j} e_{l,r}(0, \rho),$$

where $e_{j,r}(x, \rho) = \psi_j(x, \rho^2), j = \overline{1, p}$. By virtue of Lemma 1 and (6), (10) for each fixed $\varepsilon > 0$ we have

$$\lim_{r \rightarrow \infty} a_r(\rho) \exp(pi\rho) = a(\rho) \exp(pi\rho) \text{ uniformly in } \overline{\Omega_+} \setminus \{\rho : |\rho| < \varepsilon\}. \tag{58}$$

Choose $\delta > 0$ such that $a(\rho) \neq 0$ for $\rho \in [\mu - \delta, \mu) \cup (\mu, \mu + \delta]$ and consider the set $O_\delta(\mu) := \{\rho : 0 < |\rho - \mu| \leq \delta\} \cap \overline{\Omega_+}$. Denote

$$a_{kj,r}(\rho) := \frac{1}{a_r(\rho)} \prod_{l \neq k, j} e_{l,r}(0, \rho).$$

(i) Let $\mu \neq 0$, then according to (58)

$$\lim_{r \rightarrow \infty} a_{kj,r}(\rho) = a_{kj}(\rho)$$

uniformly with respect to $\rho \in O_\delta(\mu) \setminus O_\varepsilon(\mu)$ for each fixed $\varepsilon > 0$. This means, in particular, that $|a_{kj,r}(\rho)| \leq C$, $|\rho| = \delta$, where C does not depend on r . Moreover, according to Corollary 1 we have $|a_{kj,r}(\rho)| \leq 1$ for $\rho \in \mathbb{R}$. Using the maximum modulus principle we get

$$|a_{kj,r}(\rho)| \leq C, \quad \rho \in O_\delta(\mu). \tag{59}$$

Taking in (59) the limit as $r \rightarrow \infty$ we arrive at $|a_{kj}(\rho)| \leq C$, $\rho \in O_\delta(\mu)$. Thus, (57) is proved for $\mu \neq 0$.

(ii) Let $\mu = 0$. According to Lemma 5 and Theorem 3 $\rho = 0$ is not a concentration point for the set $\{\rho : \rho \in \overline{\Omega_+} \setminus \{0\}, a(\rho) = 0\}$. Hence, there exists $\delta > 0$ such that $a(\rho) \neq 0$ for all $\rho \in O_\delta(0)$. By virtue of (58) for each $\varepsilon > 0$ there exists R such that $a_r(\rho) \neq 0$ for $\rho \in O_\delta(0) \setminus O_\varepsilon(0)$, $r \geq R$. Let $\rho_{n,r}$, $n = \overline{1, N_r}$, be all zeros of the function $a_r(\rho)$ in O_ε counted with multiplicity.

Let us show that $N_r \leq K$ for all r . Since the function $e_{j,r}(0, \rho)$ has a uniformly bounded number of zeros (see [27]), it is sufficient to prove that the number of the zeros $\rho_{n,r}$ of $a_r(\rho)$, such that $e_{j,r}(0, \rho_{n,r}) \neq 0$, $j = \overline{1, m}$, is uniformly bounded too. Construct the solution $g_{k,r}(\rho) = [g_{kj,r}(x, \rho)]_{j=\overline{1, m}}$ for the potential $q(r)$ analogously to $g_k(\rho)$ for q . According to Theorem 2 and (16) $g_{k,r}(\rho_{n,r})$ is an eigenfunction of $L(q(r), H)$ related to the eigenvalue $\rho_{n,r}^2$ and

$$g_{kk,r}(x, \rho_{n,r}) = d_{kn,r} e_{k,r}(x, \rho_{n,r}), \quad d_{kn,r} \neq 0, \quad k = \overline{1, m}. \tag{60}$$

Assume to the contrary that the number of zeros N_r is not bounded. Then there exist such increasing sequences $\{r_\nu\}$, $\{n_\nu\}$, $\{n'_\nu\}$ that

$$n_\nu \neq n'_\nu, \quad d_{1n_\nu, r_\nu} d_{1n'_\nu, r_\nu} > 0, \quad e_{j, r_\nu}(0, \rho_{n_\nu, r_\nu}) e_{j, r_\nu}(0, \rho_{n'_\nu, r_\nu}) > 0, \quad j = \overline{1, m}. \tag{61}$$

Analogously to the estimates (43), (44) we obtain

$$\int_0^\infty e_{j, r_\nu}(x, \rho_{n_\nu, r_\nu}) e_{j, r_\nu}(x, \rho_{n'_\nu, r_\nu}) dx > 0, \quad j = \overline{1, m}, \tag{62}$$

for sufficiently large ν . According to (15), (60) and (61), (62) we have

$$\begin{aligned} & \sum_{j=1}^m \int_0^\infty g_{1j, r_\nu}(x, \rho_{n_\nu, r_\nu}) g_{1j, r_\nu}(x, \rho_{n'_\nu, r_\nu}) dx \\ &= d_{1n_\nu, r_\nu} d_{1n'_\nu, r_\nu} \int_0^\infty e_{1, r_\nu}(x, \rho_{n_\nu, r_\nu}) e_{1, r_\nu}(x, \rho_{n'_\nu, r_\nu}) dx \\ &+ \sum_{j=2}^m \prod_{l \neq j} e_{l, r_\nu}(0, \rho_{n_\nu, r_\nu}) e_{l, r_\nu}(0, \rho_{n'_\nu, r_\nu}) \int_0^\infty e_{j, r_\nu}(x, \rho_{n_\nu, r_\nu}) e_{j, r_\nu}(x, \rho_{n'_\nu, r_\nu}) dx > 0, \end{aligned}$$

which contradicts the orthogonality in $L_2(\Gamma)$ of the eigenfunctions $g_{1, r_\nu}(\rho_{n_\nu, r_\nu})$, $g_{1, r_\nu}(\rho_{n'_\nu, r_\nu})$. Thus, we have proved that $N_r \leq K$.

Further, put

$$A_{kj,r}(\rho) := a_{kj,r}(\rho)\varphi_r(\rho), \quad \varphi_r(\rho) = \prod_{n=1}^{N_r} \frac{\rho - \rho_{n,r}}{\rho + \rho_{n,r}}. \tag{63}$$

After removing its singularities the function $A_{kj,r}(\rho)$ is analytic in $O_\delta(0)$. It is obvious that $|\varphi_r(\rho)| \leq 1$, $\rho \in \overline{\Omega}_+$, and

$$\lim_{r \rightarrow \infty} \varphi_r(\rho) = 1, \quad \rho \neq 0. \tag{64}$$

By virtue of (58), (63), (64) and Lemma 1 we obtain

$$\lim_{r \rightarrow \infty} A_{kj,r}(\rho) = a_{kj}(\rho), \quad \rho \in O_\delta(0).$$

Using (59), (63) we get $|a_{kj}(\rho)| \leq C$, $\rho \in O_\delta(0)$. Thus, (57) is proved also for $\mu = 0$. □

Corollary 3. *For all fixed $x \geq 0$, $k = \overline{p+1, m}$, $\nu = 0, 1$ and μ such that $\mu^2 \in \Lambda_2$ the following estimates hold:*

$$f_{kk}^{(\nu)}(x, \rho) = O\left(\frac{1}{\rho}\right), \quad \rho \rightarrow 0, \quad f_{kk}^{(\nu)}(x, \rho) = O(1), \quad \rho \rightarrow \mu, \quad \rho \in \overline{\Omega}_+.$$

Proof. According to (16), (30) and Lemma 6 it is sufficient to prove that

$$\frac{1}{a(\rho)} \sum_{j \neq k} e'_j(0, \rho) \prod_{l \neq k, j} e_l(0, \rho) = O\left(\frac{1}{\rho}\right), \quad \rho \rightarrow 0, \quad \rho \in \overline{\Omega}_+, \tag{65}$$

$$\frac{1}{a(\rho)} \sum_{j \neq k} e'_j(0, \rho) \prod_{l \neq k, j} e_l(0, \rho) = O(1), \quad \rho \rightarrow \mu, \quad \rho \in \overline{\Omega}_+. \tag{66}$$

We note that one cannot apply Lemma 6 directly, because in (57) j can be chosen only from $\{p+1, \dots, m\}$, while in (65), (66) we have $j = \overline{1, m} \setminus k$. Using (20), (23) we get

$$\frac{1}{a(\rho)} \sum_{j \neq k} e'_j(0, \rho) \prod_{l \neq k, j} e_l(0, \rho) = \frac{2i\rho}{e_k(0, \rho)} - \frac{e'_k(0, \rho)}{e_k(0, \rho)a(\rho)} \prod_{l \neq k} e_l(0, \rho).$$

Since $e_k(x, \rho) \neq 0$ for $\rho \in \mathbb{R}^*$, (66) follows solely from Lemma 6. As for (65) one should also recall that (see [27])

$$\frac{\rho}{e_k(0, \rho)} = O(1), \quad \rho \rightarrow 0, \quad \rho \in \overline{\Omega}_+. \tag{67}$$

Denote

$$\varphi_k(\rho_n) := \operatorname{Res}_{\rho=\rho_n} f_k(\rho), \quad k = \overline{1, m}, \quad n = \overline{1, N}. \tag{67}$$

Theorem 4 yields that nonzero functions $\varphi_k(\rho_n) = [\varphi_{kj}(x, \rho_n)]_{j=\overline{1, m}}$ are eigenfunctions related to the eigenvalue $\lambda_n = \rho_n^2 \in \Lambda_1$ and hence we have

$$\varphi_{kk}(x, \rho_n) = \alpha_{kn} e_k(x, \rho_n), \quad k = \overline{1, m}, \quad n = \overline{1, N}. \tag{68}$$

The numbers α_{kn} are called *norming constants* associated with the edge ε_k . We note that if $\dot{\alpha}(\rho_n) \neq 0$, then

$$\alpha_{kn} = \frac{d_{kn}}{\dot{\alpha}(\rho_n)}.$$

Definition 2. The eigenvalue λ_n is called *invisible from the edge ε_j* , $j \in \{1, \dots, m\}$, if $y_j(x) \equiv 0$, for each eigenfunction $y = [y_i(x)]_{i=\overline{1,m}}$ related to λ_n ; otherwise it is called *visible from ε_j* .

Denote by Λ'_1 the set of all eigenvalues that are visible from at least one infinite edge ε_j , $j \in \{p+1, \dots, m-1\}$. By virtue of Theorem 2 $\Lambda'_1 \subset \Lambda_1$, and hence Λ'_1 has the form

$$\Lambda'_1 = \{\lambda_{n_s}\}_{s=\overline{1,N'}}.$$

For $\lambda_n \notin \Lambda'_1$ we obviously have $\alpha_{kn} = 0$, $k = \overline{p+1, m-1}$.

Definition 3. The set

$$J := \{M_v(\lambda), s_k(\rho), \lambda_{n_s}, \alpha_{kn_s}\}_{v=\overline{1,p}, k=\overline{p+1, m-1}, s=\overline{1,N'}}$$

is called *spectral-scattering data* of L .

Remark 1. Denote by Λ''_1 the subset of Λ'_1 consisting of all eigenvalues that are visible from at least one compact edge. Using (16), (19), (20) and Theorem 4 one can show that if λ_n is visible from some compact edge ε_j , $j \in \{1, \dots, p\}$, then λ_n is a pole of the Weyl function $M_j(\lambda)$. Thus, the set Λ''_1 is completely determined by specification of $\{M_v(\lambda)\}_{v=\overline{1,p}}$. Therefore in J it is sufficient to specify among λ_{n_s} , $s = \overline{1, N'}$, only those eigenvalues that belong to $\Lambda'_1 \setminus \Lambda''_1$, i.e. invisible from compact edges.

4. Inverse problem. Uniqueness theorem

The inverse problem is formulated as follows.

Inverse Problem 1. Given J , find q and H .

We agree that together with L we consider here and in the sequel a boundary value problem $\tilde{L} = L(\tilde{q}, \tilde{H})$ of the same form but with other coefficients. If a certain symbol γ denotes an object corresponding to L , then this symbol with tilde $\tilde{\gamma}$ denotes the analogous object related to \tilde{L} .

Theorem 5. *If $J = \tilde{J}$, then $L = \tilde{L}$, i.e. $q_j(x) \stackrel{\text{a.e.}}{=} \tilde{q}_j(x)$, $j = \overline{1, m}$, $H_j = \tilde{H}_j$, $j = \overline{1, p}$. Thus, the specification of the spectral-scattering data J uniquely determines the coefficients of L .*

Proof. (i) Fix $k \in \{1, \dots, p\}$. Let us show that $M_k(\lambda) \equiv \tilde{M}_k(\lambda)$ implies $H_k = \tilde{H}_k$ and $q_k(x) = \tilde{q}_k(x)$ a.e. on $(0, 1)$. Define the matrix $P(x, \lambda) = [P_{\nu j}(x, \lambda)]_{\nu, j=1,2}$ by the formula

$$P(x, \lambda) \begin{bmatrix} \tilde{\psi}_k(x, \lambda) & \tilde{\Psi}_{kk}(x, \lambda) \\ \tilde{\psi}'_k(x, \lambda) & \tilde{\Psi}'_{kk}(x, \lambda) \end{bmatrix} = \begin{bmatrix} \psi_k(x, \lambda) & \Psi_{kk}(x, \lambda) \\ \psi'_k(x, \lambda) & \Psi'_{kk}(x, \lambda) \end{bmatrix}. \tag{69}$$

Using $\langle \psi_k(x, \lambda), \Psi_{kk}(x, \lambda) \rangle \equiv -1$, we calculate

$$\left. \begin{aligned} P_{\nu 1}(x, \lambda) &= \Psi_{kk}^{(\nu-1)}(x, \lambda) \tilde{\psi}'_k(x, \lambda) - \psi_k^{(\nu-1)}(x, \lambda) \tilde{\Psi}'_{kk}(x, \lambda), \\ P_{\nu 2}(x, \lambda) &= \psi_k^{(\nu-1)}(x, \lambda) \tilde{\Psi}_{kk}(x, \lambda) - \Psi_{kk}^{(\nu-1)}(x, \lambda) \tilde{\psi}_k(x, \lambda). \end{aligned} \right\} \tag{70}$$

Moreover, formula (69) in particular gives

$$\psi_k(x, \lambda) = P_{11}(x, \lambda) \tilde{\psi}_k(x, \lambda) + P_{12}(x, \lambda) \tilde{\psi}'_k(x, \lambda). \tag{71}$$

By virtue of (6), (10), (16)–(19), (23), (25), (70) and the asymptotics of the functions $C_k^{(\nu)}(x, \lambda)$, $S_k^{(\nu)}(x, \lambda)$, $\nu = 0, 1$, (see [31]) we have

$$P_{11}(x, \lambda) = 1 + O\left(\frac{1}{\rho}\right), \quad P_{12}(x, \lambda) = O\left(\frac{1}{\rho^2}\right), \quad \rho \in \Omega_\varepsilon, \quad |\rho| \rightarrow \infty, \tag{72}$$

for each fixed $\varepsilon > 0$. On the other hand, according to (21) and (70),

$$\begin{aligned} P_{11}(x, \lambda) &= \theta_k(x, \lambda) \tilde{\psi}'_k(x, \lambda) - \tilde{\theta}'_k(x, \lambda) \psi_k(x, \lambda) + (M_k(\lambda) - \tilde{M}_k(\lambda)) \psi_k(x, \lambda) \tilde{\psi}'_k(x, \lambda), \\ P_{12}(x, \lambda) &= \psi_k(x, \lambda) \tilde{\theta}_k(x, \lambda) - \tilde{\psi}_k(x, \lambda) \theta_k(x, \lambda) + (\tilde{M}_k(\lambda) - M_k(\lambda)) \psi_k(x, \lambda) \tilde{\psi}_k(x, \lambda). \end{aligned}$$

Thus, if $M_k(\lambda) \equiv \tilde{M}_k(\lambda)$, then for each fixed x the functions $P_{11}(x, \lambda)$, $P_{12}(x, \lambda)$ are entire in λ of order $r \leq 1/2$. By virtue of the Phragmen–Lindelöf and Liouville theorems formulae (72) yield $P_{11}(x, \lambda) \equiv 1$, $P_{12}(x, \lambda) \equiv 0$. Substituting this into (71) we get $\psi_k(x, \lambda) \equiv \tilde{\psi}_k(x, \lambda)$ for all $x \in [0, 1]$ and consequently $H_k = \tilde{H}_k$ and $q_k(x) = \tilde{q}_k(x)$ a.e. on $(0, 1)$.

(ii) Fix $k \in \{p + 1, \dots, m - 1\}$. Denote

$$\Phi^+(x, \rho) = \begin{bmatrix} f_{kk}(x, \rho) & e_k(x, \rho) \\ f'_{kk}(x, \rho) & e'_k(x, \rho) \end{bmatrix}, \quad \Phi^-(x, \rho) = \begin{bmatrix} e_k(x, -\rho) & f_{kk}(x, -\rho) \\ e'_k(x, -\rho) & f'_{kk}(x, -\rho) \end{bmatrix}. \tag{73}$$

Lemma 7. For $\rho \in \mathbb{R}_1^*$ the following relation holds:

$$\Phi^-(x, \rho) = \Phi^+(x, \rho) V(\rho), \quad V(\rho) = \begin{bmatrix} 1 & s_k(-\rho) \\ -s_k(\rho) & 1 - s_k(-\rho)s_k(\rho) \end{bmatrix}. \tag{74}$$

Proof. According to (31) the functions $f_{kk}(x, \rho)$, $e_k(x, \rho)$ form a fundamental system of solutions on the edge ε_k if $f_k(x, \rho)$ exists – in particular, for $\rho \in \mathbb{R}_1^*$. Thus, we have

$$\left. \begin{aligned} e_k(x, -\rho) &= v_{11}(\rho) f_{kk}(x, \rho) + v_{21}(\rho) e_k(x, \rho), \\ f_{kk}(x, -\rho) &= v_{12}(\rho) f_{kk}(x, \rho) + v_{22}(\rho) e_k(x, \rho). \end{aligned} \right\} \tag{75}$$

Let us calculate the coefficients $v_{vj}(\rho)$. According to (11), (35), (75) we arrive at

$$\begin{aligned} 2i\rho v_{11}(\rho) &= \langle e_k(x, -\rho), e_k(x, \rho) \rangle = 2i\rho, \\ 2i\rho v_{21}(\rho) &= \langle f_{kk}(x, \rho), e_k(x, -\rho) \rangle = -2i\rho s_k(\rho), \\ 2i\rho v_{12}(\rho) &= \langle f_{kk}(x, -\rho), e_k(x, \rho) \rangle = 2i\rho s_k(-\rho), \\ 2i\rho v_{22}(\rho) &= \langle f_{kk}(x, \rho), f_{kk}(x, -\rho) \rangle = 2i\rho(1 - s_k(\rho)s_k(-\rho)), \end{aligned}$$

whence we get

$$v_{11}(\rho) = 1, \quad v_{12}(\rho) = s_k(-\rho), \quad v_{21}(\rho) = -s_k(\rho), \quad v_{22}(\rho) = 1 - s_k(-\rho)s_k(\rho),$$

which together with (75) give (74). □

Let us define the matrix $P^+(x, \rho) = [P_{vj}^+(x, \rho)]_{v,j=1,2}$, $\rho \in \overline{\Omega_+}$, by the formula

$$P^+(x, \rho)\tilde{\Phi}^+(x, \rho) = \Phi^+(x, \rho). \tag{76}$$

By virtue of (31), (73), this yields

$$\left. \begin{aligned} P_{v1}^+(x, \rho) &= \frac{1}{2i\rho} \left(f_{kk}^{(v-1)}(x, \rho)\tilde{e}'_k(x, \rho) - e_k^{(v-1)}(x, \rho)\tilde{f}'_{kk}(x, \rho) \right), \\ P_{v2}^+(x, \rho) &= \frac{1}{2i\rho} \left(e_k^{(v-1)}(x, \rho)\tilde{f}_{kk}(x, \rho) - f_{kk}^{(v-1)}(x, \rho)\tilde{e}_k(x, \rho) \right), \end{aligned} \right\} \tag{77}$$

$$e_k(x, \rho) = P_{11}^+(x, \rho)\tilde{e}_k(x, \rho) + P_{12}^+(x, \rho)\tilde{e}'_k(x, \rho). \tag{78}$$

Using (6), (10), (16), (17), (25), (30), (77) and the asymptotics of $C_k^{(v)}(x, \lambda)$, $S_k^{(v)}(x, \lambda)$, $v = 0, 1$, (see [31]) we get

$$\left. \begin{aligned} P_{11}^+(x, \rho) &= 1 + o\left(\frac{1}{\rho}\right), \\ P_{12}^+(x, \rho) &= O\left(\frac{1}{\rho^2}\right), \end{aligned} \right\} |\rho| \rightarrow \infty, \rho \in \Omega_\varepsilon. \tag{79}$$

Analogously we define the matrix $P^-(x, \rho) = [P_{vj}^-(x, \rho)]_{v,j=1,2}$, $\rho \in \overline{\Omega_-}$, by the formula

$$P^-(x, \rho)\tilde{\Phi}^-(x, \rho) = \Phi^-(x, \rho). \tag{80}$$

Obviously, we have

$$P^-(x, \rho) = P^+(x, -\rho). \tag{81}$$

Since $\tilde{s}_k(\rho) = s_k(\rho)$, formulae (74), (76), (80) give for $\rho \in \mathbb{R}_1^*$,

$$\begin{aligned} P^-(x, \rho) &= \Phi^-(x, \rho)(\tilde{\Phi}^-(x, \rho))^{-1} = \Phi^+(x, \rho)V(\rho)(\tilde{\Phi}^+(x, \rho)\tilde{V}(\rho))^{-1} = \\ &= \Phi^+(x, \rho)V(\rho)(\tilde{V}(\rho))^{-1}(\tilde{\Phi}^+(x, \rho))^{-1} = \Phi^+(x, \rho)(\tilde{\Phi}^+(x, \rho))^{-1} = P^+(x, \rho), \end{aligned}$$

and consequently for each fixed $x \geq 0$ the function $P^-(x, \rho)$ is an analytic continuation in Ω_- of $P^+(x, \rho)$. Thus, using (77), Corollary 3 and Definition 2 we conclude that the function

$$P(x, \rho) := \begin{cases} P^+(x, \rho), & \rho \in \overline{\Omega_+}, \\ P^-(x, \rho), & \rho \in \Omega_-, \end{cases}$$

is meromorphic with at most simple poles in the points $\pm \rho_{n_s}$, $s = \overline{1, N'}$, $\pm \tilde{\rho}_{n_s}$, $s = \overline{1, \tilde{N}'}$, and with an at most second-order pole in the origin. Since $\tilde{N}' = N'$, $\tilde{\rho}_{n_s} = \rho_{n_s}$, $\tilde{\alpha}_{kn_s} = \alpha_{kn_s}$, $s = \overline{1, N'}$, according to (67), (68) we have

$$\begin{aligned} \operatorname{Res}_{\rho=\rho_{n_s}} P_{v1}^+(x, \rho) &= \frac{1}{2i\rho_{n_s}} \left(\varphi_{kk}^{(v-1)}(x, \rho_{n_s}) \tilde{e}'_k(x, \rho_{n_s}) - e_k^{(v-1)}(x, \rho_{n_s}) \tilde{\varphi}'_{kk}(x, \rho_{n_s}) \right) \\ &= \frac{\alpha_{kn_s} - \tilde{\alpha}_{kn_s}}{2i\rho_{n_s}} e_k^{(v-1)}(x, \rho_{n_s}) \tilde{e}'_k(x, \rho_{n_s}) = 0, \\ \operatorname{Res}_{\rho=\rho_{n_s}} P_{v2}^+(x, \rho) &= \frac{1}{2i\rho_{n_s}} \left(\tilde{\varphi}_{kk}(x, \rho_{n_s}) e_k^{(v-1)}(x, \rho_{n_s}) - \tilde{e}_k(x, \rho_{n_s}) \varphi_{kk}^{(v-1)}(x, \rho_{n_s}) \right) \\ &= \frac{\tilde{\alpha}_{kn_s} - \alpha_{kn_s}}{2i\rho_{n_s}} \tilde{e}_k(x, \rho_{n_s}) e_k^{(v-1)}(x, \rho_{n_s}) = 0. \end{aligned}$$

Thus, according to (81) the function $\rho^2 P(x, \rho)$ is entire in ρ of order $r \leq 1$. By virtue of (79), (81), the Phragmen-Lindelöf and Liouville theorems we get

$$P_{11}^+(x, \rho) \equiv 1 + \frac{C_1}{\rho^2}, \quad P_{12}^+(x, \rho) \equiv \frac{C_2}{\rho^2}, \quad C_1, C_2 - \text{const},$$

which together with (78) give

$$e_k(x, \rho) = \tilde{e}_k(x, \rho) + \frac{1}{\rho^2} (C_1 \tilde{e}_k(x, \rho) + C_2 \tilde{e}'_k(x, \rho)). \tag{82}$$

By virtue of continuity of $e_k(x, \cdot)$ in $\overline{\Omega_+}$ we have

$$C_1 \tilde{e}_k(x, 0) + C_2 \tilde{e}'_k(x, 0) \equiv 0, \tag{83}$$

which gives $C_1 = 0$. Indeed, according to (8) we have $\tilde{e}_k(x, 0) \neq 0$ and hence $C_1 \neq 0$ yields $C_2 \neq 0$. Then (83) gives $\tilde{q}_k(x) = (C_1/C_2)^2 \neq 0$, which contradicts $\tilde{q}_k(x) \in L(0, \infty)$.

Suppose that $C_2 \neq 0$. Then by virtue of (83) we have $\tilde{e}'_k(x, 0) \equiv 0$ and consequently $\tilde{q}_k(x) = 0$. Thus, $\tilde{e}_k(x, \rho) = \exp(i\rho x)$, which together with (82) give

$$e_k(x, \rho) = \tilde{e}_k(x, \rho) + \frac{iC_2}{\rho} \exp(i\rho x).$$

Using again the continuity of $e(x, \cdot)$ we infer that $C_2 = 0$ and arrive at a contradiction. Thus, we get $C_1 = C_2 = 0$ and according to (82) $e_k(x, \rho) \equiv \tilde{e}_k(x, \rho)$, which, in turn, gives $q_k(x) = \tilde{q}_k(x)$ a.e. on $(0, \infty)$, $k = \overline{p+1, m-1}$.

It remains to prove that $q_m(x) = \tilde{q}_m(x)$. The function $M(\lambda) := e_m(0, \rho)(e'_m(0, \rho))^{-1}$ is the Weyl function of the boundary value problem for equation (1) (for $j = m$) with the boundary condition $y'_m(0) = 0$. Thus, it is sufficient to prove that $\hat{M}(\lambda) \equiv 0$ (see [31]). By virtue of (15), (19) we have

$$M(\lambda) = \frac{\Psi_{1m}(0, \lambda)}{\Psi'_{1m}(0, \lambda)}. \tag{84}$$

According to the matching conditions (2) and representation (21) we have

$$\Psi_{1m}(0, \lambda) = \Psi_{11}(0, \lambda) = \theta_1(0, \lambda) + M_1(\lambda)\psi_1(0, \lambda), \tag{85}$$

$$\Psi'_{1m}(0, \lambda) = -\theta'_1(0, \lambda) - M_1(\lambda)\psi'_1(0, \lambda) - \sum_{j=2}^{m-1} \Psi'_{1j}(0, \lambda). \tag{86}$$

By virtue of (15), (19) we get

$$\Psi_{11}(0, \lambda) = \frac{1}{\Delta(\lambda)} \prod_{l=2}^m e_l(0, \rho), \quad \Psi'_{1j}(0, \lambda) = \frac{e'_j(0, \rho)}{\Delta(\lambda)} \prod_{l \neq 1, j} e_l(0, \rho), \quad j \neq 1,$$

whence we find

$$\frac{e_m(0, \rho)}{\Delta(\lambda)} = \Psi_{11}(0, \lambda) \left(\prod_{l=2}^{m-1} e_l(0, \rho) \right)^{-1}, \quad \Psi'_{1j}(0, \lambda) = \Psi_{11}(0, \lambda) \frac{e'_j(0, \rho)}{e_j(0, \rho)},$$

which together with (86) gives

$$\Psi'_{1m}(0, \lambda) = -\theta'_1(0, \lambda) - M_1(\lambda)\psi'_1(0, \lambda) - (\theta_1(0, \lambda) + M_1(\lambda)\psi_1(0, \lambda)) \sum_{j=2}^{m-1} \frac{e'_j(0, \rho)}{e_j(0, \rho)}. \tag{87}$$

Using (84), (85), (87) we arrive at $\hat{M}(\lambda) \equiv 0$. □

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