TAMKANG JOURNAL OF MATHEMATICS
Volume 46, Number 1, 31-49, March 2015
doi:10.5556/j.tkjm.46.2015.1432

Available online at http://journals.math.tku.edu.tw/

(a,d)-CONTINUOUS MONOTONIC SUBGRAPH DECOMPOSITION OF K_{n+1} AND INTEGRAL SUM GRAPHS $G_{0,n}$

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Abstract. For $a,d,n \in \mathbb{N}$, we define $(a,d)-Continuous$ *Monotonic Subgraph Decomposition or* $(a,d)-CMSD$ *of a graph* G *of size* $\frac{(2a+(n-1)d)n}{2}$ *as the decomposition of* G *into* n *subgraphs* $G_1,G_2,\ldots,G_n$ *without isolated vertices such that each* G_i *is connected and isomorphic to a proper subgraph of* G_{i+1} *and* $|E(G_i)| = a+(i-1)d$ *for* $i = 1,2,\ldots,n$. $(1,1)-CMSD$ *of a graph* G *is called a Continuous Monotonic Subgraph Decomposition or CMSD of* G*. Harary introduced the concepts of sum and integral sum graphs and a family of integral sum graphs* $G_{-n,n}$ *over* $[-n,n]$ *and it was generalized to* $G_{-m,n}$ *where* $[r,s] = \{r,r+1,\ldots,s\}$, $r,s \in \mathbb{Z}$ *and* $m,n \in \mathbb{N}_0$*. In this paper, we study* $(a,d)-CMSD$ *of* K_{n+1} *and* $G_{0,n}$ *into families of triangular books, triangular books with book mark and Fans with handle.*

1. Introduction

Alavi [1] introduced the concept of *Ascending Subgraph Decomposition (ASD)* of a graph G with size $(n+1)C_2$ as the decomposition of G into n subgraphs $G_1,G_2,\ldots G_n$ without isolated vertices such that each G_i is isomorphic to a proper subgraph of G_{i+1} and $|E(G_i)| = i$ for $1 \le i \le n$. Nagarajan [10] generalized ASD to (a,d)-*ASD* of graph G with size $\frac{(2a+(n-1)d)n}{2}$ as the decomposition of G into n subgraphs $G_1,G_2,\ldots,G_n$ without isolated vertices such that each G_i is isomorphic to a proper subgraph of G_{i+1} and $|E(G_i)| = a+(i-1)d$ for $1 \le i \le n$. Clearly, *ASD* of a graph G and its $(1,1)$-*ASD* are the same.

Gnana Dhas [5] defined $(a,d)-Continuous$ *Monotonic Decomposition* or $(a,d)-CMD$ of a graph G of size $\frac{(2a+(n-1)d)n}{2}$ as the decomposition of G into n subgraphs $G_1,G_2,\ldots,G_n$ such that each G_i is connected and $|E(G_i)| = a+(i-1)d$ for $i = 1,2,\ldots,n$. $(1,1)$-*ASD* of a graph G of size $(n+1)C_2$ is known as *Ascending Subgraph Decomposition* or *ASD* of G. Clearly, *CMD* of a graph G and its $(1,1)-CMD$ are the same.

Received June 2, 2013, accepted May 7, 2014.
2010 *Mathematics Subject Classification*. 05C70, 05C75, 05C78.
Key words and phrases. Integral sum graph, anti-integral sum graph, triangular book with book mark, Fan graph with handle, $(a,d)-Ascending$ Subgraph Decomposition, $(a,d)-Continuous$ Monotonic Decomposition ($(a,d)-CMD$).
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*Research supported in part by DST, Government of India, under grant SR/S4/MS: 679/10 (DST- SERB) and Lerroy Wilson Foundation, India (www.WillFoundation.co.in).

Harary introduced the concept of sum graph in [7]. A graph $G = (V, E)$ is a *sum graph* if the vertices of G can be labeled with distinct positive integers so that $e = uv$ is an edge of G if and only if the sum of the labels on vertices u and v is also a label in G. Harary [8] extended the sum graph concept to *integral sum graph* to allow any integers to be used as labels. He provided examples of graphs of these types. To distinguish between the two types, we refer to sum graphs that use only positive integers as $\mathbb{N}-$ *sumgraphs* and those that use any integers as $\mathbb{Z}-$ *sumgraphs* [12].

Properties of sum graphs have been investigated by many authors, including Beineke, Chen, Harary, Mary Florida, Nicholas, Rubin Mary, Suryakala, and Vilfred [2], [7], [8], [12]-[20]. For integers r and s with $r < s$, let $[r, s]$ denote the set of integers $\{r, r+1, \ldots, s\}$ and for any non-empty set of integers S, let $G^+(S)$ denote the integral sum graph on the set S. The $\mathbb{Z}-sum$ graphs of Harary are therefore $G_{-r,r} = G^+([-r, r])$ for $r \in \mathbb{N}$ [12]. The extension of Harary graphs $G_{-r,r}$ to all intervals of integers was introduced by Vilfred and Mary Florida in [13] and [14]. In $G^+(S)$, the set of all edges, each with edge sum k is called an *edge sum class* of $G^+(S)$ and is denoted by $[k]_{G^+(S)}$ or simply $[k]$, $k \in S$ [15]. Integral sum graphs $G_{-4,4}$, $G_{-4,5}$ and $G_{-5,5}$ are given in Figures 1 to 3, respectively.

Two vertices with label j and k of a sum graph $G^+(S)$ with n as its maximum vertex label, are called *supplementary vertices* if $j + k = n+1$ and the corresponding labels are called *supplementary labels*, $1 \le j, k \le n$, $j \ne k$ and $n \ge 2$ [12]. In G_n, $|E(G_n)| = \frac{1}{2}(n(n-1)/2 - \lfloor \frac{n}{2} \rfloor)$, $d(v_j) = n-1-j$ if $1 \le j \le \lfloor \frac{n+1}{2} \rfloor$ and $d(v_j) = n-j$ if $\lfloor \frac{n+1}{2} \rfloor + 1 \le j \le n$ where $\lfloor x \rfloor$ is the floor of x, $V(G_n) = \{v_1, v_2, \ldots, v_n\}$ and j is the vertex sum label of v_j in G_n, $1 \le j \le n$ and $2 \le n$.

Theorem 1.1 ([13]). *If* $-r, s \in \mathbb{N}$ *with* $r < 0 < s$, *then* $G_{r,s} = K_1 + (G_{-r} + G_S)$.

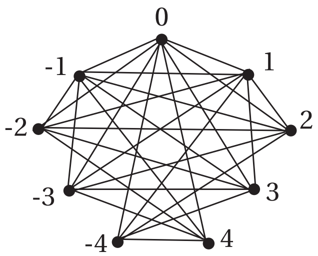

Fig. 1. $G_{-4,4}$

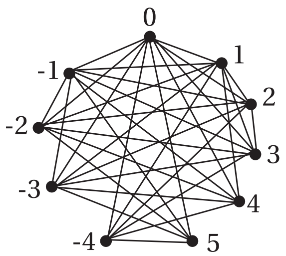

Fig. 2. $G_{-4,5}$

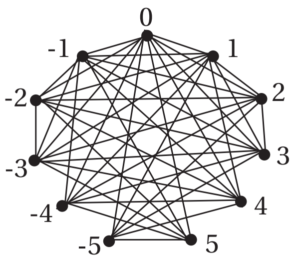

Fig. 3. $G_{-5,5}$

A graph G is called an *anti-sum graph* if its vertices can be labeled with distinct positive integers in such a way that two vertices are adjacent in G if and only if the sum of their labels is not the label of another vertex. Obviously, a graph G is an anti-sum graph if and only if its complement is a sum graph. Thus, many results on anti-sum graphs are simply analogues of the corresponding results on sum graphs. An *anti-integral sum graph* is also defined just as anti-sum graph, the difference being that the labels may be any distinct integers [15].

A graph G is a *split graph* if its vertices can be partitioned into a clique and a stable set or independent set. A *clique* in a graph is a set of pair-wise adjacent vertices and a *stable set* or *independent set* in a graph is a set of pair-wise non-adjacent vertices [3]. G_n is a split graph for $3 \le n$, $n \in \mathbb{N}$.

When k copies of C_n share a common edge, it will form an *n-gon book of k pages* and is denoted by $B(n,k)$. The common edge is called the *spine* or *base of the book*. A *triangular book* $B(3,n)$ or $B_{3,n}$ consists of n triangles with a common edge and can be described as $ST(n)+K_1 = P_2 + nK_1$ where $ST(n)$ denotes the star with n leaves. Let us denote the triangular book $B(3,n)$ with the spine (u,v) by $TB_n(u,v) = P_2(u,v)+nK_1$. Clearly $TB_0 = K_2$ *represent a book without pages or the trivial book* [20].

An $n-gon$ book of k pages $B(n,k)$ with a pendant edge terminating from any one of the end vertices of the spine is called an *n-gon book with a book mark*. Triangular book $TB_n(u,v)$ with book mark (u,w) is denoted by $TB_n(u,v)(u,w)$ or $TB^*_n(u,v)$ where w is the pendant vertex adjacent to u. $TB^*_n(u,v)$ is of order $n+3$ and size $2n+2$ [20]. $TB_4(u_0,v_0)(u_0,w_0)$ with pages $(u_0\ v_0\ v_j)$ for $j = 1,2,3,4$ is shown in Figure 4.

A *fan graph* F_{n-1} is the graph obtained by taking $n-3$ concurrent chords at a vertex in a cycle C_n, $n \ge 3$ [17]. The vertex at which all the $n-3$ chords are concurrent is called the *apex vertex*. Fan graph F_n can be described as $F_n = P_n + K_1$ where P_n is a path on n vertices, $n \ge 2$. If a fan graph F_n has a pendant edge attached with the apex vertex, then the graph is called a *fan with a handle* or *a palm fan* and is denoted by F^*_n [20]. Fan graph F^*_5 with a handle u_0v_0 is shown in Figure 5.

Among the family of graphs some graphs may have (a,d)-*ASD*, some may have (a,d)-*CMD*, some may have both (a,d)-*ASD* and (a,d)-*CMD* and the others have neither (a,d)-*ASD* nor (a,d)-*CMD*. Huaitand [9] studied (a,d)-*ASD* of regular graphs and proved that every regular bipartite graph as *ASD*. Nagarajan [10] studied $(a,d)-ASD$ of wheels. Finding graphs having either (a,d)-*ASD* or (a,d)-*CMD* is difficult and finding graphs having both (a,d)-*ASD* and (a,d)-*CMD* seem to be more difficult, $a,d \in \mathbb{N}$. While studying decomposition of integral sum graphs we come across graphs having both (a,d)-*ASD* and $(a,d)-CMD$ and this motivated us to define $CMSD$ and $(a,d)-CMSD$ of graphs as follows.

Definition 1.2. A decomposition of graph G that is both (a,d)-*ASD* and (a,d)-*CMD* is called a $(a,d)-Continuous\ Monotonic\ Subgraph\ Decomposition\ or$ (a,d)-*CMSD of* G, $a,d \in \mathbb{N}$. *Thus* (a,d)-*CMSD of graph G with size* $\frac{(2a+(n-1)d)n}{2}$ *is the decomposition of G into n subgraphs* $G_1, G_2, \ldots, G_n$ *without isolated vertices such that each* G_i *is connected and isomorphic to a proper subgraph of* G_{i+1} *and* $|E(G_i)| = a+(i-1)d$ *for* $1 \le i \le n$.

In this paper we prove that (i) for $n \ge 3$, K_n admits (a,d)-*CMSD* into triangular books for some a and d, $a,d \in \mathbb{N}$; (ii) for $n \in \mathbb{N}$, $G_{0,2n}$, $G_{0,4n+2}$ and $G_{0,4n+3}$ admit (a,d)-*CMSD* into triangular books with book mark for some a and d, $a, d \in \mathbb{N}$; (iii) $G_{0,4n+1}$ admits *ASD* but doesn't admit (a, d)-*ASD* and (a, d)-*CMD* into triangular books with book mark for any $a, d \in \mathbb{N}$; (iv) for $n \in \mathbb{N}$, $G_{0,4n+2}$, $G_{0,4n}$ and $G_{0,4n-1}$ admit (a, d)-*CMSD* into Fans with a handle for some a and d, $a, d \in \mathbb{N}$ and (v) $G_{0,4n+1}$ admits *ASD* into Fans with a handle and one P_2 but doesn't admit (a, d)-*ASD* and (a, d)-*CMD* into Fans with a handle for any $a, d \in \mathbb{N}$.

For all basic notation and definitions in graph theory, we follow [6]. For additional material on graph labeling problems, we refer to [4]. In this paper the *underlying graph* of a sum graph or an integral sum graph is obtained by removing all vertex labels; *comparison of sum graphs or integral sum graphs of the same order* means comparison of the corresponding underlying graphs only. All graphs in this paper are simple graphs. To present our results, we need a few known results.

Theorem 1.3 ([13]). *For $m + n \geq 3$, $|E(G_{-m,n})| = \frac{1}{4}(m^2 + n^2 + 3(m + n) + 4mn)$ - $\frac{1}{2}(\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor)$ where $\lfloor x \rfloor$ denotes the floor of x, $m, n \in \mathbb{N}_0$. In particular, $|E(G_{0,n})| = \frac{n(n+3)}{4}$ - $\frac{1}{2}(\lfloor \frac{n}{2} \rfloor)$, $|E(G_{-n,n})| = \frac{3n(n+1)}{2}$ - $\lfloor \frac{n}{2} \rfloor$ and $|E(G_{-(n-1),n})| = \frac{n(3n-1)}{2}$, $n \in \mathbb{N}$.*

Theorem 1.4 ([13]). *For $m, n \geq 2$, $G_{0,n}$ and $G_{-m,n}$ contain exactly one vertex of degree n and $m + n$, respectively. For $2 \leq n$, $G_{-1,n}$ has exactly two vertices of degree $n + 1$. $G_{-1,1}$ is the only integral sum graph G having more than two vertices of degree 2.*

Theorem 1.5 ([20]). *Let k and n be such that $2 \leq 2k < n$. If k pairs of supplementary vertices are removed from (i) Harary graph G_n, then the result is isomorphic to G_{n-2k} without the vertex labels and (ii) the graph G^c_n, then the result is isomorphic to G^c_{n-2k} without the vertex labels.*

Theorem 1.6 ([20]). *For $n \geq 3$, the underlying graphs of $G_{0,n} - \{0, n\}$ and $G_{0,n-2}$ are isomorphic and for $n \geq 2r+3$ and $r \in \mathbb{N}$, the underlying graphs of $G_{0,n} - (\{0, n, n-1, n-2, \cdots, n-2r+1, n-2r\} \cup ([n] \cup [n-1] \cup \cdots \cup [n-2r+1]))$ and $G_{0,n-2r-2}$ are isomorphic.*

Theorem 1.7 ([17]). *For $n \geq 2$, Fan graph $F_n = P_n + K_1$ is an integral sum graph.*

Integral sum labeling of F_5 is shown in Figure 6.

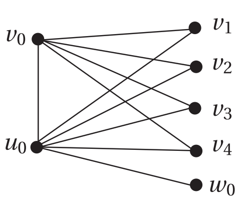

Fig. 4.

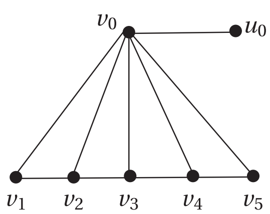

Fig. 5.

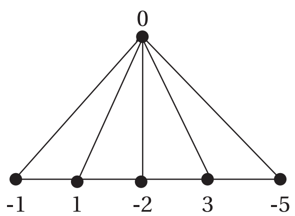

Fig. 6.

Theorem 1.8 ([20]). *For $n \in \mathbb{N}$,* (i) *$TB_n(u_0, v_0)(u_0, w_0)$ and* (ii) *F^*_n are integral sum graphs.*

Proof.

(i) $TB_n(u_0,v_0)(u_0,w_0)$ is of order $n+3$, size $2n+2$ and (u_0,w_0) is the pendant edge terminating at u_0 and let $V(TB_n(u_0,v_0)(u_0,w_0)) = \{w_0,u_0,v_0,v_1,\dots,v_n\}$. Define mapping $f: V(TB_n(u_0,v_0)(u_0,w_0)) \to \mathbb{N}_0$ such that $f(u_0)=0$, $f(v_0)=2m$, $f(v_i)=2mi+1$ for $i=1,2,\dots,n$ and $f(w_0)=2m(n+1)+1$, $m\in\mathbb{N}$.
Consider the integral sum graph $G^+(S)$ where $S=\{0,2m,2m+1,4m+1,6m+1,\dots,2mn+1,2m(n+1)+1 : m\in\mathbb{N}\} = f(V(TB_n(u_0,v_0)(u_0,w_0)))$. Our aim is to prove that f is an integral sum labeling of $TB_n(u_0,v_0)(u_0,w_0)$ and $G^+(S)=TB_n(u_0,v_0)(u_0,w_0)$.
$f(u_0)=0$ implies, $f(u_0)+f(v_i)=f(v_i)$ and $f(u_0)+f(w_0)=f(w_0)$ for $i=0,1,2,\dots,n$. This implies, u_0 is adjacent to w_0, v_0 and v_i for $i=1,2,\dots,n$. For $i=1,2,\dots,n-1$, $f(v_0)+f(v_i)=f(v_{i+1})$, $f(v_0)+f(v_n)=f(w_0)$, $f(v_0)+f(u_0)=f(v_0)$, $f(v_0)+f(w_0)\neq f(u_0), f(v_0), f(w_0), f(v_j)$ for $j=1,2,\dots,n$. This implies, v_0 is adjacent to u_0 and v_i and non-adjacent to w_0 for $i=1,2,\dots,n$. Also $f(w_0)+f(u_0)=f(w_0)$ and $f(w_0)+f(v_j)\neq f(w_0), f(u_0), f(v_j)$ for $j=0,1,\dots,n$. This implies, w_0 is a pendant vertex adjacent only to u_0. For $i,j=0,1,2,\dots,n$, $f(v_i)+f(w_0)\neq f(u_0), f(v_j)$. Also for $1\le i,j,k\le n$, $f(v_i)+f(v_j)\neq f(v_k)$ since $f(v_i)+f(v_j)$ is an even number and $f(v_k)$ is an odd number. This implies, v_i and v_j are non-adjacent in $TB_n(u_0,v_0)(u_0,w_0)$ when $i\neq j$ and $1\le i,j\le n$. Thus v_j is adjacent only to u_0 and v_0 for $j=1,2,\dots,n$.
From all the above conditions integral sum graph $G^+(S)$ is same as $TB_n(u_0,v_0)(u_0,w_0)$ and f is an integral sum labeling of $TB_n(u_0,v_0)(u_0,w_0)$ where $S=\{0,2m,2m+1,4m+1,\dots,2mn+1,2m(n+1)+1 : m\in\mathbb{N}\}$. Integral sum labeling of TB_7^* is shown in Figure 7.

(ii) $F_n = P_n + K_1$ and F_n^* is of order $n+2$ and size $2n$ where P_n is a path on n vertices. Let $V(F_n^*)=\{u_0,v_0,v_1,\dots,v_n\}$ where u_0 is the pendant vertex, v_0 is the apex vertex and $d(v_0)=n+1=\Delta(F_n^*)$. Define mapping $f: V(F_n^*)\to\mathbb{N}_0$ such that $f(v_0)=0$, $f(v_1)=p_m$, the m^{th} Fibonacci number, $m\ge 2$, $f(v_i)=p_{m+i-1}$ for $i=2,\dots,n$ and $f(u_0)=p_{m+n}$. Here, $f(v_0)=0<f(v_1)=p_m<f(v_2)=p_{m+1}<\dots<f(v_n)=p_{m+n-1}<f(u_0)=p_{m+n}$ and for $i-j\neq 1$ and $1\le i,j,k\le n$, $f(v_i)+f(v_j)\neq f(v_k)$. Also $f(v_i)+f(v_{i+1})=f(v_{i+2})$ for $i=1,2,\dots,n-2$ and $f(v_{n-1})+f(v_n)=f(u_0)$, $m\ge 2$. Hence the labeling f is an integral sum labeling of graph F_n^* and thereby F_n^* is an integral sum graph. Integral sum labeling of F_9^* is shown in Figure 8. □

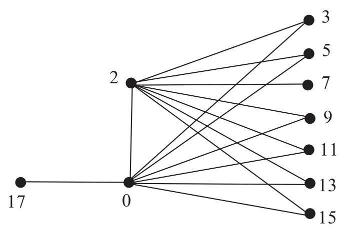

Fig. 7.

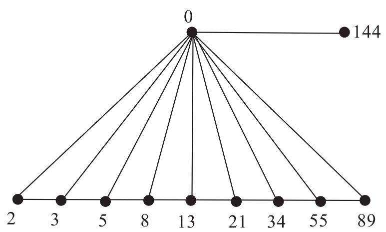

Fig. 8.

2. CMSD and (a,d)-CMSD of K_n and $G_{0,n}$

Motivated by the studies of Alavi [1],Nagarajan [10] and Gnana Dhas [5], we define *CMSD* and (a,d)-*CMSD* of graphs and in particular we study *CMSD* and (a,d)-*CMSD* of K_n and $G_{0,n}$ into families of triangular books, triangular books with book mark and Fans with a handle. Throughout this section, vertices of K_n as well as vertices of $G_{0,n-1}$ are considered as the vertices of an $n-gon$ ordered in the anti-clockwise direction.

Theorem 2.1. For $n \geq 3$, K_n admits $(1,4)$-*CMSD* or $(3,4)$-*CMSD* into triangular books when n is even or odd, respectively.

Proof. Let $V(K_n) = \{0,1,\ldots,n-1\}$. $|E(K_n)| = nC_2$. Consider (a,d)-*CMSD* of K_n for even and odd values of n, separately
Case (i) n is even, $n \geq 3$.
Let $n = 2m$, $m \geq 2$. Then decomposition of $K_n = K_{2m}$ into triangular books of $(1,4)$-*CMSD* is obtained as follows.

$K_{2m} = TB_{2m-2}(0,1) \cup TB_{2m-4}(2,3) \cup \ldots \cup TB_2(2m-4,2m-3) \cup TB_0(2m-2,2m-1)$ where $TB_{2m-2j}(2j-2,2j-1)$ in K_{2m} represents triangular book with spine $(2j-2,2j-1)$ and $(2j-2,2j-1,2j),(2j-2,2j-1,2j+1),\cdots,(2j-2,2j-1,2m-1)$ as the $(2m-2j)$ number of triangular pages and is a connected subgraph, $j = 1,2,\ldots,m$. In K_{2m}, $(0,1)$ is the spine for $TB_{2m-2}(0,1)$, both the vertices 0 and 1 are adjacent to the remaining 2m-2 vertices, $2,3,\ldots,2m-1$ and each one is of degree $2m-1$ in $TB_{2m-2}(0,1)$; $(2,3)$ is the spine for $TB_{2m-4}(2,3)$, both the vertices 2 and 3 are adjacent to the $2m-4$ vertices, $4,5,\ldots,2m-1$ and each one is of degree $2m-1$ in $TB_{2m-2}(0,1) \cup TB_{2m-4}(2,3)$; $(4,5)$ is the spine for $TB_{2m-6}(4,5)$, both the vertices 4 and 5 are adjacent to the $2m-6$ vertices, $6,7,\ldots,2m-1$ and each one is of degree $2m-1$ in $TB_{2m-2}(0,1) \cup TB_{2m-4}(2,3) \cup TB_{2m-6}(4,5);\ldots;(2m-4,2m-3)$ is the spine for $TB_2(2m-4,2m-3)$, both the vertices $2m-4$ and $2m-3$ are adjacent to the 2 vertices, $2m-2$ and $2m-1$ and each one is of degree $2m-1$ in $TB_{2m-2}(0,1) \cup TB_{2m-4}(2,3) \cup TB_{2m-6}(4,5) \cup \ldots \cup TB_2(2m-4,2m-3)$; $(2m-2,2m-1)$ is the spine for $TB_0(2m-2,2m-1)$ which is a triangular book without pages and each one of the vertices $2m-2$ and $2m-1$ is of degree $2m-1$ in $TB_{2m-2}(0,1) \cup TB_{2m-4}(2,3) \cup TB_{2m-6}(4,5) \cup \ldots \cup TB_2(2m-4,2m-3) \cup TB_0(2m-2,2m-1) = K_{2m}$. Also $|E(TB_0(2m-2,2m-1))| = 1 < |E(TB_2(2m-4,2m-3))| = 5 < |E(TB_4(2m-6,2m-5))| = 9 < \cdots < |E(TB_{2m-4}(2,3))| = 4m-7 < |E(TB_{2m-2}(0,1))| = 4m-3$. And clearly, $TB_0(2m-2,2m-1)$ is a connected subgraph of $TB_2(2m-4,2m-3)$ which is a connected subgraph of $TB_4(2m-6,2m-5)$ which is a connected subgraph of ... which is a connected subgraph of $TB_{2m-4}(2,3)$ which is a connected subgraph of $TB_{2m-2}(0,1)$, without vertex labels. Thus K_{2m} admits $(1,4)$-*CMSD* into triangular books for $m \geq 2$. In different colors $(1,4)$-*CMSD* of K_4, K_6 and K_8 are shown in Figures 9, 10 and 11, respectively and $K_4 = TB_2(0,1) \cup TB_0(2,3)$, $K_6 = TB_4(0,1) \cup TB_2(2,3) \cup TB_0(4,5)$ and $K_8 = TB_6(0,1) \cup TB_4(2,3) \cup TB_2(4,5) \cup TB_0(6,7)$.

Case (ii): n is odd, $n \geq 3$.
Let $n = 2m+1$, $m \in \mathbb{N}$. Then $K_n = K_{2m+1}$ can be decomposed into triangular books of $(3,4)$-*CMSD* as follows.

$K_{2m+1} = TB_{2m-1}(0,1) \cup TB_{2m-3}(2,3) \cup \ldots \cup TB_3(2m-4,2m-3) \cup TB_1(2m-2,2m-1)$ where $TB_{2m+1-2j}(2j-2,2j-1)$ in K_{2m+1} represents triangular book with spine $(2j-2,2j-1)$ and $(2j-2,2j-1,2j),(2j-2,2j-1,2j+1),\ldots,(2j-2,2j-1,2m)$ as the $(2m+1-2j)$ number of triangular pages and is a connected subgraph, $j = 1,2,\ldots,m$. The above decomposition of K_{2m+1} is similar to the decomposition given in case (i) except K_{2m+1} admits $(3,4)$-*CMSD* into triangular books since $|E(TB_1(2m-2,2m-1))| = 3 < |E(TB_3(2m-4,2m-3))| = 7 < |E(TB_5(2m-6,2m-5))| = 11 < \cdots < |E(TB_{2m-3}(2,3))| = 4m-5 < |E(TB_{2m-1}(0,1))| = 4m-1$ and $TB_1(2m-2,2m-1)$ is a connected subgraph of $TB_3(2m-4,2m-3)$ which is a connected subgraph of $TB_5(2m-6,2m-5)$ which is a connected subgraph of $\ldots$ which is a connected subgraph of $TB_{2m-3}(2,3)$ which is a connected subgraph of $TB_{2m-1}(0,1)$, without vertex labels. $(3,4)$-*CMSD* of K_3, K_5 and K_7 are shown in different colors in Figures 12, 13 and 14, respectively and $K_3 = TB_1(0,1), K_5 = TB_3(0,1) \cup TB_1(2,3)$ and $K_7 = TB_5(0,1) \cup TB_3(2,3) \cup TB_1(4,5)$. Hence the result. □

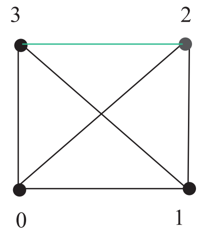

Fig. 9.

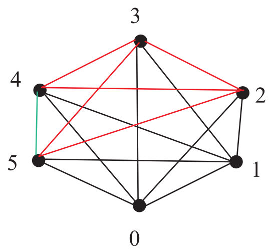

Fig. 10.

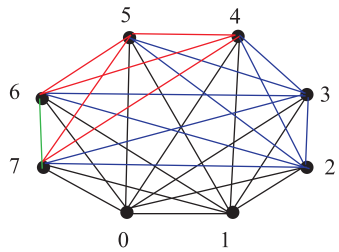

Fig. 11.

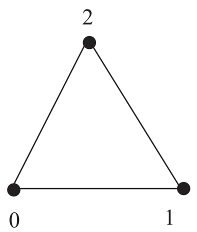

Fig. 12.

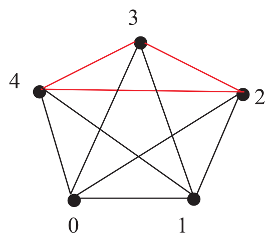

Fig. 13.

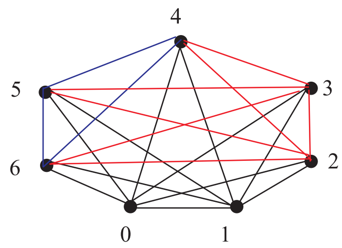

Fig. 14.

Corollary 2.2. *K_n admits (a,d)-CMSD into triangular books for some a and d, $a,d \in \mathbb{N}$.*

Theorem 2.3. *For $n \geq 3$, K_n admits $(1,1)$-CMSD into stars.*

Proof. The $(1,1)$-*CMSD* of K_n into stars is obtained as follows. $K_{1,1}(0;1) \cup K_{1,2}(2;0,1) \cup K_{1,3}(3;0,1,2) \cup \ldots \cup K_{1,n-1}(n-1;0,1,2,\ldots,n-2) \cup K_(1,n)(n;0,1,2,\ldots,n-1)$ where $K_{1,j}(j;0,1,\ldots,j-1)$ is the star $K_{1,n}$ with internal vertex j and leaves $0,1,\ldots,j-1$, $1 \leq j \leq n$. □

Theorem 2.4. *For $m \in \mathbb{N}$, $G_{0,2m}$ admits $(2,2)$-CMSD into triangular books with book mark.*

Proof. In the sum graph G_m, $|E(G_m)| = \frac{1}{2}(\frac{m(m-1)}{2} - \lfloor \frac{m}{2} \rfloor)$, $d(v_j) = m-1-j$ if $1 \le j \le \lfloor \frac{m+1}{2} \rfloor$ and $d(v_j) = m-j$ if $\lfloor \frac{m+1}{2} \rfloor + 1 \le j \le m$ where $\lfloor x \rfloor$ is the floor of x and $2 \le m$. Therefore $|E(G_{0,2m})| = 2m + |E(G_{2m})| = 2m + \frac{1}{2}(\frac{2m(2m-1)}{2} - \lfloor \frac{2m}{2} \rfloor) = m(m+1)$ where $G_{0,2m} = K_1 + G_{2m}$. The proof of the theorem is similar to the proof given to Theorem 2.1. Let $V(G_{0,2m}) = \{v_0, v_1, v_2, \ldots, v_{2m}\}$ where j is the integral sum label of vertex v_j in the integral sum graph $G_{0,2m}$, $0 \le j \le 2m$. $(2,2)$-*CMSD* of $G_{0,2m}$ into triangular books with book mark is obtained as follows.

$G_{0,2m} = TB_0(0,2m-1)(0,2m) \cup TB_1(1,2m-2;0)(1,2m-1) \cup TB_2(2,2m-3;0,1)(2,2m-2) \cup TB_3(3,2m-4;0,1,2)(3,2m-3) \cup \ldots \cup TB_{m-1}(m-1,m;0,1,2,\ldots,m-2)(m-1,m+1)$ where $TB_j(j,2m-(j+1);0,1,2,\ldots,j-1)(j,2m-j)$ represents triangular book with spine $(j,2m-(j+1))$, book mark $(j,2m-j)$ and leaves $0,1,2,\ldots,j-1$ for $j = 1,2,\ldots,m-1$ and $TB_0(0,2m-1)(0,2m)$ is the triangular book with spine $(0,2m-1)$, book mark $(0,2m)$ and without any leaf. This implies $G_{0,2m}$ admits $(2,2)$-*CMSD* into triangular books with book mark since $|E(TB_0(0,2m-1)(0,2m))| = 2 < |E(TB_1(1,2m-2;0)(1,2m-1))| = 4 < |E(TB_2(2,2m-3;0,1)(2,2m-2))| = 6 < \cdots < |E(TB_{m-2}(m-2,m+1)(m-2,m+2))| = 2m-2 < |E(TB_{m-1}(m-1,m;0,1,2,\ldots,m-1)(m-1,m+1))| = 2m$ and $TB_0(0,2m-1)(0,2m)$ is a connected subgraph of $TB_1(1,2m-2;0)(1,2m-1)$ which is a connected subgraph of $TB_2(2,2m-3;0,1)(2,2m-2)$ which is a connected subgraph of $TB_3(3,2m-4;0,1,2)(3,2m-3)$ which is a connected subgraph of ... which is a connected subgraph of $TB_{m-1}(m-1,m;0,1,2,\ldots,m-2)(m-1,m+1)$. Hence the result is proved. $(2,2)$-*CMSD* of $G_{0,6}, G_{0,8}$ and $G_{0,10}$ are shown in different colors in Figures 15, 16 and 17, respectively and $G_{0,6} = TB_0(0,5)(0,6) \cup TB_1(1,4;0)(1,5) \cup TB_2(2,3;0,1)(2,4)$, $G_{0,8} = TB_0(0,7)(0,8) \cup TB_1(1,6;0)(1,7) \cup TB_2(2,5;0,1)(2,6) \cup TB_3(3,4;0,1,2)(3,5)$ and $G_{0,10} = TB_0(0,9)(0,10) \cup TB_1(1,8;0)(1,9) \cup TB_2(2,7;0,1)(2,8) \cup TB_3(3,6;0,1,2)(3,7) \cup TB_4(4,5;0,1,2,3)(4,6)$. □

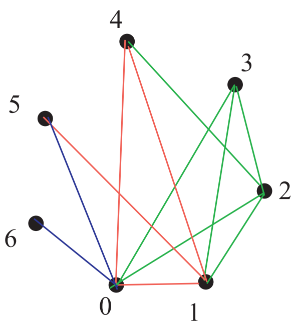

Fig. 15.

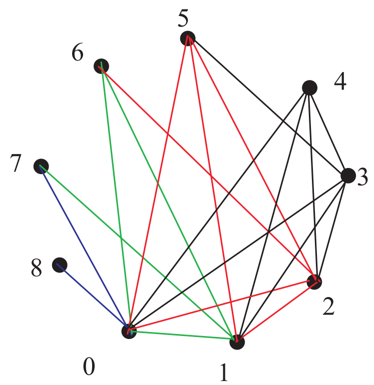

Fig. 16.

Theorem 2.5. *For $n \in \mathbb{N}$, $G_{0,4n+1}$ doesn't admit (a,d)-ASD and (a,d)-CMD into triangular books with book mark for any $a,d \in \mathbb{N}$.*

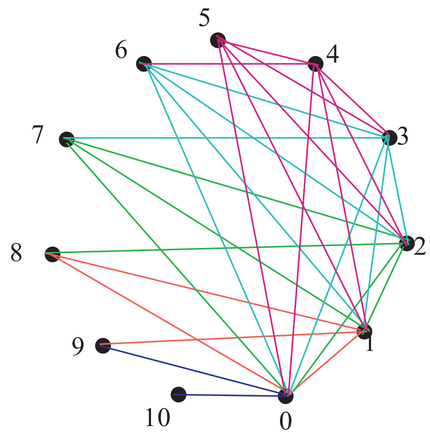

Fig. 17.

Proof. For $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$, we have $|E(G_{0,4n+1})| = \frac{1}{2}(\frac{(4n+1)(4n+4)}{2} - \lfloor \frac{4n+1}{2} \rfloor) = (n+1)(4n+1) - n = (2n+1)^2$ and $|E(TB_k(u,v)(u,w))| = 2k+2$. For $n \in \mathbb{N}$, if $G_{0,4n+1}$ admits (a,d)-*ASD* or (a,d)-*CMD* into triangular books with book mark for any $a,d \in \mathbb{N}$, then let $G_{0,4n+1} = TB^*_{k_1}(u_1,v_1) \cup TB^*_{k_2}(u_2,v_2) \cup \ldots \cup TB^*_{k_m}(u_m,v_m)$ where $TB^*_{k_1}(u_1,v_1), TB^*_{k_2}(u_2,v_2), \ldots, TB^*_{k_m}(u_m,v_m)$ are edge disjoint triangular books with book mark in $G_{0,4n+1}, u_1,u_2,\ldots,u_m,v_1,v_2,\ldots,v_m \in V(G_{0,4n+1})$, $0 \le k_1 < k_2 < \cdots < k_m$, $k_1,k_2,\ldots,k_m \in \mathbb{N}_0$ and $m \in \mathbb{N}$. Then $|E(G_{0,4n+1})| = |E(TB^*_{k_1}(u_1,v_1))| + |E(TB^*_{k_2}(u_2,v_2))| + \cdots + |E(TB^*_{k_m}(u_m,v_m))|$ which implies, $(2n+1)^2 = (2k_1+2)+(2k_2+2)+\cdots+(2k_m+2)$ which is not possible since the L.H.S. is an odd number whereas the R.H.S. is an even number. Hence the result is true by the method of contradiction. □

Corollary 2.6. *For $n \in \mathbb{N}$, $G_{0,4n+1}$ doesn't admit (a,d)-CMSD into triangular books with book mark for any $a,d \in \mathbb{N}$.*

Theorem 2.7. For $n \in \mathbb{N}$,

(i) $G_{0,4n}$ admits $(6,8)$-*CMSD* into triangular books with book mark;
(ii) $G_{0,4n+1}$ can be decomposed into triangular books with book mark;
(iii) $G_{0,4n+2}$ admits $(2,8)$-*CMSD* into triangular books with book mark and
(iv) $G_{0,4n+3}$ admits $(4,8)$-*CMSD* into triangular books with book mark.

Proof. Let $V(G_{0,n}) = \{0,1,2,\ldots,n\}$. In the sum graph G_n, $|E(G_n)| = \frac{1}{2}(\frac{n(n-1)}{2} - \lfloor \frac{n}{2} \rfloor)$, $d(v_j) = n-1-j$ if $1 \le j \le \lfloor \frac{n+1}{2} \rfloor$ and $d(v_j) = n-j$ if $\left\lfloor \frac{(n+1)}{2} \right\rfloor + 1 \le j \le n$ where $\lfloor x \rfloor$ is the floor of x, j is the vertex sum label of v_j and $n \in \mathbb{N}$. Therefore $|E(G_{0,n})| = n + |E(G_n)| = \frac{1}{2}(\frac{n(n+3)}{2} - \lfloor \frac{n}{2} \rfloor)$. Consider the following four cases of n and the proof is similar to the proof given to Theorem 2.1.

Case (i) $n = 4m$, $m \in \mathbb{N}$.

In this case $(6,8)$-*CMSD* of $G_{0,n} = G_{0,4m}$ into triangular books with book mark is obtained as follows.

$G_{0,4m} = TB_{4m-2}(0,1;2,3,\ldots,4m-1)(0,4m) \cup TB_{4m-6}(2,3;4,5,\ldots,4m-3)(2,4m-2) \cup TB_{4m-10}(4,5;6,7,\ldots,4m-5)(4,4m-4) \cup TB_{4m-14}(6,7;8,9,\ldots,4m-7)(6,4m-6) \cup \ldots \cup TB_6(2m-4,2m-3;2m-2,2m-1,\ldots,2m+3)(2m-4,2m+4) \cup TB_2(2m-2,2m-1;2m,2m+1)(2m-2,2m+2)$ where $TB_{4m-(2+4j)}(2j,2j+1;2j+2,2j+3,\ldots,4m-2j-1)(2j,4m-2j)$ represents triangular book in $G_{0,4m}$ with spine $(2j,2j+1)$, *pendant vertex* with label $4m-2j$ and leaves $2j+2,2j+3,\ldots,4m-2j-1$ and is a connected subgraph for $j = 0,1,2,\ldots,m-1$. In this decomposition all the edges of $G_{0,4m}$ are partitioned into the edges of triangular books with book mark and $|E(TB_2(2m-2,2m-1;2m,2m+1)(2m-2,2m+2))| = 6 < |E(TB_6(2m-4,2m-3;2m-2,2m-1,\ldots,2m+3)(2m-4,2m+4))| = 14 < |E(TB_{10}(2m-6,2m-5;2m-4,2m-3,\ldots,2m+5)(2m-6,2m+6))| = 22 < \cdots < |E(TB_{4m-6}(2,3;4,5,\ldots,4m-3)(2,4m-2))| = 8m-10 < |E(TB_{4m-2}(0,1;2,3,\ldots,4m-1)(0,4m))| = 8m-2$ and $TB_{4m-2}(0,1;2,3,\ldots,4m-1)(0,4m)$ is a connected subgraph of $TB_{4m-6}(2,3;4,5,\ldots,4m-3)(2,4m-2)$ which is a connected subgraph of $TB_{4m-10}(4,5;6,7,\ldots,4m-5)(4,4m-4)$ which is a connected subgraph of $TB_{4m-14}(6,7;8,9,\ldots,4m-7)(6,4m-6)$ which is a connected subgraph of $\ldots$ which is a connected subgraph of $TB_6(2m-4,2m-3;2m-2,2m-1,\ldots,2m+3)(2m-4,2m+4)$ which is a connected subgraph of $TB_2(2m-2,2m-1;2m,2m+1)(2m-2,2m+2)$. Thus $G_{0,4m}$ admits $(6,8)$-*CMSD* into triangular books with book mark. Thus $G_{0,4m}$ admits $(6,8)$-*ASD* into triangular books with book mark. $(6,8)$-*CMSD* of $G_{0,4} = TB_2(0,1;2,3)(0,4)$, $G_{0,8} = TB_6(0,1;2,3,4,5,6,7)(0,8) \cup TB_2(2,3;4,5)(2,6)$ and $G_{0,12} = TB_{10}(0,1;2,3,\ldots,11)(0,12) \cup TB_6(2,3;4,5,\ldots,9)(2,10) \cup TB_2(4,5;6,7)(4,8)$ are shown in Figures 18, 19 and 20, respectively.

Case (ii): $n = 4m+1$, $m \in \mathbb{N}$.
In this case decomposition of $G_{0,4m+1}$ into triangular books with book mark is obtained as follows.

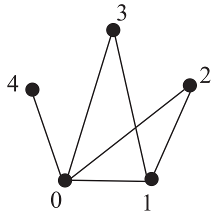

Fig. 18.

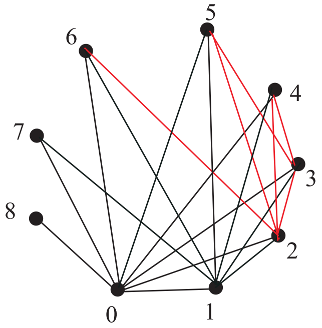

Fig. 19.

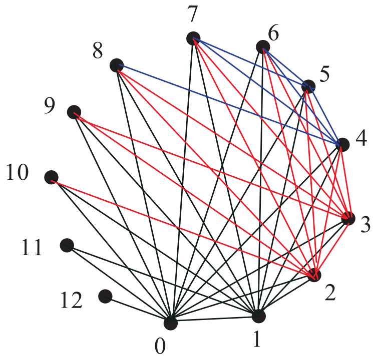

Fig. 20.

$G_{0,4m+1} = TB_{4m-1}(0,1;2,3,\ldots,4m)(0,4m+1) \cup TB_{4m-5}(2,3;4,5,\ldots,4m-2)(2,4m-1) \cup$

$TB_{4m-9}(4,5;6,7,\ldots,4m-4)(4,4m-3) \cup TB_{4m-13}(6,7;8,9,\ldots,4m-6)(6,4m-5) \cup \ldots \cup TB_7(2m-4,2m-3;2m-2,2m-1,\ldots,2m+4)(2m-4,2m+5) \cup TB_3(2m-2,2m-1;2m,2m+1,2m+2)(2m-2,2m+3) \cup TB_0(2m,2m+1)$ where $TB_{4m+1-(2+4j)}(2j,2j+1;2j+2,2j+3,\ldots,4m-2j)(2j,4m+1-2j)$ represents triangular book in $G_{0,4m+1}$ with spine $(2j,2j+1)$, pendant vertex with label $4m+1-2j$ and leaves $2j+2,2j+3,\ldots,4m-2j$ and is a connected subgraph for $j=0,1,2,\ldots,m-1$ and $TB_0(2m,2m+1)$ is a triangular book with spine $(2m,2m+1)$ and without any leaf. All the edges of $G_{0,4m+1}$ are covered under this decomposition and $|E(TB_0(2m,2m+1))| = 1 < |E(TB_3(2m-2,2m-1;2m,2m+1,2m+2)(2m-2,2m+3))| = 8 < |E(TB_7(2m-4,2m-3;2m-2,2m-1,\ldots,2m+4)(2m-4,2m+5))| = 16 < \cdots < |E(TB_{4m-5}(2,3;4,5,\ldots,4m-2)(2,4m-1))| = 8m-8 < |E(TB_{4m-1}(0,1;2,3,\ldots,4m)(0,4m+1))| = 8m$ and $TB_0(2m,2m+1)$ is a connected subgraph of $TB_3(2m-2,2m-1;2m,2m+1,2m+2)(2m-2,2m+3)$ which is a connected subgraph of $TB_7(2m-4,2m-3;2m-2,2m-1,\ldots,2m+4)(2m-4,2m+5)$ which is a connected subgraph of ... which is a connected subgraph of $TB_{4m-5}(2,3;4,5,\ldots,4m-2)(2,4m-1)$ which is a connected subgraph of $TB_{4m-1}(0,1;2,3,\ldots,4m)(0,4m+1)$, without vertex labels. Thus $G_{0,4m+2}$ is decomposed into triangular books with book mark. The following decomposition of $G_{0,5} = TB_3(0,1;2,3,4)(0,5) \cup TB_0(2,3)$, $G_{0,9} = TB_7(0,1;2,3,\ldots,8)(0,9) \cup TB_3(2,3;4,5,6)(2,7) \cup TB_0(4,5)$ and $G_{0,13} = TB_{11}(0,1;2,3,\ldots,12)(0,13) \cup TB_7(2,3;4,5,\ldots,10)(2,11) \cup TB_3(4,5;6,7,8)(4,9) \cup TB_0(6,7)$ are shown in Figures 21, 22 and 23, respectively.

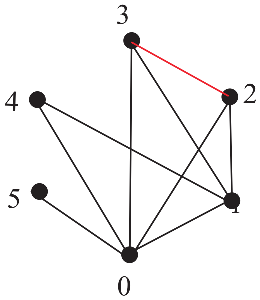

Fig. 21.

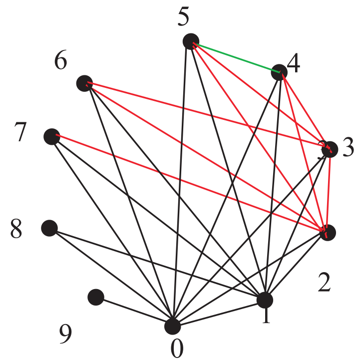

Fig. 22.

Case (iii): $n = 4m+2$, $m \in \mathbb{N}$.

In this case $(2,8)$-*CMSD* of $G_{0,4m+2}$ into triangular books with book mark is obtained as follows.

$G_{0,4m+2} = TB_{4m}(0,1;2,3,\ldots,4m+1)(0,4m+2) \cup TB_{4m-4}(2,3;4,5,\ldots,4m-1)(2,4m) \cup TB_{4m-8}(4,5;6,7,\ldots,4m-3)(4,4m-2) \cup TB_{4m-12}(6,7;8,9,\ldots,4m-5)(6,4m-4) \cup \ldots \cup TB_8(2m-4,2m-3;2m-2,2m-1,\ldots,2m+5)(2m-4,2m+6) \cup TB_4(2m-2,2m-1;2m,2m+1,2m+2,2m+3)(2m-2,2m+4) \cup TB_0(2m,2m+1)(2m,2m+2)$. Here $TB_{4m-4j}(2j,2j+1;2j+2,2j+3,\ldots,4m-2j+1)(2j,4m-2j+2)$ represents triangular book in $G_{0,4m+2}$ with spine $(2j,2j+1)$,

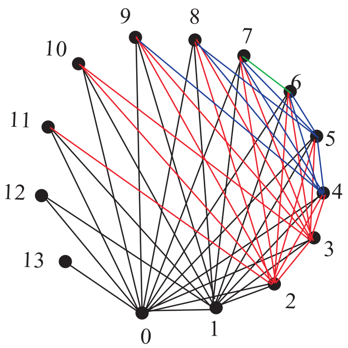

Fig. 23.

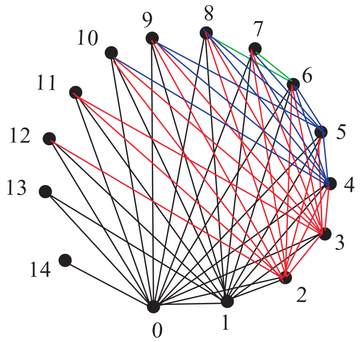

Fig. 24.

pendant vertex $4m-2j+2$ and leaves $2j+2, 2j+3, \ldots, 4m-2j+1$ and is a connected subgraph for $j = 0, 1, 2, \ldots, m-1$ and $TB_0(2m, 2m+1)(2m, 2m+2)$ is a triangular book with spine $(2m, 2m+1)$, pendant vertex with label $2m+2$ and without any leaf. In this decomposition all the edges of $G_{0,4m+2}$ are partitioned into edges of triangular books with book mark and $|E(TB_0(2m, 2m+1)(2m, 2m+2))| = 2 < |E(TB_4(2m-2, 2m-1; 2m, 2m+1, 2m+2, 2m+3)(2m-2, 2m+4))| = 10 < |E(TB_8(2m-4, 2m-3; 2m-2, 2m-1, \ldots, 2m+5)(2m-4, 2m+6))| = 18 < \cdots < |E(TB_{4m-4}(2, 3; 4, 5, \ldots, 4m-1)(2, 4m))| = 8m-6 < |E(TB_{4m}(0, 1; 2, 3, \ldots, 4m+1)(0, 4m+2))| = 8m+2$ and $TB_0(2m, 2m+1)(2m, 2m+2)$ is a connected subgraph of $TB_4(2m-2, 2m-1; 2m, 2m+1, 2m+2, 2m+3)(2m-2, 2m+4)$ which is a connected subgraph of $TB_8(2m-4, 2m-3; 2m-2, 2m-1, \ldots, 2m+5)(2m-4, 2m+6)$ which is a connected subgraph of ... which is a connected subgraph of $TB_{4m-4}(2, 3; 4, 5, \ldots, 4m-1)(2, 4m)$ which is a connected subgraph of $TB_{4m}(0, 1; 2, 3, \ldots, 4m+1)(0, 4m+2)$, without vertex labels. Thus $G_{0,4m+2}$ admits $(2, 8)$-*CMSD* into triangular books with book mark. $(2, 8)$-*CMSD* of $G_{0,14} = TB_{12}(0, 1; 2, 3, \ldots, 13)(0, 14) \cup TB_8(2, 3; 4, 5, \ldots, 11)(2, 12) \cup TB_4(4, 5; 6, 7, 8, 9)(4, 10) \cup TB_0(6, 7)(6, 8)$ is shown in Figure 24.

Case (iv): $n = 4m+3$, $m \in \mathbb{N}$.
In this case, $(4, 8)$-*CMSD* of $G_{0,4m+3}$ into triangular books with book mark is obtained as follows. $G_{0,4m+3} = TB_{4m+1}(0, 1; 2, 3, \ldots, 4m+2)(0, 4m+3) \cup TB_{4m-3}(2, 3; 4, 5, \ldots, 4m)(2, 4m+1) \cup TB_{4m-7}(4, 5; 6, 7, \ldots, 4m-2)(4, 4m-1) \cup TB_{4m-11}(6, 7; 8, 9, \ldots, 4m-4)(6, 4m-3) \cup \ldots \cup TB_9(2m-4, 2m-3; 2m-2, 2m-1, \ldots, 2m+6)(2m-4, 2m+7) \cup TB_5(2m-2, 2m-1; 2m, 2m+1, \ldots, 2m+4)(2m-2, 2m+5) \cup TB_1(2m, 2m+1; 2m+2)(2m, 2m+3)$ where $TB_{4m+1-4j}(2j, 2j+1; 2j+2, 2j+3, \ldots, 4m-2j+2)(2j, 4m-2j+3)$ represents triangular book in $G_{0,4m+3}$ with spine $(2j, 2j+1)$, pendant vertex $4m-2j+3$ and leaves $2j+2, 2j+3, \ldots, 4m-2j+2$ and is a connected subgraph for $j = 0, 1, 2, \ldots, m$. In this decomposition all the edges of $G_{0,4m+3}$ are partitioned into edges of triangular books with book mark and $|E(TB_1(2m, 2m+1; 2m+2)(2m, 2m+3))| = 4 < |E(TB_5(2m-2, 2m-1; 2m, 2m+1, \ldots, 2m+4)(2m-2, 2m+5))| = 12 < |E(TB_9(2m-4, 2m-3; 2m-2, 2m-1, \ldots, 2m+6)(2m-4, 2m+7))| = 20 < \cdots < |E(TB_{4m-3}(2, 3; 4, 5, \ldots, 4m)(2, 4m+1))| = 8m-4 < |E(TB_{4m+1}(0, 1; 2, 3, \ldots, 4m+2)(0, 4m+3))| = 8m+4$ and $TB_1(2m, 2m+1; 2m+$

$2)(2m,2m+3)$ is a connected subgraph of $TB_5(2m-2,2m-1;2m,2m+1,\ldots,2m+4)(2m-2,2m+5)$ which is a connected subgraph of $TB_9(2m-4,2m-3;2m-2,2m-1,\ldots,2m+6)(2m-4,2m+7)$ which is a connected subgraph of ... which is a connected subgraph of $TB_{4m-3}(2,3;4,5,\ldots,4m)(2,4m+1)$ which is a connected subgraph of $TB_{4m+1}(0,1;2,3,\ldots,4m+2)(0,4m+3)$, without vertex labels. Thus $G_{0,4m+3}$ admits $(4,8)$-*CMSD* into triangular books with book mark. $(4,8)$-*CMSD* of $G_{0,15} = TB_{13}(0,1;2,3,\ldots,14)(0,15) \cup TB_9(2,3;4,5,\ldots,12)(2,13) \cup TB_5(4,5;6,7,8,9,10)(4,11) \cup TB_1(6,7;8)(6,9)$ is shown in Figure 25. Hence the result. $\square$

Theorem 2.8. For $m \in \mathbb{N}$, $G_{0,4m+1}$ does not admit (a,d)-*ASD* and (a,d)-*CMD* into Fans with a handle for any $a,d \in \mathbb{N}$.

Proof. If possible, let $G_{0,4m+1}$ admit $(a,d)-ASD$ into Fans with a handle for some $a,d \in \mathbb{N}$. Then let $G_{0,4m+1} \cong F^*_{n_1} \cup F^*_{n_2} \cup \ldots \cup F^*_{n_k}$ where $F^*_{n_1}, F^*_{n_2}, \ldots, F^*_{n_k}$ are edge disjoint fans with handle for some $n_1, n_2, \ldots, n_k \in \mathbb{N}$ and $2 \le n_1 < n_2 < \cdots < n_k$. Then $|E(G_{0,4m+1})| = |E(F^*_{n_1})| + |E(F^*_{n_2})| + \cdots + |E(F^*_{n_k})|$ which implies $(2m+1)^2 = 2n_1 + 2n_2 + \cdots + 2n_k$ which is a contradiction since the L.H.S. is an odd number whereas the R.H.S. is an even number. Hence the result. $\square$

Corollary 2.9. *For $m \in \mathbb{N}$, $G_{0,4m+1}$ does not admit (a,d)-CMSD into Fans with a handle for any $a,d \in \mathbb{N}$.*

Theorem 2.10. *For $n \in \mathbb{N}$,*

(i) $G_{0,4n+1}$ *can be decomposed into Fans with a handle and one P_2;*
(ii) $G_{0,4n+2}$ *admits $(2,8)$-CMSD into Fans with a handle;*
(iii) $G_{0,4n-1}$ *admits $(4,8)$-CMSD into Fans with a handle and*
(iv) $G_{0,4n}$ *admits $(6,8)$-CMSD into Fans with a handle.*

Proof. For $n \ge 3$, F^*_{n-1}, fan with a handle has $n+1$ vertices and $2(n-1)$ edges. Let $V(G_{0,n}) = \{v_0, v_1, v_2, \ldots, v_n\}$ where v_j is the vertex with integral sum label j in $G_{0,n}$, $0 \le j \le n$. In the sum graph G_n, $|E(G_n)| = \frac{1}{2}(\frac{n(n-1)}{2} - \lfloor \frac{n}{2} \rfloor)$, $d(v_j) = n-1-j$ if $1 \le j \le \lfloor \frac{n+1}{2} \rfloor$ and $d(v_j) = n-j$ if $\lfloor \frac{n+1}{2} \rfloor + 1 \le j \le n$ where $\lfloor x \rfloor$ is the floor of x and v_j is the vertex with sum label j in G_n. Now consider decomposition of $G_{0,n}$ into Fans with a handle for different values of n separately.

In $G_{0,n}$, the subset $\{v_i v_j : i+j = n \text{ or } n-1,\ 0 \le i,j \le n\} \cup \{v_0 v_i : i = 1,2,\ldots,n-2\}$ of $E(G_{0,n})$ forms F^*_{n-1}, fan graph with cycle $(v_0 v_{n-1} v_1 v_{n-2} \ldots v_{\lfloor \frac{n}{2} \rfloor})$, pendant edge $v_0 v_n$ attached at the apex vertex v_0 and $n-3$ concurrent edges, $v_0 v_j s$ for $j = 1,2,\ldots,\lfloor \frac{n}{2} \rfloor -1, \lfloor \frac{n}{2} \rfloor +1, \lfloor \frac{n}{2} \rfloor +2, \ldots, n-2$. Using the definition of integral sum labeling, $G_n - (\{v_n, v_{n-1}\} \cup \{v_i v_j : i+j = n \text{ or } n-1, 1 \le i,j \le n-2\}) = G_n - \{n, n-1, [n], [n-1]\} = G_{n-2}$. Also using Theorem 1.5, $G_{n-2} - \{v_1, v_{n-2}\}$ is isomorphic to unlabeled graph G_{n-4}. Therefore $G_{0,n} - (\{v_0, v_n, v_{n-1}, v_{n-2}\} \cup \{v_i v_0 : i+j = n \text{ or } n-1, 1 \le i,j \le n-2\})$ is isomorphic to unlabeled graph $G_{0,n-4}$. This also follows from Theorem 1.6. Relabeling the vertices $v_1, v_2, \ldots, v_{n-3}$ in the resultant graph $G_{0,n} - (\{v_0, v_n, v_{n-1}, v_{n-2}\} \cup \{v_i v_j :$

$i+j=n$ or $n-1, 1 \le i,j \le n-2\}$) as $v_0, v_1, \ldots, v_{n-4}$ using the bijection $i \to i-1$ among the vertex labels and continuing the same technique of choosing the vertex subset $\{v_0, v_{n-4}, v_{n-5}, v_{n-6}\}$ and the relabeled edge subset $\{v_iv_j : i+j = n-4$ or $n-5$ for $0 \le i,j \le n-4\} \cup \{v_0v_i : i = 1,2,\ldots,n-6\}$ (in the relabeled graph) which form a fan F^*_{n-5}. The underlying graph of the subgraph $G_{0,n-4} - (\{v_0, v_{n-4}, v_{n-5}, v_{n-6}\} \cup \{v_iv_j : i+j = n-4$ or $n-5, 1 \le i,j \le n-6\})$ of the relabeled graph $G_{0,n-4}$ is isomorphic to the underlying graph of $G_{0,n-8}$. Continue the above process. And to complete the proof, we consider the following four cases of n.

Case (i): $n = 4m+1$, $m \in \mathbb{N}$.
In this case, $G_{0,n} = G_{0,4m+1} = F^*_{4m} \cup F^*_{4(m-1)} \cup F^*_{4(m-2)} \cup \ldots \cup F^*_8 \cup F^*_4 \cup P_2(m,m+1) = P_2(m,m+1) \cup (\bigcup_{j=0}^{m-1} F^*_{4(m-j)})$ where $F^*_{4m}, F^*_{4(m-1)}, F^*_{4(m-2)}, \ldots, F^*_8, F^*_4, P_2(m,m+1)$ are edge disjoint subgraphs of $G_{0,4m+1}$; here F^*_{4m} is the Fan with the handle (v_0, v_{4m+1}), apex vertex v_0 and $P_{4m} = v_{4m}v_1v_{4m-1}v_2 \ldots v_{2m+2}v_{2m-1}v_{2m+1}v_{2m}$; $|E(G_{0,4m+1})| = 4m+1+|E(G_{4m+1})| = 4m+1+\frac{(4m+1)(4m)}{4} - \frac{2m}{2} = (2m+1)^2$; $|E(P_2)| = 1 < |E(F^*_4)| = 8 < |E(F^*_8)| = 16 < \cdots < |E(F^*_{n-5})| = 2(n-5) < |E(F^*_{n-1})| = 2(n-1) = 8m$; $|E(P_2)| + |E(F^*_4)| + |E(F^*_8)| + \cdots + |E(F^*_{n-1})| = 1+8+16+\cdots+8m = 4m^2+4m+1 = (2m+1)^2$ and v_j is the vertex with integral sum label j in $G_{0,4m+1}$, $j \in [0,4m+1]$. Moreover, P_2 is a connected subgraph of F^*_4 which is a connected subgraph of F^*_8 which is a connected subgraph of … which is a connected subgraph of F^*_{n-5} which is a connected subgraph of F^*_{n-1}, without vertex labels. Thus $G_{0,4m+1}$ admits *CMSD* into Fans with a handle and one P_2. The decomposition of $G_{0,13}$ into Fans with a handle and one P_2 is given in Figure 26 and its subgraph decomposition is shown separately in Figures 26.1 to 26.4.

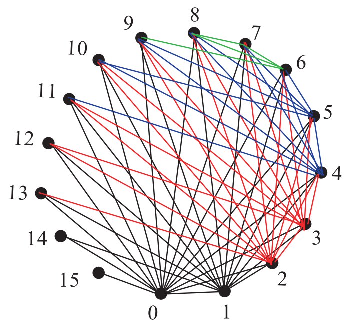

Fig. 25.

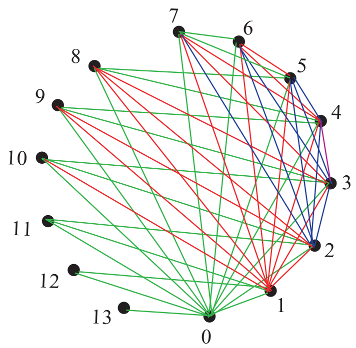

Fig. 26.

Case (ii): $n = 4m+2$, $m \in \mathbb{N}$.
In this case, $G_{0,n} = G_{0,4m+2} = F^*_{4m+1} \cup F^*_{4(m-1)+1} \cup F^*_{4(m-2)+1} \cup \ldots \cup F^*_5 \cup P_3(m,m+1,m+2) = P_3(m,m+1,m+2) \cup (\bigcup_{j=0}^{m-1} F^*_{4(m-j)+1})$ where $F^*_{4m+1}, F^*_{4(m-1)+1}, F^*_{4(m-2)+1}, \ldots, F^*_5, P_3(m,m+1,m+2)$ are edge disjoint subgraphs of $G_{0,4m+2}$; F^*_{4m+1} is the Fan with the handle (v_0, v_{4m+2}), apex vertex v_0 and $P_{4m+1} = v_{4m+1}v_1v_{4m}v_2 \ldots v_{2m-1}v_{2m+2}v_{2m}v_{2m+1}$; $P_3(v_m, v_{m+1}, v_{m+2})$ is the path

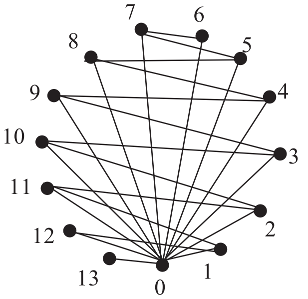

Fig. 26.1.

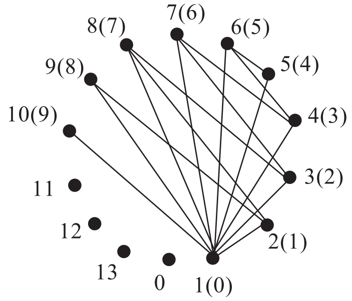

Fig. 26.2.

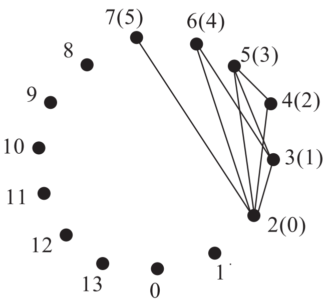

Fig. 26.3.

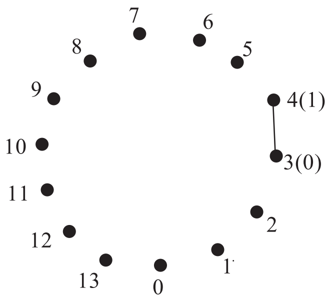

Fig. 26.4.

$v_m v_{m+1} v_{m+2}$ in $G_{0,4m+2}$ and v_j is the vertex with integral sum label j in $G_{0,4m+2}$, $j \in [0, 4m+2]$. Also $|E(G_{0,4m+2})| = 4m+2+|E(G_{4m+2})| = 4m+2+\frac{(4m+2)(4m+1)}{4} - \frac{(2m+1)}{2} = 4m^2+6m+2 = 2(2m+1)(m+1)$, $|E(P_3)| = 2 < |E(F_5^*)| = 10 < |E(F_9^*)| = 18 < \cdots < |E(F_{n-5}^*)| = 2(n-5) < |E(F_{n-1}^*)| = 2(n-1) = 2(4m+1) = 8m+2$ and $2+10+18+\cdots+(2+8m) = 2(2m+1)(m+1)$. Thus $G_{0,4m+2}$ admits $(2,8)$-*CMSD* into Fans with a handle. Here P_3 is the trivial fan with a handle. $(2,8)$-*CMSD* of $G_{0,10}$ into Fans with a handle is shown in Figure 27 and its subgraph decomposition is shown separately in Figures 27.1 to 27.3.

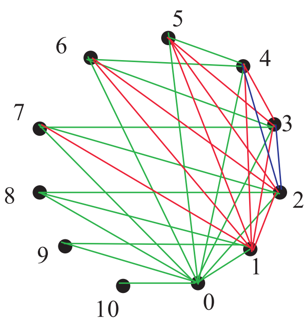

Fig. 27.

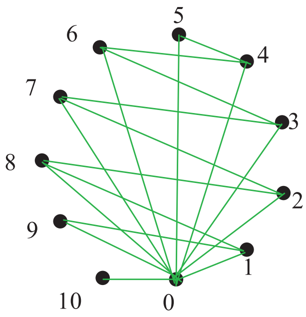

Fig. 27.1.

Case (iii) : $n = 4m-1$, $m \in \mathbb{N}$.

In this case, $G_{0,n} = G_{0,4m-1} = F_{4m-2}^* \cup F_{4m-6}^* \cup F_{4m-10}^* \cup \ldots \cup F_6^* \cup F_2^* = \bigcup_{j=0}^{m-1} F_{4(m-j)-2}^*$,

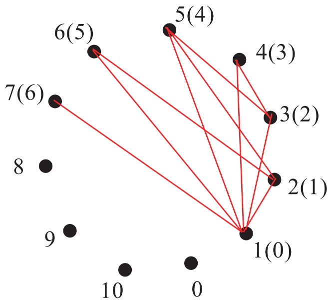

Fig. 27.2.

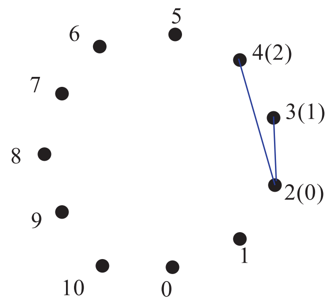

Fig. 27.3.

$|E(G_{0,4m-1})| = 4m-1+|E(G_{4m-1})| = 4m-1+\frac{(4m-1)(4m-2)}{4}-\frac{(2m-1)}{2} = 4m^2, |E(F_2^*)| = 4 < |E(F_6^*)| = 12 < |E(F_{10}^*)| = 20 < \cdots < |E(F_{4m-6}^*)| = 2(4m-6) < |E(F_{4m-2}^*)| = 2(4m-2)$ where $F_{4m-2}^*, F_{4(m-1)-2}^*, F_{4(m-2)-2}^*, \ldots, F_6^*, F_2^*$ are edge disjoint subgraphs of $G_{0,4m-1}$; F_{4m-2}^* is the Fan with the handle (v_0, v_{4m-1}) and v_j is the vertex with integral sum label j in $G_{0,4m-1}$, $j \in [0, 4m-1]$. This implies $G_{0,4m-1}$ admits $(4,8)$-*CMSD* into Fans with a handle. $(4,8)$-*CMSD* of $G_{0,11}$ into Fans with a handle is shown in Figure 28 and its subgraph decomposition is shown separately in Figures 28.1 to 28.3.

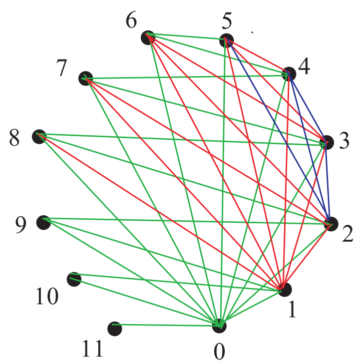

Fig. 28.

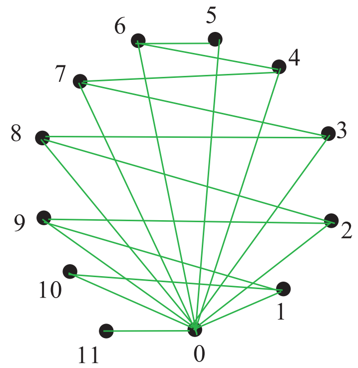

Fig. 28.1.

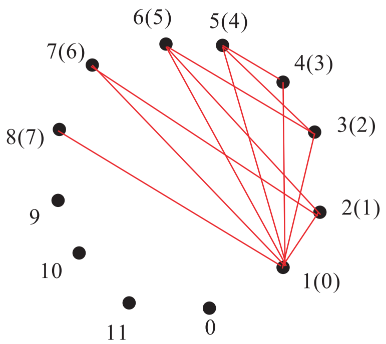

Fig. 28.2.

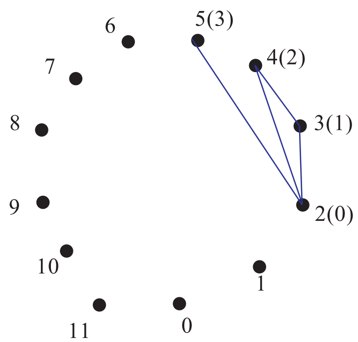

Fig. 28.3.

Case (iv): $n = 4m$, $m \in \mathbb{N}$.
In this case, $G_{0,n} = G_{0,4m} = F^*_{4m-1} \cup F^*_{4m-5} \cup F^*_{4m-9} \cup \ldots \cup F^*_7 \cup F^*_3 = \bigcup_{j=0}^{m-1} F^*_{4(m-j)-1}$, $|E(G_{0,4m})| = 4m + |E(G_{4m})| = 4m + \frac{4m(4m-1)}{4} - \frac{2m}{2} = 2m(2m+1)$, $|E(F^*_3)| = 6 < |E(F^*_7)| = 14 < |E(F^*_{11})| = 22 < \cdots < |E(F^*_{4m-5})| = 2(4m-5) < |E(F^*_{4m-1}) = 2(4m-1)$ and $6+14+\cdots+(6+8(m-1)) = 2m(2m+1)$ where $F^*_{4m-1}, F^*_{4(m-1)-1}, F^*_{4(m-2)-1}, \ldots, F^*_7, F^*_3$ are edge disjoint subgraphs of $G_{0,4m}$, F^*_{4m-1} is the Fan with the handle (v_0, v_{4m}) and v_j is the vertex with integral sum label j in $G_{0,4m}$, $j \in [0, 4m]$. Thus $G_{0,4m}$ admits $(6,8)$-*CMSD* into Fans with a handle. $(6,8)$-*CMSD* of $G_{0,12}$ into Fans with a handle is shown in Figure 29 and its subgraph decomposition is shown separately in Figures 29.1 to 29.3.

Thus in all the above four cases we could prove the result. □

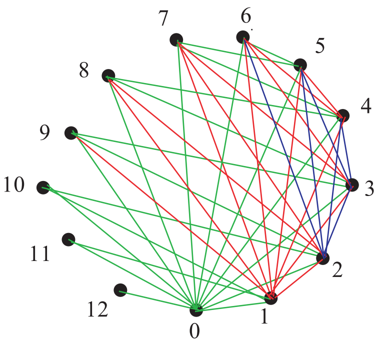

Fig. 29.

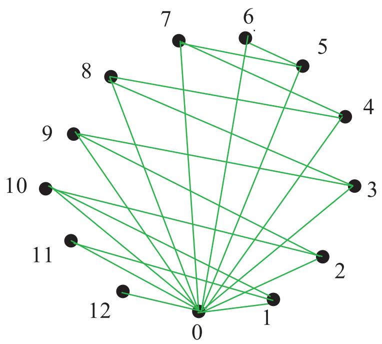

Fig. 29.1.

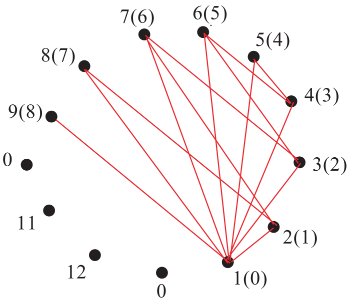

Fig. 29.2.

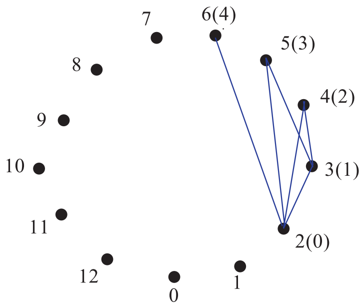

Fig. 29.3.

Theorem 2.11. *The necessary condition for the existence of (a,d)-CMSD of K_n into families of Fans with a handle is $n \equiv 0, 1 \; mod(4)$.*

Proof. Let K_n admit (a,d)-*CMSD* into families of Fans with a handle for some $a, d \in \mathbb{N}$. And let $K_n = F^*_{n_1} \cup F^*_{n_2} \cup \ldots \cup F^*_{n_k}$ where $F^*_{n_1}, F^*_{n_2}, \ldots, F^*_{n_k}$ are edge disjoint Fans with a handle for some $n_1, n_2, \ldots, n_k \in \mathbb{N}$ and $1 \le n_1 < n_2 < \cdots < n_k$. Then $|E(K_n)| = |E(F^*_{n_1})| + |E(F^*_{n_2})| + \cdots + |E(F^*_{n_k})|$

which implies, $nC_2 = 2n_1 + 2n_2 + \cdots + 2n_k$. This implies, $n(n-1) = 4(n_1 + n_2 + \cdots + n_k)$ which implies $n \equiv 0, 1 \ mod(4)$. Hence the result. $\square$

Conjecture For $n \in \mathbb{N}$, K_{4n} admits $(2n,2)$-*CMSD* of Fans with a handle and K_{4n+1} admits $(2(n+1),2)$-*CMSD* of Fans with a handle.

The above conjecture is verified true for $n = 1$ and 2. Figures 30, 31, 32, 33 show $(2,2)$-*CMSD*, $(4,2)$-*CMSD*, $(4,2)$-*CMSD*, $(6,2)$-*CMSD* of K_n into Fans with a handle for $n = 4,5,8,9$, respectively.

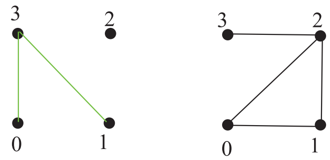

Fig. 30. $K_4 = F_1^* \cup F_2^*$.

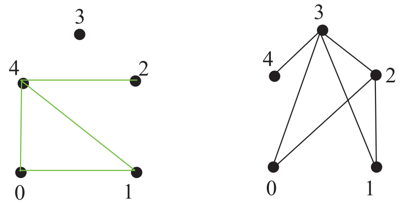

Fig. 31. $K_5 = F_2^* \cup F_3^*$.

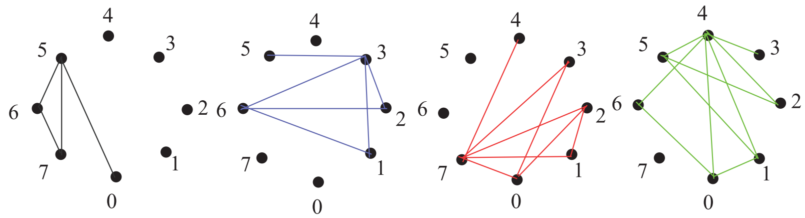

Fig. 32. $K_8 = F_2^* \cup F_3^* \cup F_4^* \cup F_5^*$.

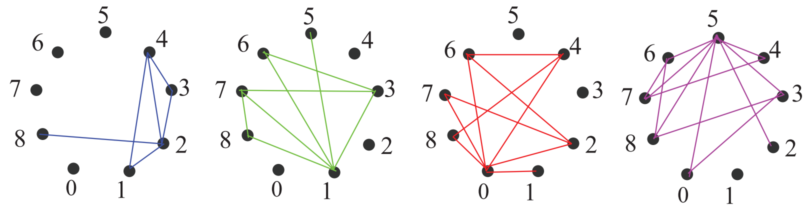

Fig. 33. $K_9 = F_3^* \cup F_4^* \cup F_5^* \cup F_6^*$.

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