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THE VARIANT OF THE GOURSAT PROBLEM FOR THE SIXTH ORDER EQUATION

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Abstract. Our study is considered as a continuation of results in [1-8]. More precisely we investigate the Goursat problem in class $C^{2+2+2}(D)$ for the equation (1) below by Riemann method variant from the works [8] and [9].

1. Introduction

In the domain $D = \{(x, y, z);\ x_0 < x < x_1,\ y_0 < y < y_1,\ z_0 < z < z_1\}$ we consider the equation with variable coefficients:

$$L(u) = \sum_{\alpha,\beta,\gamma=0}^{2} a_{\alpha\beta\gamma}(x, y, z) \frac{\partial^{\alpha+\beta+\gamma} u}{\partial x^\alpha \partial y^\beta \partial z^\gamma} = F(x, y, z). \tag{1}$$

It may be considered as a special generalization of the investigation in [1-8]. Many authors studied this problem in the different directions (e.g.,[1-8]). The most general results were obtained in [7]. On the other hand, our aim in this paper is to consider the equation (1) which is a complication to the works [8-11] with higher derivative u_{xyz}. This equation plays an important role in approximation and transformation theory and during the description of vibration processes (see, [12]). The problem of integral transformation representation of type of ordinary linear differential operator into others is reduced to the similar equation in [13].

2. Main Results

Let X, Y, Z be bounds D for $x = x_0$, $y = y_0$, $z = z_0$, respectively. We consider that $a_{222} \equiv 1$, and the smoothness of other coefficients (1) is defined by the inclusions:

$$a_{\alpha\beta\gamma} \in C^{\alpha+\beta+\gamma}(\overline{D}), \quad F \in C^{2+2+2}(\overline{D}). \tag{2}$$

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Here, $C^{\alpha+\beta+\gamma}$ describes the class of continuous in $\overline{D}$ together with derivatives $\partial^{r+s+t}/\partial x^r \partial y^s \partial z^t$ $(r=0,\dots,\alpha;\ s=0,\dots,\beta;\ t=0,\dots,\gamma)$.

Problem (Goursat). To find, in class $C^{2+2+2}(\overline{D})$, the solution of equation (1) satisfying the following boundary conditions:

$$u|_X = \varphi(y,z), \qquad u|_Y = \psi(x,z), \qquad u|_Z = \theta(x,y);$$
$$u_x|_X = \varphi_1(y,z), \quad u_y|_Y = \psi_1(x,z), \quad u_z|_Z = \theta_1(x,y). \tag{3}$$

where
$$\varphi,\varphi_1 \in C^{2+2}(\overline{X}) \quad \psi,\psi_1 \in C^{2+2}(\overline{Y}) \quad \theta,\theta_1 \in C^{2+2}(\overline{Z}).$$

We suppose the satisfaction of the condition in ribs D

$$\varphi(y_0,z) = \psi(x_0,z), \quad \varphi(y,z_0) = \theta(x_0,y), \quad \psi(x,z_0) = \theta(x,y_0). \tag{4}$$

Here the agreed values are continuously differentiated. We solve this problem by means of the development of (using) the Riemann method variant from the works [8] and [9].

Riemann function $R(x,y,z,\zeta,\eta,\varsigma)$ is called the solution of following integral equation:

$$\begin{aligned}
V(x,y,z) &- \int_{\varsigma}^{z} [a_{221}(x,y,\gamma)V(x,y,\gamma) - (z-\gamma)a_{220}(x,y,\gamma)V(x,y,\gamma)]d\gamma \\
&- \int_{\eta}^{y} [a_{212}(x,\beta,z)V(x,\beta,z) - (y-\beta)a_{202}(x,\beta,z)V(x,\beta,z)]d\beta \\
&- \int_{\zeta}^{x} [a_{122}(\alpha,y,z)V(\alpha,y,z) - (x-\alpha)a_{022}(\alpha,y,z)V(\alpha,y,z)]d\alpha \\
&+ \int_{\zeta}^{x}\int_{\eta}^{y} [a_{112}(\alpha,\beta,z)V(\alpha,\beta,z) - (y-\beta)a_{102}(\alpha,\beta,z)V(\alpha,\beta,z) \\
&-(x-\alpha)a_{012}(\alpha,\beta,z)V(\alpha,\beta,z) + (x-\alpha)(y-\beta)a_{002}(\alpha,\beta,z)V(\alpha,\beta,z)]d\beta d\alpha \\
&+ \int_{\zeta}^{x}\int_{\varsigma}^{z} [a_{121}(\alpha,y,\gamma)V(\alpha,y,\gamma) - (z-\gamma)a_{120}(\alpha,y,\gamma)V(\alpha,y,\gamma) \\
&-(x-\alpha)a_{021}(\alpha,y,\gamma)V(\alpha,y,\gamma) + (x-\alpha)(z-\gamma)a_{020}(\alpha,,y,\gamma)V(\alpha,y,\gamma)]d\gamma d\alpha \\
&+ \int_{\eta}^{y}\int_{\varsigma}^{z} [a_{211}(x,\beta,\gamma)V(x,\beta,\gamma) - (y-\beta)a_{201}(x,\beta,\gamma)V(x,\beta,\gamma) \\
&-(z-\gamma)a_{210}(x,\beta,\gamma)V(x,\beta,\gamma) + (y-\beta)(z-\gamma)a_{200}(x,\beta,\gamma)V(x,\beta,\gamma)]d\gamma d\beta \\
&- \int_{\zeta}^{x}\int_{\eta}^{y}\int_{\varsigma}^{z} \Big[a_{111}(\alpha,\beta,\gamma)V(\alpha,\beta,\gamma) - (y-\beta)a_{101}(\alpha,\beta,\gamma)V(\alpha,\beta,\gamma) \\
&-(x-\alpha)a_{011}(\alpha,\beta,\gamma)V(\alpha,\beta,\gamma) - (z-\gamma)a_{110}(\alpha,\beta,\gamma)V(\alpha,\beta,\gamma) \\
&+(y-\beta)(z-\gamma)a_{100}(\alpha,\beta,\gamma)V(\alpha,\beta,\gamma) + (x-\alpha)(z-\gamma)a_{010}(\alpha,\beta,\gamma)V(\alpha,\beta,\gamma) \\
&+(x-\alpha)(y-\beta)a_{001}(\alpha,\beta,\gamma)V(\alpha,\beta,\gamma) \\
&-(x-\alpha)(y-\beta)(z-\gamma)a_{000}(\alpha,\beta,\gamma)V(\alpha,\beta,\gamma)\Big] d\alpha d\beta d\gamma = 1.
\end{aligned} \tag{5}$$

which exists and unique (see [14], pp.180). We may verify that the realization of the identity for any function u from the class $C^{2+2+2}(\overline{D})$ directly by the inspection:

$$\begin{aligned}(uR)_{xxyyzz} = {} & RL(u) + (uA_1)_{xxyyz} + (uA_2)_{xxyzz} + (uA_3)_{xyyzz} - (uB_1)_{xyzz} \\ & -(uB_2)_{xxyz} - (uB_3)_{xyyz} - (uB_4)_{xxyy} - (uB_5)_{xxzz} - (uB_6)_{yyzz} + (uC_1)_{xyz} \\ & +(uC_2)_{xzz} + (uC_3)_{xyy} + (uC_4)_{xxy} + (uC_5)_{xxz} + (uC_6)_{yyz} + (uC_7)_{yzz} \\ & -(uD_1)_{xy} - (uD_2)_{xz} - (uD_3)_{yz} - (uD_4)_{zz} - (uD_5)_{yy} - (uD_6)_{xx} + (uE_1)_x \\ & +(uE_2)_y + (uE_3)_z + \{u(a_{002}R)_z + u_x(a_{102}R)_z + u_y(a_{012}R)_z + u_{xy}(a_{112}R)_z \\ & +u_{xx}(a_{202}R)_z + u_{yy}(a_{022}R)_z + u_{xxy}(a_{212}R)_z + u_{xyy}(a_{122}R)_z + u_{xxyy}R_z\}_z \\ & +\{u(a_{202}R)_{xz} + u_y(a_{212}R)_{xz} + u_{yy}R_{xz}\}_{xz} + \{u(a_{022}R)_{yz} + u_x(a_{122}R)_{yz} \\ & +u_{xx}R_{yz}\}_{yz} + \{uR_{xyz}\}_{xyz} + \{u(a_{200}R)_x + u_y(a_{210}R)_x + u_{yy}(a_{220}R)_x \\ & +u_z(a_{201}R)_x + u_{zz}(a_{202}R)_x + u_{yz}(a_{211}R)_x + u_{yyz}(a_{221}R)_x + u_{yzz}(a_{212}R)_x \\ & +u_{yyzz}R_x\}_x + \{u(a_{020}R)_y + u_x(a_{120}R)_y + u_z(a_{021}R)_y + u_{xz}(a_{121}R)_y \\ & +u_{xx}(a_{220}R)_y + u_{zz}(a_{022}R)_y + u_{xzz}(a_{122}R)_y + u_{xxz}(a_{221}R)_y \\ & +u_{xxzz}R_y\}_y + \{u(a_{220}R)_{xy} + u_z(a_{221}R)_{xy} + u_{zz}R_{xy}\}_{xy}, \end{aligned} \tag{6}$$

where the notation

$$A_1 = R_z - a_{221}R, \quad A_2 = R_y - a_{212}R, \quad A_3 = R_x - a_{122}R,$$

$$B_1 = R_{xy} - (a_{212}R)_x - (a_{122}R)_y + a_{112}R, \quad B_2 = R_{yz} - (a_{221}R)_y - (a_{212}R)_z + a_{211}R,$$

$$B_3 = R_{xz} - (a_{221}R)_x - (a_{122}R)_z + a_{121}R, \quad B_4 = R_{zz} - (a_{221}R)_z + a_{220}R,$$

$$B_5 = R_{yy} - (a_{212}R)_y + a_{202}R, \quad B_6 = R_{xx} - (a_{122}R)_x + a_{022}R,$$

$$\begin{aligned} C_1 = {} & R_{xyz} - (a_{221}R)_{xy} - (a_{212}R)_{xz} - (a_{122}R)_{yz} + (a_{112}R)_z \\ & +(a_{211}R)_x + (a_{121}R)_y - a_{111}R, \\ C_2 = {} & R_{xyy} - (a_{212}R)_{xy} - (a_{122}R)_{yy} - (a_{112}R)_y + (a_{202}R)_x - a_{102}R, \\ C_3 = {} & R_{xzz} - (a_{221}R)_{xz} - (a_{122}R)_{zz} - (a_{121}R)_z + (a_{220}R)_x - a_{120}R, \\ C_4 = {} & R_{yzz} - (a_{221}R)_{yz} - (a_{212}R)_{zz} + (a_{211}R)_z + (a_{220}R)_y - a_{210}R, \\ C_5 = {} & R_{yyz} - (a_{221}R)_{yy} - (a_{212}R)_{yz} + (a_{211}R)_y + (a_{202}R)_z - a_{201}R, \\ C_6 = {} & R_{xxz} - (a_{221}R)_{xx} - (a_{122}R)_{xz} + (a_{121}R)_x + (a_{022}R)_y - a_{021}R, \\ C_7 = {} & R_{xxy} - (a_{212}R)_{xx} - (a_{122}R)_{xy} + (a_{112}R)_x + (a_{022}R)_y - a_{012}R, \\ D_1 = {} & R_{xyzz} - (a_{221}R)_{xyz} - (a_{212}R)_{xzz} - (a_{122}R)_{yzz} + (a_{112}R)_{zz} + (a_{211}R)_{xz} \\ & +(a_{121}R)_{yz} + (a_{220}R)_{xy} - (a_{111}R)_z - (a_{120}R)_y - (a_{210}R)_x + a_{110}R, \\ D_2 = {} & R_{xyyz} - (a_{221}R)_{xyy} - (a_{212}R)_{xyz} - (a_{122}R)_{yyz} + (a_{112}R)_{yz} + (a_{211}R)_{xy} \\ & +(a_{121}R)_{yy} + (a_{202}R)_{xz} - (a_{111}R)_y - (a_{102}R)_z - (a_{201}R)_x + a_{101}R, \end{aligned}$$

$$\begin{aligned}
D_3 &= R_{xxyz} - (a_{221}R)_{xxy} - (a_{212}R)_{xxz} - (a_{122}R)_{xyz} + (a_{112}R)_{xz} + (a_{211}R)_{xx} \\
&\quad + (a_{121}R)_{xy} + (a_{022}R)_{yz} - (a_{111}R)_x - (a_{021}R)_y - (a_{012}R)_z + a_{011}R, \\
D_4 &= R_{xxyy} - (a_{212}R)_{xxy} - (a_{122}R)_{xyy} + (a_{112}R)_{xy} + (a_{202}R)_{xx} \\
&\quad + (a_{022}R)_{yy} - (a_{102}R)_x - (a_{012}R)_y + a_{002}R, \\
D_5 &= R_{xxzz} - (a_{221}R)_{xxz} - (a_{122}R)_{xzz} + (a_{121}R)_{xz} + (a_{220}R)_{xx} \\
&\quad + (a_{022}R)_{zz} - (a_{120}R)_x - (a_{012}R)_z + a_{020}R, \\
D_6 &= R_{yyzz} - (a_{221}R)_{yyz} - (a_{212}R)_{yzz} + (a_{211}R)_{yz} + (a_{220}R)_{yy} \\
&\quad + (a_{202}R)_{zz} - (a_{210}R)_y - (a_{201}R)_z + a_{200}R, \\
E_1 &= R_{xyyzz} - (a_{221}R)_{xyyz} - (a_{212}R)_{xyzz} - (a_{122}R)_{yyzz} + (a_{112}R)_{yzz} + (a_{211}R)_{xyz} \\
&\quad + (a_{121}R)_{yyz} + (a_{220}R)_{xyy} + (a_{202}R)_{xzz} - (a_{111}R)_{yz} - (a_{102}R)_{zz} - (a_{120}R)_{yy} \\
&\quad - (a_{210}R)_{xy} - (a_{201}R)_{xz} + (a_{110}R)_y + (a_{101}R)_z + (a_{200}R)_x - a_{100}R, \\
E_2 &= R_{xxyzz} - (a_{221}R)_{xxyz} - (a_{212}R)_{xxzz} - (a_{122}R)_{xyzz} + (a_{112}R)_{xzz} + (a_{211}R)_{xxz} \\
&\quad + (a_{121}R)_{xyz} + (a_{220}R)_{xxy} + (a_{022}R)_{yzz} - (a_{111}R)_{xz} - (a_{120}R)_{xy} - (a_{210}R)_{xx} \\
&\quad - (a_{021}R)_{yz} - (a_{012}R)_{zz} + (a_{110}R)_x + (a_{011}R)_z + (a_{020}R)_y - a_{010}R, \\
E_3 &= R_{xxyyz} - (a_{221}R)_{xxyy} - (a_{212}R)_{xxyz} - (a_{122}R)_{xyyz} + (a_{112}R)_{xyz} + (a_{211}R)_{xxy} \\
&\quad + (a_{121}R)_{xyy} + (a_{202}R)_{xxz} + (a_{022}R)_{yyz} - (a_{111}R)_{xy} - (a_{102}R)_{xz} - (a_{201}R)_{xx} \\
&\quad - (a_{021}R)_{yy} - (a_{012}R)_{yz} + (a_{101}R)_x + (a_{011}R)_y + (a_{002}R)_z - a_{001}R. \qquad (7)
\end{aligned}$$

Here $a_{\alpha\beta\gamma}$ depend on (x, y, z), while R and its derivatives depend on $(x, y, z, \zeta, \eta, \varsigma)$. The derivation (6) is a heuristics problem. The easiest way to check the identity itself is the direct calculation. Here we consider that R with its first there arguments satisfies the adjoint equation (1)

$$\begin{aligned}
L^*(V) &\equiv V_{xxyyzz} - (a_{221}V)_{xxyyz} - (a_{211}V)_{xxyzz} - (a_{121}V)_{xyyzz} + (a_{112}V)_{xyzz} \\
&\quad + (a_{211}V)_{xxyz} + (a_{121}V)_{xyyz} + (a_{220}V)_{xxyy} + (a_{202}V)_{xxzz} + (a_{022}V)_{yyzz} \\
&\quad - (a_{110}V)_{xyz} - (a_{102}V)_{xzz} - (a_{120}V)_{xyy} - (a_{210}V)_{xxy} - (a_{201}V)_{xxz} \\
&\quad - (a_{021}V)_{yyz} + (a_{012}V)_{yzz} + (a_{110}V)_{xy} + (a_{101}V)_{xz} + (a_{011}V)_{yz} + (a_{002}V)_{zz} \\
&\quad + (a_{020}V)_{yy} + (a_{200}V)_{xx} - (a_{100}V)_x - (a_{010}V)_y - (a_{001}V)_z - a_{000}V = 0. \qquad (8)
\end{aligned}$$

It directly follows from (5)that

$$\begin{aligned}
&A_1(x, y, z, x, y, z) \equiv A_2(x, y, z, x, y, z) \equiv A_3(x, y, z, x, y, z) \\
&\equiv B_1(x, y, z, x, y, z) \equiv B_2(x, y, z, x, y, z) \equiv B_3(x, y, z, x, y, z) \\
&\equiv B_4(x, y, z, x, y, z) \equiv B_5(x, y, z, x, y, z) \equiv C_1(x, y, z, x, y, z) \\
&\equiv C_2(x, y, z, x, y, z) \equiv C_3(x, y, z, x, y, z) \equiv C_4(x, y, z, x, y, z) \\
&\equiv C_5(x, y, z, x, y, z) \equiv C_6(x, y, z, x, y, z) \equiv C_7(x, y, z, x, y, z) \\
&\equiv D_1(x, y, z, x, y, z) \equiv D_2(x, y, z, x, y, z) \equiv D_3(x, y, z, x, y, z) \\
&\equiv D_4(x, y, z, x, y, z) \equiv D_5(x, y, z, x, y, z) \equiv D_6(x, y, z, x, y, z) \\
&\equiv E_1(x, y, z, x, y, z) \equiv E_2(x, y, z, x, y, z) \equiv E_3(x, y, z, x, y, z) \equiv 0. \qquad (9)
\end{aligned}$$

Now, we put for (6) $x=\alpha$, $y=\beta$, $z=\gamma$, $\zeta=x$, $\eta=y$. and $\varsigma=z$, and calculate the triple integral in the bounds $x_0<\alpha<x$, $y_0<\beta<y$, $z_0<\gamma<z$ from the right and left parts. Here, taking into account the identity (9), boundary conditions (3) and the fact that $u(x,y,z)$ is the solution of the equation (1) we obtain the formula:

$$\begin{aligned}
u_{xyz} = & \int_{x_0}^{x}\int_{y_0}^{y}\int_{z_0}^{z} R(\alpha,\beta,\gamma,x,y,z)d\gamma d\beta d\alpha + \varphi_{1yz}(y,z)R(x_0,y,z,x,y,z)\\
&+\psi_{1xz}(x,z)R(x,y_0,z,x,y,z) - \varphi_{1yz}(y_0,z)R(x_0,y_0,z,x,y,z)\\
&+\theta_{1xy}(x,y)R(x,y,z_0,x,y,z) - \varphi_{1yz}(y,z_0)R(x_0,y,z_0,x,y,z)\\
&-\psi_{1xz}(x,z_0)R(x,y_0,z_0,x,y,z) + \varphi_{1yz}(y_0,z_0)R(x_0,y_0,z_0,x,y,z)\\
&-A_1(x_0,y,z)\varphi_{1y}(y,z) - A_1(x,y_0,z)\psi_{1y}(x,z) + A_1(x_0,y_0,z)\varphi_{1y}(y_0,z)\\
&-A_1(x,y,z_0)\theta_{xy}(x,y) + A_1(x_0,y,z_0)\varphi_{1y}(y,z_0) + A_1(x,y_0,z_0)\psi_{1x}(x,z_0)\\
&-A_1(x_0,y_0,z_0)\varphi_{1y}(y_0,z_0) - A_2(x_0,y,z)\varphi_{1z}(y,z) - A_2(x,y_0,z)\psi_{xy}(x,z)\\
&+A_2(x_0,y_0,z)\varphi_{1z}(y_0,z) - A_2(x,y,z_0)\theta_{1x}(x,y) + A_2(x_0,y,z_0)\varphi_{1y}(y,z_0)\\
&+A_2(x,y_0,z_0)\theta_{1x}(x,y_0) - A_2(x_0,y_0,z_0)\varphi_{1z}(y_0,z_0) - A_3(x_0,y,z)\varphi_{yz}(y,z)\\
&-A_3(x,y_0,z)\psi_{1z}(x,z) + A_3(x_0,y_0,z)\psi_{1z}(x_0,z) - A_3(x,y,z_0)\theta_{1y}(x,y)\\
&+A_3(x_0,y,z_0)\theta_{1y}(x_0,y) + A_3(x,y_0,z_0)\psi_{1z}(x,z_0) - A_3(x_0,y_0,z_0)\psi_{1z}(x_0,z_0)\\
&+B_1(x_0,y,z)\varphi_z(y,z) + B_1(x,y_0,z)\psi_z(x,z) - B_1(x_0,y_0,z)\varphi_z(y_0,z)\\
&+B_1(x,y,z_0)\theta_1(x,y) - B_1(x_0,y,z_0)\theta_1(x_0,y) - B_1(x,y_0,z_0)\theta_1(x,y_0)\\
&+B_1(x_0,y_0,z_0)\theta_1(x_0,y_0) + B_2(x_0,y,z)\varphi_1(y,z) + B_2(x,y_0,z_0)\psi_{1x}(x,z)\\
&-B_2(x_0,y_0,z)\varphi_1(y_0,z) + B_2(x,y,z_0)\theta(x,y) - B_2(x_0,y,z_0)\varphi_1(y,z_0)\\
&-B_2(x,y_0,z_0)\theta_x(x,y_0) + B_2(x_0,y_0,z_0)\varphi_1(y_0,z_0) + B_3(x_0,y,z)\varphi_y(y,z)\\
&+B_3(x,y_0,z)\psi_1(x,z) - B_3(x_0,y_0,z)\psi_1(x_0,z) + B_3(x,y,z_0)\theta_y(x,y)\\
&-B_3(x_0,y,z_0)\varphi_y(y,z_0) - B_3(x,y_0,z_0)\psi_1(x,z_0) + B_3(x_0,y_0,z_0)\psi_1(x_0,z_0)\\
&-C_1(x_0,y,z)\varphi(y,z) - C_1(x,y_0,z)\psi(x,z) + C_1(x_0,y_0,z)\varphi(y_0,z)\\
&-C_1(x,y,z_0)\theta(x,y) + C_1(x_0,y,z_0)\varphi(y,z_0) + C_1(x,y_0,z_0)\psi(x,z_0)\\
&-C_1(x_0,y_0,z_0)\varphi(y_0,z_0)\\
&+\int_{x_0}^{x}\{B_6(\alpha,y_0,z)\psi_{1z}(\alpha,z) + B_6(\alpha,y,z_0)\theta_{1y}(\alpha,y) + B_6(\alpha,y_0,z_0)\psi_{1z}(\alpha,z_0)\\
&\quad -C_6(\alpha,y_0,z)\psi_1(\alpha,z) - C_6(\alpha,y,z_0)\theta_y(\alpha,y) + C_6(\alpha,y_0,z_0)\psi_1(\alpha,z_0)\\
&\quad -C_7(\alpha,y_0,z)\psi_z(\alpha,z) - C_7(\alpha,y,z_0)\theta_1(\alpha,y) + C_7(\alpha,y_0,z_0)\theta_1(\alpha,y_0)\\
&\quad +D_3(\alpha,y_0,z)\psi(\alpha,z) + D_3(\alpha,y,z_0)\theta(\alpha,y) - D_3(\alpha,y_0,z_0)\psi(\alpha,z_0)\}d\alpha
\end{aligned}$$

$$
\begin{aligned}
&+\int_{y_0}^{y}\{B_5(x,\beta,z_0)\theta_{1x}(x,\beta)+B_5(x_0,\beta,z)\varphi_{1z}(\beta,z)-B_5(x_0,\beta,z_0)\varphi_{1z}(\beta,z_0)\\
&\quad -C_2(x_0,\beta,z)\varphi_z(\beta,z)-C_2(x,\beta,z_0)\theta_1(x,\beta)+C_2(x_0,\beta,z_0)\theta_1(x_0,\beta)\\
&\quad -C_5(x_0,\beta,z)\varphi_1(\beta,z)-C_5(x,\beta,z_0)\theta_x(x,\beta)+C_5(x_0,\beta,z_0)\varphi_1(\beta,z_0)\\
&\quad +D_2(x_0,\beta,z)\varphi(\beta,z)+D_2(x,\beta,z_0)\theta(x,\beta)-D_2(x_0,\beta,z_0)\varphi(\beta,z_0)\}d\beta\\
&+\int_{z_0}^{z}\{B_4(x_0,y,\gamma)\varphi_{1y}(y,\gamma)+B_4(x,y_0,\gamma)\psi_{1x}(x,\gamma)-B_4(x_0,y_0,\gamma)\varphi_{1y}(y_0,\gamma)\\
&\quad -C_3(x_0,y,\gamma)\varphi_y(y,\gamma)-C_3(x,y_0,\gamma)\psi_1(x,\gamma)+C_3(x_0,y_0,\gamma)\psi_1(x_0,\gamma)\\
&\quad -C_4(x_0,y,\gamma)\varphi_1(y,\gamma)-C_4(x,y_0,\gamma)\psi_x(x,\gamma)+C_4(x_0,y_0,\gamma)\varphi_1(y_0,\gamma)\\
&\quad +D_1(x_0,y,\gamma)\varphi(y,\gamma)+D_1(x,y_0,\gamma)\psi(x,\gamma)-D_1(x_0,y_0,\gamma)\varphi(y_0,\gamma)\}d\gamma\\
&+\int_{x_0}^{x}\int_{y_0}^{y}\{D_4(\alpha,\beta,z_0)\theta_1(\alpha,\beta)-E_1(\alpha,\beta,z_0)\theta(\alpha,\beta)\}d\beta d\alpha\\
&+\int_{x_0}^{x}\int_{z_0}^{z}\{D_5(\alpha,y_0,\gamma)\psi_1(\alpha,\gamma)-E_2(\alpha,,y_0,\gamma)\psi(\alpha,\gamma)\}d\gamma d\alpha\\
&+\int_{y_0}^{y}\int_{z_0}^{z}\{D_6(x_0,\beta,\gamma)\varphi_1(\beta,\gamma)-E_1(x_0,\beta,\gamma)\varphi(\beta,\gamma)\}d\gamma d\beta. \qquad (10)
\end{aligned}
$$

Hence, marking the right part (10) without the first addend as $H(x,y,z,x,y,z)$, we come to the solution of the problem under our consideration (1) and (3):

$$
\begin{aligned}
u(x,y,z)&=\varphi(y,z)+\psi(x,z)+\theta(x,y)-\varphi(y_0,z)-\psi(x,z_0)-\theta(x_0,y)+\varphi(y_0,z_0)\\
&\quad+\int_{x_0}^{x}\int_{y_0}^{y}\int_{z_0}^{z}H(\alpha,\beta,\gamma,\alpha,\beta,\gamma)d\gamma d\beta d\alpha\\
&\quad+\int_{x_0}^{x}\int_{x_0}^{\alpha}\int_{y_0}^{y}\int_{y_0}^{\beta}\int_{z_0}^{z}\int_{z_0}^{\gamma}R(t_1,t_2,t_3,\alpha,\beta,\gamma)F(t_1,t_2,t_3)dt_3d\gamma dt_2d\beta dt_1d\alpha. \quad (11)
\end{aligned}
$$

Smoothness (differentiable) conditions (2) for coefficients of the equation (1) provide belonging to the solution class $C^{2+2+2}(D)$. If we consider the functions $\varphi,\varphi_1,\psi,\psi_1,\theta,\theta_1$as derivatives we many consider (10) and (11) as a general representation of the solution for the equation (1), in the similar way as it is done in [15] for the equation with the higher derivative u_{xy}. Therefore, our results may be formulated as follows:

Theorem. *The Goursat problem for the equation* (1) *with the boundary conditions* (3) *and conditions* (4) *has the solution which is written by means of the formula* (11).

Special cases

Formulas (10) and (11) provide for the solution of the problem (1)under the conditions (3) in quadratures if the explicit form R is known. We shall mention some of these cases that were obtained by means of direct solution (5).

Let us consider seven coefficients of the equation (1) a_{221}, a_{212}, a_{122}, a_{112}, a_{211}, a_{121}, a_{111} as different from zero while all the others are identically equal to zero. Here (5) has the same form as the equation (4) from the paper [11]. We consider that the following identities are satisfied:

$$\begin{aligned} &\frac{\partial a_{220}}{\partial y} + a_{220}a_{121} - a_{120} \equiv 0, \quad \frac{\partial a_{220}}{\partial y} + a_{220}a_{211} - a_{210} \equiv 0, \\ &\frac{\partial a_{121}}{\partial y} + a_{121}a_{211} - a_{111} \equiv 0, \quad \frac{\partial a_{120}}{\partial y} + a_{120}a_{211} - a_{110} \equiv 0. \end{aligned} \tag{12}$$

Then, it follows from [11] that the Riemann function has the form:

$$R(x,y,z,\zeta,\eta,\varsigma) = \exp\Big[\int_{\varsigma}^{z} a_{220}(\zeta,\eta,\gamma)d\gamma + \int_{\zeta}^{x} a_{121}(\alpha,\beta,z)d\alpha + \int_{\eta}^{y} a_{211}(\alpha,\beta,z)d\beta\Big].$$

If the last identity (12) is non-zero but of the product $\lambda(x)\mu(y)\nu(z)$ and when

$$a_{220} = A(z) + \delta xy, \quad a_{121} = B(x) + \delta yz, \quad a_{211} = C(y) + \delta xz,$$

by using the formulas (18) and (25) from [11] we obtain the following

$$R(x,y,z,\zeta,\eta,\varsigma) = R_0(x,y,z,\zeta,\eta,\varsigma)_0F_2(1,1,\omega),$$

where ${}_0F_2$- is the generalized (distributed) hyper-geometrical function (see, [16] pp,20) while

$$\begin{aligned} &R_0 \equiv \exp[\int_{\zeta}^{x} B(\alpha)d\alpha + \int_{\eta}^{y} C(\beta)d\beta + \int_{\varsigma}^{z} A(\gamma)d\gamma + \delta(xyz - \zeta\eta\varsigma)], \\ &\omega = \int_{\zeta}^{x} \lambda(\alpha)d\alpha \int_{\beta}^{y} \mu(\beta)d\beta \int_{\varsigma}^{z} \nu(\gamma)d\gamma \end{aligned}$$

In the similar manner we can also calculate V in other cases when in (5) only one of coefficients $a_{\alpha\beta\gamma}$, having a defined structure, is different form the identical zero.

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