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SOME LANDAU TYPE INEQUALITIES FOR FUNCTIONS WHOSE DERIVATIVES ARE OF LOCALLY BOUNDED VARIATION

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Abstract. Some inequalities of the Landau type for functions whose derivatives are of locally bounded variation are pointed out.

1. Introduction

The following version of Ostrowski's inequality for functions of bounded variation was obtained by the second author in [2] (see also [3]):

Theorem 1. *Let $\varphi : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$. Then for any $x \in [a,b]$ one has the inequality:*

$$\left|\varphi(x) - \frac{1}{b-a}\int_a^b \varphi(t)\,dt\right| \le \left[\frac{1}{2} + \frac{\left|x - \frac{a+b}{2}\right|}{b-a}\right] \bigvee_a^b (\varphi), \tag{1.1}$$

where $\bigvee_a^b(\varphi)$ denotes the total variation of φ on $[a,b]$. The constant $\frac{1}{2}$ is the best possible.

We now recall some classical results due to Landau [8].

Let $I = \mathbb{R}_+$ or $I = \mathbb{R}$. If $f : I \to \mathbb{R}$ is twice differentiable and $f, f'' \in L_p(I)$, $p \in [1,\infty]$, then $f' \in L_p(I)$. Moreover, there exists a constant $C_p(I) > 0$ independent of the function f, such that

$$\|f'\|_{I,p} \le C_p(I)\, \|f\|_{I,p}^{\frac{1}{2}} \|f''\|_{I,p}^{\frac{1}{2}}, \tag{1.2}$$

where $\|\cdot\|_{I,p}$ is the $p-$norm on the interval I, i.e., we recall

$$\|h\|_{I,\infty} := ess\sup_{t\in I} |h(t)|,$$

and

$$\|h\|_{I,p} := \left(\int_I |h(t)|^p\,dt\right)^{\frac{1}{p}} \quad \text{if} \quad p \in [1,\infty).$$

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Landau considered the case $p = \infty$ and proved that

$$C_\infty(\mathbb{R}_+) = 2 \text{ and } C_\infty(\mathbb{R}) = \sqrt{2} \tag{1.3}$$

are the best constants for which (1.2) holds.

In 1932, G.H. Hardy and J. E. Littlewood [5] proved (1.2) for $p = 2$, with the best constants

$$C_2(\mathbb{R}_+) = \sqrt{2} \text{ and } C_2(\mathbb{R}) = 1. \tag{1.4}$$

In 1935, G. H. Hardy, E. Landau and J. E. Littlewood [6] showed that the best constant $C_p(\mathbb{R}_+)$ in (1.2) satisfies the estimate

$$C_p(\mathbb{R}_+) \le 2 \text{ for } p \in [1,\infty), \tag{1.5}$$

which yields $C_p(\mathbb{R}) \le 2$ for $p \in [1,\infty)$. Actually, as shown in [7] and [1], $C_p(\mathbb{R}) \le \sqrt{2}$.

In this paper, by the use of the inequality (1.1), we point out some Landau type results for arbitrary subintervals I of $\mathbb{R}$ and under more relaxed assumptions on the derivative f'.

2. A Technical Lemma

The following technical lemma, that is important in the sequel, holds [4]. For the sake of completeness, a short proof is outlined below.

Lemma 1. *Let $C, D > 0$ and $r, u \in (0,1]$. Consider the function $g_{r,u} : (0,\infty] \to \mathbb{R}$ given by*

$$g_{r,u}(\lambda) = \frac{C}{\lambda^u} + D\lambda^r. \tag{2.1}$$

Define

$$\lambda_0 := \left(\frac{uC}{rD}\right)^{\frac{1}{r+u}} \in (0,\infty),$$

then for $\lambda_1 \in (0,\infty)$ we have,

$$\inf_{\lambda \in (0,\lambda_1]} g_{r,u}(\lambda) = \begin{cases} \dfrac{r+u}{u^{\frac{u}{r+u}} \cdot r^{\frac{u}{r+u}}} C^{\frac{r}{r+u}} D^{\frac{r}{r+u}} & \text{if } \lambda_1 \ge \lambda_0, \\ \frac{C}{\lambda_1^u} + D\lambda_1^r & \text{if } 0 < \lambda_1 < \lambda_0. \end{cases} \tag{2.2}$$

Proof. We observe that

$$g'_{r,u}(\lambda) = \frac{rD\lambda^{r+u} - Cu}{\lambda^{u+1}}, \quad \lambda \in (0,\infty).$$

The unique solution of the equation $g'_{r,u}(\lambda) = 0$, $\lambda \in (0,\infty)$ is

$$\lambda_0 = \left(\frac{uC}{rD}\right)^{\frac{1}{r+u}} \in (0,\infty).$$

The function $g_{r,u}$ is decreasing on $(0,\lambda_0)$ and increasing on (λ_0,∞). The global minimum for $g_{r,u}$ on $(0,\infty)$ is

$$\begin{aligned} g_{r,u}(\lambda_0) &= \frac{C}{\left(\frac{uC}{rD}\right)^{\frac{u}{r+u}}} + D\cdot\left(\frac{uC}{rD}\right)^{\frac{r}{r+u}} \\ &= \frac{r+u}{u^{\frac{u}{r+u}} r^{\frac{r}{r+u}}}\cdot C^{\frac{r}{r+u}} D^{\frac{u}{r+u}} \end{aligned}$$

which proves (2.2).

The following particular cases are useful.

Corollary 1. *Let $C,D>0$ and $r\in(0,1]$. Consider the function $g_r:(0,\infty)\to\mathbb{R}$, given by*

$$g_r(\lambda)=\frac{C}{\lambda}+D\lambda^r.$$

Define

$$\overline{\lambda_0}=\left(\frac{C}{rD}\right)^{\frac{1}{r+1}}\in(0,\infty),$$

then for $\lambda_1\in(0,\infty)$,

$$\inf_{\lambda\in(0,\lambda_1]} g_r(\lambda)=\begin{cases} \frac{r+1}{r^{\frac{r}{r+u}}} C^{\frac{r}{r+1}} D^{\frac{1}{r+1}} & \text{if } \ \lambda_1\geq\overline{\lambda_0}, \\ \frac{C}{\lambda_1}+D\lambda_1^r & \text{if } \ 0<\lambda_1<\overline{\lambda_0}. \end{cases} \tag{2.3}$$

Corollary 2. *Let $C,D>0$ and $u\in(0,1]$. Consider the function $g_u:(0,\infty)\to\mathbb{R}$ given by*

$$g_u(\lambda)=\frac{C}{\lambda^u}+D\lambda.$$

Define

$$\widetilde{\lambda_0}=\left(\frac{uC}{D}\right)^{\frac{1}{1+u}}\in(0,\infty),$$

then for $\lambda_1\in(0,\infty)$,

$$\inf_{\lambda\in(0,\lambda_1]} g_u(\lambda)=\begin{cases} \frac{1+u}{u^{\frac{u}{1+u}}} C^{\frac{1}{u+1}} D^{\frac{u}{u+1}} & \text{if } \ \lambda_1\geq\widetilde{\lambda_0}, \\ \frac{C}{\lambda_1^u}+D\lambda_1 & \text{if } \ 0<\lambda_1<\widetilde{\lambda_0}. \end{cases} \tag{2.4}$$

Remark 1. If $r=u=1$, then the following result holds:

$$\inf_{\lambda\in(0,\lambda_1]}\left(\frac{C}{\lambda}+D\lambda\right)=\begin{cases} 2\sqrt{CD} & \text{if } \ \lambda_1\geq\sqrt{\frac{C}{D}}, \\ \frac{C}{\lambda_1}+D\lambda_1 & \text{if } \ 0<\lambda_1<\sqrt{\frac{C}{D}}. \end{cases}$$

3. The Case When $f \in L_\infty(I)$

The following theorem holds.

Theorem 2. *Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ a locally absolutely continuous function on I. If $f \in L_\infty(I)$, the derivative $f' : I \to \mathbb{R}$ is of locally bounded variation and there exists a constant $V_I > 0$ and $r \in (0,1]$ such that*

$$\left|\bigvee_a^b (g')\right| \leq V_I |a-b|^r \quad \text{for any } a, b \in I; \tag{3.1}$$

then $f' \in L_\infty(I)$ and

$$\|f'\|_{I,\infty} \leq \begin{cases} \dfrac{2^{\frac{r}{r+1}}(r+1)}{r^{\frac{r}{r+1}}} \|f\|_{I,\infty}^{\frac{r}{r+1}} V_I^{\frac{1}{r+1}} & \text{if } m(I) \geq \dfrac{2^{\frac{r+2}{r+1}} \|f\|_{I,\infty}^{\frac{r}{r+1}}}{r^{\frac{1}{r+1}} V_I^{\frac{1}{r+1}}}, \\ \\ \dfrac{4\|f\|_{I,\infty}}{m(I)} + \dfrac{V_I (m(I))^r}{2^r} & \text{if } 0 < m(I) < \dfrac{2^{\frac{r+2}{r+1}} \|f\|_{I,\infty}^{\frac{r}{r+1}}}{r^{\frac{1}{r+1}} V_I^{\frac{1}{r+1}}}. \end{cases} \tag{3.2}$$

Proof. Applying (1.1) for $\varphi = f'$, we deduce

$$|f'(x)| \leq \left|\frac{f(b)-f(a)}{b-a}\right| + \left[\frac{1}{2} + \frac{\left|x - \frac{a+b}{2}\right|}{b-a}\right] \left|\bigvee_a^b (f')\right|$$

for any $a, b \in I$, $a \neq b$ and x between them, giving, for $x = a$,

$$|f'(a)| \leq \frac{|f(b)-f(a)|}{|b-a|} + \left|\bigvee_a^b (f')\right| \tag{3.3}$$

for any $a, b \in I$, $a \neq b$.

Using the hypothesis (3.1) and the fact that $f \in L_\infty(I)$, we conclude that

$$\begin{aligned} |f'(a)| &\leq \frac{|f(b)-f(a)|}{|b-a|} + V_I |b-a|^r \\ &= \frac{2\|f\|_{I,\infty}}{|b-a|} + V_I |b-a|^r \end{aligned} \tag{3.4}$$

for almost every $a, b \in I$, $a \neq b$.

Now, observe that for any $a \in I$ and any $s \in \left(0, \frac{m(I)}{2}\right)$, there exists $b \in I$ such that $s = |b-a|$ and then, by (3.4),

$$|f'(a)| \leq \frac{2\|f\|_{I,\infty}}{s} + V_I s^r \tag{3.5}$$

for almost any $a \in I$ and every $s \in \left(0, \frac{m(I)}{2}\right)$. By taking the infimum over s on $\left(0, \frac{m(I)}{2}\right)$, we have,

$$|f'(a)| \leq \inf_{s \in \left(0, \frac{m(I)}{2}\right)} \left[\frac{2\|f\|_{I,\infty}}{s} + V_I s^r \right] = K \tag{3.6}$$

for almost any $a \in I$.

If we take the essential supremum over $a \in I$ in (3.6), we conclude that

$$\|f'\|_{I,\infty} \leq K. \tag{3.7}$$

Making use of Corollary 1, we get

$$K = \begin{cases} \dfrac{r+1}{r^{\frac{r}{r+1}}} \left(2\|f\|_{I,\infty}\right)^{\frac{r}{r+1}} \cdot V_I^{\frac{1}{r+1}} & \text{if } \dfrac{m(I)}{2} \geq \left(\dfrac{2\|f\|_{I,\infty}}{rV_I}\right)^{\frac{1}{r+1}}, \\[2ex] \dfrac{2\|f\|_{I,\infty}}{\frac{m(I)}{2}} + V_I \left(\dfrac{m(I)}{2}\right)^r & \text{if } \dfrac{m(I)}{2} < \left(\dfrac{2\|f\|_{I,\infty}}{rV_I}\right)^{\frac{1}{r+1}} \end{cases}$$

$$= \begin{cases} \dfrac{2^{\frac{r}{r+1}}(r+1)}{r^{\frac{r}{r+1}}} \|f\|_{I,\infty}^{\frac{r}{r+1}} V_I^{\frac{1}{r+1}} & \text{if } m(I) \geq \dfrac{2^{\frac{r+2}{r+1}} \|f\|_{I,\infty}^{\frac{r}{r+1}}}{r^{\frac{1}{r+1}} V_I^{\frac{1}{r+1}}}, \\[2ex] \dfrac{4\|f\|_{I,\infty}}{m(I)} + \dfrac{V_I (m(I))^r}{2^r} & \text{if } 0 < m(I) < \dfrac{2^{\frac{r+2}{r+1}} \|f\|_{I,\infty}^{\frac{r}{r+1}}}{r^{\frac{1}{r+1}} V_I^{\frac{1}{r+1}}} \end{cases}$$

and the inequality (3.2) is obtained.

4. The Case when f is Hölder Continuous

The following theorem holds.

Theorem 3. *Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ a locally absolutely continuous function on I. If f satisfies the Hölder condition*

$$|f(b) - f(a)| \leq K |b-a|^{\ell} \quad \textit{for any} \quad a, b \in I, \tag{4.1}$$

where $K > 0$ and $\ell \in (0,1)$ are given, and the derivative $f' : I \to \mathbb{R}$ is of locally bounded variation and the condition (3.1) holds, then f' is bounded in I and

$$\|f'\|_{I,\infty} \leq \begin{cases} \dfrac{r+1-\ell}{(1-\ell)^{\frac{1-\ell}{r+1-\ell}} r^{\frac{r}{r+1-\ell}}} K^{\frac{r}{r+1-\ell}} V_I^{\frac{1-\ell}{r+1-\ell}} & \textit{if } m(I) \geq 2\left[\frac{(1-\ell)K}{rV_I}\right]^{\frac{1}{r+1-\ell}}, \\[2ex] \dfrac{2^{1-\ell}K}{[m(I)]^{1-\ell}} + \dfrac{V_I [m(I)]^r}{2^r} & \textit{if } 0 < m(I) < 2\left[\frac{(1-\ell)K}{rV_I}\right]^{\frac{1}{r+1-\ell}}. \end{cases} \tag{4.2}$$

Proof. We know, from the proof of Theorem 2, that

$$|f'(a)| \le \frac{|f(b)-f(a)|}{|b-a|} + V_I |b-a|^r, \quad \text{for all } a,b \in I,\ a \ne b. \tag{4.3}$$

Using the hypothesis (4.1), we conclude that

$$|f'(a)| \le \frac{K}{|b-a|^{1-\ell}} + V_I |b-a|^r \tag{4.4}$$

for any $a,b \in I$, $a \ne b$.

By a similar argument to the one used in proving Theorem 2, we conclude that

$$|f'(a)| \le \inf_{s\in\left(0,\frac{m(I)}{2}\right)} \left[\frac{K}{s^{1-\ell}} + V_I s^r\right] = M \tag{4.5}$$

for any $a \in I$.

If we now apply Lemma 1 for $C = K$, $u = 1-\ell$, $D = V_I$, we observe that

$$\inf_{s\in\left(0,\frac{m(I)}{2}\right)} \left[\frac{K}{s^{1-\ell}} + V_I s^r\right] = \begin{cases} \dfrac{r+1-\ell}{(1-\ell)^{\frac{1-\ell}{r+1-\ell}} r^{\frac{r}{r+1-\ell}}} K^{\frac{r}{r+1-\ell}} V_I^{\frac{1-\ell}{r+1-\ell}} & \text{if } \dfrac{m(I)}{2} \ge \left(\frac{(1-\ell)K}{rV_I}\right)^{\frac{1}{r+1-\ell}}, \\ \dfrac{K}{\left(\frac{m(I)}{2}\right)^{1-\ell}} + V_I \left(\dfrac{m(I)}{2}\right)^r & \text{if } \dfrac{m(I)}{2} < \left(\frac{(1-\ell)K}{rV_I}\right)^{\frac{1}{r+1-\ell}}. \end{cases}$$

and the inequality (4.2) is obtained.

The following corollary holds.

Corollary 3. *Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a locally absolutely continuous function on I. If $f' \in L_p(I)$, $p > 1$ and if f' is of locally bounded variation and the condition* (3.1) *holds, then $f' \in L_\infty(I)$ and*

$$\|f'\|_{I,\infty} \le \begin{cases} \dfrac{pr+1}{p^{\frac{pr}{pr+1}} r^{\frac{pr}{pr+1}}} \|f\|_{I,p}^{\frac{pr}{pr+1}} V_I^{\frac{1}{pr+1}} & \text{if } m(I) \ge 2\left(\dfrac{\|f\|_{I,p}}{prV_I}\right)^{\frac{p}{pr+1}}, \\ \dfrac{2^{\frac{1}{p}} \|f\|_{I,p}}{[m(I)]^{1-\ell}} + \dfrac{V_I [m(I)]^r}{2^r} & \text{if } 0 < m(I) < 2\left(\dfrac{\|f\|_{I,p}}{prV_I}\right)^{\frac{p}{pr+1}}. \end{cases} \tag{4.6}$$

Proof. If $f' \in L_p(I)$, then we have

$$\begin{aligned} |f(b) - f(a)| &= \left| \int_a^b f'(s)\,ds \right| \le \left| \int_a^b |f'(s)|\,ds \right| \\ &\le |b-a|^{\frac{1}{q}} \left| \int_a^b |f'(s)|^p\,ds \right|^{\frac{1}{p}} \\ &\le |b-a|^{1-\frac{1}{p}} \|f'\|_{I,p}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1, \end{aligned}$$

for a.e. $a, b \in I$.

Using Theorem 3 for $\ell = 1 - \frac{1}{p}$ and $K = \|f'\|_{I,p}$, we deduce the desired result (4.6).

The following result may be proved as well.

Corollary 4. *With the assumptions in Corollary* 3*, and if* $f' \in L_1(I)$*, then* $f' \in L_\infty(I)$ *and*

$$\|f'\|_{I,\infty} \le \begin{cases} \dfrac{r+1}{r^{\frac{r}{r+1}}} \|f'\|_{I,1}^{\frac{r}{r+1}} V_I^{\frac{1}{r+1}} & \text{if } m(I) \ge 2\left(\dfrac{\|f'\|_{I,1}}{rV_I}\right)^{\frac{1}{r+1}}, \\ \dfrac{2\|f'\|_{I,1}}{m(I)} + \dfrac{V_I[m(I)]^r}{2^r} & \text{if } 0 < m(I) < 2\left(\dfrac{\|f'\|_{I,1}}{rV_I}\right)^{\frac{1}{r+1}}. \end{cases} \tag{4.7}$$

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