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SINGULAR RIGHT FOCAL BOUNDARY VALUE PROBLEM WITH GIVEN MAXIMAL VALUES

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Abstract. In this paper, we prove existence results for the singular problem $(-1)^{n-p}(\Phi_m x^{(n-1)})'(t) = \mu f(t, x(t), \ldots, x^{(n-1)}(t))$, for $0 < t < 1$, $x^{(i)}(0) = 0$, $i = 0, 1, \ldots, p-1$, $x^{(i)}(1) = 0$, $i = p, p+1, \ldots, n-1$, $\max\{x(t) : t \in [0,1]\} = A$. The paper presents conditions which guarantee that for any $A > 0$ there exists $\mu_A > 0$ such that the above problem with $\mu = \mu_A$ has a solution $x \in C^{n-1}([0,1])$ with $\Phi_m(x^{(n-1)}) \in AC([0,1])$ which is positive on $(0,1)$. Here the positive Carathédory function f may be singular at the zero value of all its phase variables. Proofs are based on the Leray-Schauder degree and Vitali's convergence theorem.

1. Introduction

The right focal boundary value problems has been widely studied by a number of authors in recent years. For details, see [1, 7, 8, 9, 10, 15] and the references therein. However the boundary value problems treated in the above mentioned references are not allowable to process singularity. For studies about higher-order singular boundary value problem, we refer to [2, 3, 4, 5, 6, 17].

Agarwal, O'Regan and Lakshmikantham studied the existence of solutions for right focal boundary value problem in [3]:

$$\begin{cases} (-1)^{n-p} y^{(n)} = \phi(t) f(t, y, \ldots, y^{(n-1)}), & n \geq 2,\ t \in (0,1), \\ y^{(i)}(0) = 0, & 0 \leq i \leq p-1, \\ y^{(i)}(1) = 0, & p \leq i \leq n-1, \end{cases} \tag{1.1}$$

where $f \in C([0,1] \times (0,\infty)^p, (0,\infty))$, $f(t, y_0, \ldots, y_{n-1})$ may be singular at $y_i = 0$, $0 \leq i \leq p-1$, $\phi \in C(0,1)$ with $\phi > 0$ on $(0,1)$ and $\phi \in L^1[0,1]$, ϕ may be singular at $t = 0$ and/or 1. However, by assuming that f has the following increasing condition

$$\sum_{i=0}^{p-1} h_i(u_i) \leq f(t, u_0, \ldots, u_{p-1}) \leq \sum_{i=0}^{p-1} g_i(u_i) + r(u) \tag{1.2}$$

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on $[0,1] \times (0,\infty)^p$ with $h_i > 0$ continuous and non-increasing on $(0,\infty)$ for each $i = 0,\ldots,p-1$, $g_i > 0$ continuous and non-increasing on $(0,\infty)$ for each $i = 0,\ldots,p-1$, and $r \geq 0$ continuous, nondecreasing on $[0,\infty)$, here $|u| = \max\{u_0, u_1, \ldots, u_{n-1}\}$.

$$\int_0^1 \phi(s) g_i(k_i s^{p-i}) ds < \infty \text{ for each } i = 0,\ldots,p-1, \tag{1.3}$$

here $k_i > 0 (i = 0,\ldots,p-1)$ are constant, and

$$\text{if } z > 0 \text{ satisfies } z \leq a_0 + b_0 r(z) \text{ for constants } a_0 \geq 0 \text{ and } b_0 \geq 0, \tag{1.4}$$

then there exists a constant K (which may depend only on a_0 and b_0) with $z \leq K$.

The authors obtain an existence result. In fact, condition (1.4) implies the degree of variable u in the term $r(u)$ must be smaller than 1.

In [6], the singular problem $(-1)^n x^{(2n)}(t) = \mu f(t, x, \ldots, x^{(2n-2)})$, $x^{(2j)}(0) = x^{(2j)}(T) = 0$, $(0 \leq j \leq n-1)$, $\max\{x(t) : 0 \leq t \leq T\} = A$ depending on the parameter μ is considered. The existence of at least one positive solution was obtained under the assumption

$$f(t, x_0, \ldots, x_{n-2}) \leq \phi(t) + \sum_{j=0}^{2n-2} q_j(t) \omega_j(|x_j|) + \sum_{j=0}^{2n-2} h_j(t) |x_j|^{\alpha_j}$$

for a.e. $t \in J$ and for each $(x_0, \ldots, x_{2n-2}) \in D$, where $\phi, h_j \in L_1(J)$ and $q_j \in L_\infty(J)$ are nonnegative, $\omega_j : R_+ \to R_+$ are non-increasing, $\alpha_j \in (0,1)$.

Motivated by the above results, we consider the right focal boundary value problem in the following form

$$(-1)^{n-p} (\Phi_m(x^{(n-1)}))'(t) = \mu f(t, x(t), \ldots, x^{(n-1)}(t)), \quad 0 < t < 1, \tag{1.5}$$

$$x^{(i)}(0) = 0, \quad i = 0, 1, \ldots, p-1, \quad x^{(i)}(1) = 0, \quad i = p, p+1, \ldots, n-1. \tag{1.6}$$

Together with the boundary conditions (1.6), we discuss the condition

$$\max\{x(t) : t \in J\} = A, \tag{1.7}$$

where $\Phi_m x := |x|^{m-2} x, m > 1$, $\Phi_{m'}$ is the inverse operator of Φ_m, where $\frac{1}{m} + \frac{1}{m'} = 1, n \geq 2$.
Let $J = [0,1], R_- = (-\infty, 0), R_+ = (0,\infty), R_0 = R \backslash \{0\}$,

$$D = \begin{cases} \underbrace{R_+ \times \cdots \times R_+}_{p} \times \underbrace{R_+ \times R_- \times \cdots \times R_+}_{n-p}, & n-p = 2k+1, \\ \underbrace{R_+ \times \cdots \times R_+}_{p} \times \underbrace{R_+ \times R_- \times \cdots \times R_-}_{n-p}, & n-p = 2k. \end{cases}$$

Nonlinearity term f satisfies local Carathédory conditions on $J \times D (f \in Car(J \times D))$ and may be singular at the zero value of all its phase variables. By using Leray-Schauder degree theory we get a new result on the existence of solutions to boundary value problem (1.5)-(1.7). The method of obtaining *priori* bound of solution is different from [3, 6] In addition, the maximum degree of some variables among $x_0, \dots, x_{n-1}$ in function $f(t, x_0, \dots, x_{n-1})$ are allowable to be 1.

Let $A \in R^+$. By a *solution* of BVP (1.5)-(1.7) we understand a function $x \in AC^{n-1}(J)$ (i.e., x has an absolutely continuous $(n-1)$st derivative on J) such that

(i) $x^{(i)}(t) > 0$ on $(0, 1]$ for $i = 0, \dots, p-1$ and $(-1)^{2n-p-i}x^{(i)}(t) > 0$ on $[0, 1)$ for $i = p, \dots, n-1$,

(ii) x satisfies boundary conditions (1.6)(1.7),

(iii) there exists $\mu_A \in R_+$ such that x fulfills (1.5) with $\mu = \mu_A$ for a.e. $t \in J$.

By a *solution* of BVP (1.5), (1.6) we understand a function $x \in AC^{2n-1}(J)$ such that $x^{(i)}(t) > 0$ on $(0, 1]$ for $i = 0, \dots, p-1$ and $(-1)^{n-i+1}x^{(2n-p-i)}(t) > 0$ on $[0, 1)$ for $i = p, \dots, n-1$, x satisfies boundary conditions (1.6) and (1.5) holds a.e. $t \in J$.

The purpose of this paper is to give conditions which guarantee the existence of a solution to BVP (1.5)-(1.7) for each given $A \in R_+$.

From now on, $\|x\| = \max\{|x(t)| : t \in J\}$, $\|x\|_1 = \int_0^1 |x(t)|dt$ and $\|x\|_\infty = ess \max\{ |x(t)| : 0 \le t \le 1\}$ stands for the norm in $C^0(J)$, $L_1(J)$, and $L_\infty(J)$, respectively. For any measurable set $\mathcal{M} \subset R$, $\mu(\mathcal{M})$ denotes the Lebesgue measure of $\mathcal{M}$.

The assumptions imposed upon the function f in (1.5) are listed as follows:

(H_1) $f \in Car(J \times D)$ and there exists nonnegative functions $\phi \in L_1(J), q_i \in L_\infty(J)$, and continuous functions $g_i : [0, 1] \times R^n \to R^+ (i = 0, \dots, n-1)$ and non-increasing nonnegative continuous function $\omega_i : R_+ \to R_+$ such that for $(t, x) \in J \times D$,

$$f(t, x_0, \dots, x_{n-1}) = \phi(t) + \sum_{i=0}^{n-1} q_i(t)\omega_i(|x_i|) + \sum_{i=0}^{n-1} g_i(t, x_i), \tag{1.8}$$

where

$$\lim_{|x_i|\to\infty} \sup_{t\in[0,1]} \frac{g_i(t, x_i)}{(\Phi_m(|x_i|))^{k_i}} = \alpha_i \ge 0, \quad k_i \text{ are any constants in } (0, 1), i = 0, \dots, p-1, \tag{1.9}$$

and

$$\lim_{|x_i|\to\infty} \sup_{t\in[0,1]} \frac{g_i(t, x_i)}{\Phi_m(|x_i|)} = \beta_i \ge 0, \quad i = p, \dots, n-1, \tag{1.10}$$

and ω_i satisfies

$$\int_0^1 \omega_i(s^{p-i})ds < \infty,\ 0 \le i \le p-1, \qquad \int_0^1 \omega_i(P_i(s))ds < \infty,\ p \le i \le n-1, \tag{1.11}$$

where

$$P_i(t) = \frac{1}{(n-2-i)!} \int_t^1 (\theta - t)^{n-2-i} \Phi_{m'} \left(\int_\theta^1 \phi(r)dr \right) d\theta,$$

and there exists $\lambda > 0$ such that

$$\omega_i(xy) \le \lambda \omega_i(x)\omega_i(y) \ \text{ for } x, y \in (0, \infty). \tag{1.12}$$

The paper is organized as follows. Section 2 presents the *priori* bound of BVP (1.5)-(1.7). Besides, we prove that some sets of functions containing solutions of our auxiliary regular BVPs are uniformly absolutely continuous on J. Section 3 deals with auxiliary regular BVPs of problem (1.5), (1.6), (1.7). First we prove the existence of solution by applying the Borsuk antipodal theorem and the Leray-Schauder degree (see, e.g. [12]). Then we prove the existence of solution for problem (1.5), (1.6), (1.7). Proof is based on the Arzelà-Ascoli theorem and the Vitali's convergence theorem, see, e.g. [11, 12, 14].

2. Auxiliary Results

Lemma 2.1. *If y is a solution of BVP* (1.5), (1.6), *then $y(t)$ is a fixed point of the operator*

$$(Ty)(t) = (-1)^{n-p-1} \int_0^1 G(t,s)\Phi_{m'}\left(\int_s^1 f(\theta, y(\theta), \ldots, y^{(n-1)}(\theta))d\theta\right) ds, \tag{2.1}$$

where $G(t,s)$ is the Green's function of the following BVP

$$\begin{cases} x^{(n-1)}(t) = 0, \quad t \in (0,1), \\ x^{(i)}(0) = 0, \quad i = 0, \ldots, p-1, \quad x^{(i)}(1) = 0, \quad i = p, \ldots, n-2, \end{cases}$$

and $G(t,s)$ can be expressed as

$$G(t,s) = \frac{1}{(n-2)!} \begin{cases} \sum\limits_{i=0}^{p-1} \binom{n-2}{i} t^i (-s)^{n-i-2}, & 0 \le s \le t \le 1; \\ -\sum\limits_{i=p}^{n-2} \binom{n-2}{i} t^i (-s)^{n-i-2}, & 0 \le t \le s \le 1. \end{cases}$$

Furthermore,

$$\begin{aligned} &(-1)^{n-p-1} \frac{\partial^i}{\partial t^i} G(t,s) \ge 0, \ i = 0, \ldots, p-1, \\ &(-1)^{n-i-1} \frac{\partial^i}{\partial t^i} G(t,s) \ge 0, \ \ i = p, \ldots, n-2, \ (t,s) \in J \times J. \end{aligned} \tag{2.2}$$

Proof. By integrating the equation in (1.5) from $t \in [0,1)$ to 1 and using $x^{(n-1)}(1) = 0$, we obtain that

$$(-1)^{n-p}\Phi_m(y^{(n-1)}(t)) = -\int_t^1 f(\theta, y(\theta), \ldots, y^{(n-1)}(\theta))d\theta,$$

i.e.

$$y^{(n-1)}(t) = (-1)^{n-p-1}\Phi_{m'}\left(\int_t^1 f(\theta, y(\theta), \ldots, y^{(n-1)}(\theta))d\theta\right).$$

By [15] we have the result is true.

Remark 2.1. It follows from (2.1) (2.2) that

$$\begin{cases} y^{(i)}(t) > 0, & i = 0, \dots, p-1, \ \ t \in (0,1] \\ (-1)^{i-p} y^{(i)}(t) > 0, & i = p, \dots, n-1, \ \ t \in [0,1). \end{cases} \tag{2.3}$$

As in [5], for each $m \in N$, define $\mathcal{X}_m, \varphi_m \in C^0(R), R_m \subset R$ and $f_m \in Car(J \times R^n)$ by the formulas

$$\mathcal{X}_m(u) = \begin{cases} u, & \text{for } \ u \geq \frac{1}{m}, \\ \frac{1}{m}, & \text{for } \ u < \frac{1}{m}, \end{cases} \qquad \varphi_m(u) = \begin{cases} -\frac{1}{m}, & \text{for } \ u > -\frac{1}{m}, \\ u, & \text{for } \ u \leq -\frac{1}{m}, \end{cases}$$

$$\tau_m(u) = \begin{cases} \mathcal{X}_m(u), & \text{for } \ n-p = 2k+1, \\ \varphi_m(u), & \text{for } \ n-p = 2k, \end{cases}$$

and

$$\begin{aligned} & f_m(t, x_0, \dots, x_{n-1}) \\ = {} & \phi(t) + \sum_{i=0}^{p-1} q_i(t)\omega_i(|\mathcal{X}_m(x_i)|) + q_p(t)\omega_p(|\mathcal{X}_m(x_p)|) + q_{p+1}(t)\omega_{p+1}(|\varphi_m(x_{p+1})|) \\ & + \cdots + q_{n-1}(t)\omega_{n-1}(|\tau_m(x_{n-1})|) + \sum_{i=0}^{n-1} g_i(t, x_i) \end{aligned}$$

for $(t, x_0, \dots, x_{n-1}) \in J \times R^n$. Hence

$$\begin{aligned} 0 < \phi(t) \leq {} & f_m(t, x_0, \dots, x_{n-1}) \\ & \leq \phi(t) + \sum_{i=0}^{n-1} q_i(t)\omega_i(|x_i|) + \sum_{i=0}^{n-1} g_i(t, x_i) \end{aligned} \tag{2.4}$$

for a.e. $t \in J$ and each $(x_0, \dots, x_{n-1}) \in R_0^n$.

Consider auxiliary regular differential equation

$$(\Phi_m x^{(n-1)})'(t) = \mu f_m(t, x(t), \dots, x^{(n-1)}(t)) \tag{2.5}$$

depending on the parameters $\mu \in R$ and $m \in N$.

Lemma 2.2. *Let $m \in N$, then*

$$x^{(i)}(t) \geq t^{p-i}\Gamma, \quad i = 0, \dots, p-1; \qquad (-1)^{2n-p-i} x^{(i)}(t) \geq P_i(t), \quad i = p, \dots, n-1, \tag{2.6}$$

on J for any solution x of BVP (2.6), (1.6), *where* $\Gamma = (-1)^{n-p-1} \int_0^1 G(1,s)\Phi_{m'} \left(\int_s^1 \phi(\theta) d\theta \right) ds$.

Proof. By [2] we have

$$x^{(i)}(t) \geq t^{p-i} x^{(i)}(1) \text{ for } t \in J,\ i = 0, \dots, p-1. \tag{2.7}$$

Applying the inequality $\|x^{(i)}\| \geq \|x\|, i = 0, \dots, n-1$, and (2.1), (2.2) to (2.7) we get

$$\begin{aligned} x^{(i)}(t) \geq t^{p-i}\|x\| = t^{p-i}x(1) &\geq (-1)^{n-p-1} t^{p-i} \int_0^1 G(1,s)\Phi_{m'}\left(\int_s^1 \mu\phi(\theta)d\theta\right) ds \\ &= \Phi_{m'}(\mu) t^{p-i}\Gamma \end{aligned}$$

for $t \in J,\ i = 0, \dots, p-1$.

On the other hand, by (2.3) we have $(-1)^{n-p}x^{(n)}(t) \geq \phi(t), t \in J$. Integrating the above inequality from t to 1, we get step by step

$$(-1)^{2n-p-i}x^{(i)}(t) \geq \int_t^1 \frac{(\theta - t)^{n-i-2}}{(n-i-2)!}\Phi_{m'}\left(\int_\theta^1 \phi(r)dr\right) d\theta = P_i(t), \quad i = p, \dots, n-1.$$

Lemma 2.3. *Suppose that assumption (H_1) is satisfied, $m \in N$ and $A \in R^+$. Denote $\mu_* = \Phi_m\left(\frac{A}{\Gamma}\right)$. Then there is no solution in BVP (2.5), (1.6), (1.7) for $\mu > \mu_*$.*

Proof. Suppose $x(t)$ is a solution of BVP (2.5), (1.6), (1.7). By (2.1) and (2.4) we have

$$\begin{aligned} x(t) &= (-1)^{n-p-1} \int_0^1 G(t,s)\Phi_{m'}\left(\mu \int_s^1 f_m(r, x(r), \dots, x_{n-1}(r))dr\right) ds \\ &\geq (-1)^{n-p-1} \int_0^1 G(t,s)\Phi_{m'}\left(\mu \int_s^1 \phi(r)dr\right) ds, \end{aligned}$$

i.e.

$$\begin{aligned} \max\{x(t) : t \in J\} &\geq (-1)^{n-p-1}\Phi_{m'}(\mu) \max_{t\in J} \int_0^1 G(t,s)\Phi_{m'}\left(\int_s^1 \phi(r)dr\right) ds \\ &> (-1)^{n-p-1}\Phi_{m'}(\mu_*) \int_0^1 G(1,s)\Phi_{m'}\left(\int_s^1 \phi(r)dr\right) ds \\ &= \Phi_{m'}(\mu_*)\Gamma = A, \end{aligned}$$

which contradicts to (1.7).

Lemma 2.4. *Suppose $0 < u \in L^1[0,T], 0 \leq \psi \in L^\infty[0,T]$ and*

$$u(t) \leq K + \int_t^T u(s)\psi(s)ds, t \in [0,T], K > 0.$$

Then $u(t) \leq K \exp \int_t^T \psi(s)ds, \quad \forall t \in [0,T]$.

Proof. Let $G(t) = K + \int_t^T u(s)\psi(s)ds$, then $G'(t) = -u(t)\psi(t) \geq -\psi(t)G(t)$, i.e.

$$\frac{G'(t)}{G(t)} \geq -\psi(t).$$

Integrating the above inequality from t to T we have

$$\ln K - \ln G(t) \geq -\int_t^T \psi(s)ds,$$

i.e. $G(t) \leq K \exp \int_t^T \psi(s)ds$. Then $u(t) \leq G(t) \leq K \exp \int_t^T \psi(s)ds$.

Lemma 2.5. *Let assumption* (H_1) *be satisfied and* $A \in R_+$. *Then there exists a positive constant* P *depending only on* A *such that for any solution* x *of BVP* (2.5), (1.6) *with a* $\mu \in R_+$ *satisfying*

$$\max\{x(t) : t \in J\} = \lambda A, \quad \lambda \in (0, 1], \tag{2.8}$$

we have

$$\mu \leq \mu_* \text{ and } \|x^{(j)}\| \leq P \text{ for } 0 \leq j \leq n-1, \tag{2.9}$$

where μ_* *is defined in Lemma* 2.3.

Proof. Let x be a solution of BVP (2.5), (1.6) with $\mu \in R_+$ satisfying (2.8) for some $\lambda \in (0, 1]$. Then by Lemma 2.3, $\mu \leq \Phi_m(\lambda)\mu_*$ and so $\mu \leq \mu_*$.

Following we will show $\|x^{(j)}\| \leq P$, $j = 0, \dots, n-1$. We finish the proof by three steps.

Step 1. It follows from boundary condition that

$$x^{(i)}(t) = \int_0^t \frac{(t-s)^{p-i-1}}{(p-i-1)!} \left(\int_s^1 \frac{(\theta - s)^{n-p-2}}{(n-p-2)!} |x^{(n-1)}(\theta)| d\theta \right) ds, \quad i = 0, \dots, p-1. \tag{2.10}$$

$$(-1)^{2n-p-i} x^{(i)}(t) = \int_t^1 \frac{(\theta - t)^{n-i-2}}{(n-i-2)!} |x^{(n-1)}(\theta)| d\theta, \quad i = p, \dots, n-2. \tag{2.11}$$

It follows from (2.11) that

$$|x^{(i)}(t)| \leq \frac{(1-t)^{n-i-1}}{(n-i-1)!} |x^{(n-1)}(t)|, \quad i = p, \dots, n-1. \tag{2.12}$$

From (2.10) we have

$$\|x^{(i)}\| \leq \frac{1}{(p-i-1)!(n-p-1)!} \|x^{(n-1)}\|, \quad i = 0, \dots, p-1. \tag{2.13}$$

Step 2. Prove $|x^{(n-1)}(t)| \leq P, t \in [0, 1]$.

For any small $\varepsilon > 0$, there is $\delta > 0$ so that

$$|g_i(t,x_i)| < (\alpha_i+\varepsilon)(\Phi_m(|x_i|))^{k_i} \text{ uniformly for } t\in[0,1],\ k_i\in(0,1) \text{ and } |x_i|>\delta, \quad i=0,\dots,p-1,$$

and

$$|g_i(t,x_i)| < (\beta_i+\varepsilon)\Phi_m(|x_i|) \text{ uniformly for } t\in[0,1], \text{ and } |x_i|>\delta,\ \ i=p,\dots,n-1.$$

Let, for $i=0,\dots,n-1$,

$$\Delta_{1,i} = \{t : t\in[0,1], |x_i(t)|\le\delta\},$$

$$\Delta_{2,i} = \{t : t\in[0,1], |x_i(t)|>\delta\},$$

$$g_{\delta,i} = \max_{t\in[0,1],|x_i|\le\delta} g_i(t,x_i).$$

For some $m>0, t\in[0,1]$,

$$(-1)^{n-p}(\Phi_m x^{(n-1)})'(t) = \mu f_m(t,x(t),\dots,x_{n-1}(t)). \tag{2.14}$$

Integrating the above equality from t to 1, noticing Lemma 2.2, (1.9), (1.10), (2.4) and (2.12) (2.13) we have

$$\begin{aligned}
\Phi_m(|x^{(n-1)}(t)|) &\le \mu_* \int_t^1 \left[\phi(s)+\sum_{i=0}^{p-1} q_i(s)\omega_i\left(s^{p-i}\Gamma\right)+\sum_{i=p}^{n-1} q_i(s)\omega_i(P_i(s))\right] ds \\
&\quad + \sum_{i=0}^{n-1}\int_{\delta_{1,i}\cap[t,1]} g_i(s,x^{(i)}(s))ds + \sum_{i=0}^{n-1}\int_{\delta_{2,i}\cap[t,1]} g_i(s,x^{(i)}(s))ds \\
&\le \mu_*\left(\Lambda+\sum_{i=0}^{n-1} g_{\delta,i}+\sum_{i=0}^{n-1}\int_{\delta_{2,i}\cap[t,1]} g_i(s,x^{(i)}(s))ds\right) \\
&\le \mu_*\left[\Lambda+\sum_{i=0}^{n-1} g_{\delta,i}+\sum_{i=0}^{p-1}(\alpha_i+\varepsilon)\left(\Phi_m\left(\frac{|x^{(n-1)}(0)|}{(p-i-1)!(n-p-1)!}\right)\right)^{k_i}\right. \\
&\quad \left. +\int_t^1 \sum_{i=p}^{n-1}(\beta_i+\varepsilon)\Phi_m\left(\frac{(1-s)^{n-i-1}}{(n-i-1)!}\right)\Phi_m(|x^{(n-1)}(s)|)ds\right],
\end{aligned}$$

i.e.

$$|\Phi_m(x^{(n-1)}(t))| \le \left(C+D\left(\Phi_m(|x^{(n-1)}(0)|)\right)^{k_i}\right)+\int_t^1 E(s)\Phi_m(|x^{(n-1)}(s)|)ds.$$

where

$$\Lambda = \int_0^1\left[\phi(s)+\sum_{i=0}^{p-1}\|q_i\|_\infty\lambda\omega_i\left(s^{p-i}\right)\omega_i(\Gamma)+\sum_{i=p}^{n-1}\|q_i\|_\infty\omega_i(P_i(s))\right] ds,$$

$$C = 2\mu_* \left[\Lambda + \sum_{i=0}^{n-1} g_{\delta,i} \right],$$

$$D = 2\mu_* \sum_{i=0}^{p-1} (\alpha_i + \varepsilon) \left(\Phi_m \left(\frac{1}{(p-i-1)!(n-p-1)!} \right) \right)^{k_i},$$

$$E(t) = 2\mu_* \sum_{i=p}^{n-1} (\beta_i + \varepsilon) \Phi_m \left(\frac{(1-t)^{n-i-1}}{(n-i-1)!} \right).$$

By Lemma 2.4 and keep in mind $k_i \in (0,1)$, so there exists P(which does not independent on λ) such that $|x^{(n-1)}(0)| = \|x^{(n-1)}\| \le P$.

Step 3. Prove $\|x^{(i)}\| \le P$ for $i = 0, 1, \ldots, n-1$.

By (2.12) (2.15) and Step 2, we have

$$\|x^{(i)}\| \le \frac{P}{(n-i-1)!} \le P, \qquad \text{for } i = p, \ldots, n-2,$$

$$\|x^{(i)}\| \le \frac{P}{(n-p-1)!(p-i-1)!} \le P, \text{ for } i = 0, \ldots, p-1.$$

Thus $\|x^{(i)}\| \le P$ for $i = 0, 1, \ldots, n-1$.

Lemma 2.6. *Let assumption* (H_1) *be satisfied and* $A \in R_+$*. Let BVP* (2.5), (1.6), (1.7) *has a solution* x_m *for each* $m \in N$ *with* $\mu = \mu_m$ *in* (2.5)*. Then the sequence*

$$\{\mu_m f_m(t, x_m(t), \ldots, x_m^{(n-1)}(t))\} \subset L_1(J)$$

is uniformly absolutely continuous on J*, that is for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\mu_m \int_{\mathcal{M}} f_m(t, x_m(t), \ldots, x_m^{(n-1)}(t))dt < \varepsilon$$

for any measurable set $\mathcal{M} \subset J, \mu(\mathcal{M}) < \delta$.

Proof. With respect to (2.5) and properties of measurable sets, it is sufficient to verify that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any at most countable set $\{(a_j, b_j)\}_{j \in J}$ of mutually disjoint intervals $\{(a_j, b_j)\}_{j \in J}$ with $\sum_{j \in J} (b_j - a_j) < \delta$, we have for each $m \in N$,

$$\sum_{j \in J} \int_{a_j}^{b_j} \left[\phi(t) + \sum_{i=0}^{n-1} q_i(t) \omega_i(|x_m^{(i)}|) + \sum_{i=0}^{n-1} g_i(t, x^{(i)}(t)) \right] dt < \varepsilon. \tag{2.15}$$

By Lemma 2.2 we have

$$\begin{aligned} x_m^{(i)}(t) &\ge t^{p-i} \Phi(t), \quad i = 0, \ldots, p-1, \quad t \in J, \\ |x_m^{(i)}(t)| &\ge P_i(t), \quad i = p, \ldots, n-1, \quad t \in J. \end{aligned} \tag{2.16}$$

In addition by Lemma 2.4

$$\|x_m^{(i)}\| \le P, \qquad i = 0, \dots, p-1. \tag{2.17}$$

From (1.12), (2.16), (2.17) we have

$$\sum_{j\in J}\int_{a_j}^{b_j}\left[\phi(t)+\sum_{i=0}^{n-1}q_i(t)\omega_i(|x_m^{(i)}|)+\sum_{i=0}^{n-1}g_i(t,x^{(i)}(t))\right]dt$$
$$\le \sum_{j\in J}\int_{a_j}^{b_j}\left[\phi(t)+\sum_{i=0}^{p-1}q_i(t)\lambda\omega_i(t^{p-i})\omega_i(\Gamma)+\sum_{i=p}^{n-1}q_i(t)\omega_i(P_i(t))+\sup_{(t,x)\in[0,1]\times[0,P]}g_i(t,x_i)\right]dt.$$

By (H_1), we know that $\phi, h_i \in L_1(J), q_i \in L_\infty(J), \int_0^1 \omega_i(t^{p-i})dt < \infty, i = 0, \dots, p-1, \int_0^1 \omega_j(P_j(s))ds < \infty, j = p, \dots, n-1$. Consequently, for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any at most countable set $\{(a_j, b_j)\}_{j\in J}$ of mutually disjoint intervals $(a_j, b_j) \subset J$ with $\sum_{j\in J}(b_j - a_j) < \delta$. So (2.17) holds.

3. Existence results

Theorem 3.1. *Suppose that the assumption (H_1) is satisfied and $A \in R_+$. Then for each $m \in N$ there exists a solution x_m of BVP (2.5), (1.6), (1.7) with $\mu = \mu_m$ in (2.5), and*

$$\|x_m^{(j)}\| \le P \text{ for } m \in N, j = 0, \dots, n-1 \tag{3.1}$$

and

$$0 < \mu_m \le \mu_*. \tag{3.2}$$

Proof. Fix $m \in N$. Set

$$\Omega = \{(x,\mu) : (x,\mu) \in C^n(J)\times R, \|x^{(j)}\| < P+1 \text{ for } j = 0, \dots, n-1, |\mu| < \mu_* + 1\}.$$

Then Ω is a bounded, open and symmetric with respect to $(0,0)$ subset of the Banach space $C^n(J)\times R$ endowed with the norm $\|(x,\mu)\| = \sum_{i=0}^{n-1}\|x^{(j)}\| + |\mu|$. Define the operator $\mathcal{F}_1 : \overline{\Omega} \to C^n(R)\times R$ by

$$\mathcal{F}_1(x,\mu) = \left((-1)^{n-p-1}\int_0^1 G(t,s)\Phi_{m'}\left(\mu\int_s^1 f_m(r,x(r),\dots,x^{(n-1)}(r))dr\right)ds,\right.$$
$$\left.\max\{x(t) : t\in J\} + \min\{x(t) : t \in J\} + \mu\right),$$

where G is defined in Lemma 2.1. We first show that

$$D(\mathcal{I} - \mathcal{F}_1, \Omega, 0) \ne 0, \tag{3.3}$$

where D stands for the Leray-Schauder degree and $\mathcal{I}$ is the identity operator on $C^n(J)\times R$. To prove (3.3) we define the operator $\mathcal{H}:[0,1]\times\overline{\Omega}\to C^{n-1}(J)\times R$,

$$\mathcal{H}(\lambda,x,\mu)=\Big((-1)^{n-p-1}\int_0^1 G(t,s)\Phi_{m'}\Big[\mu(1-\lambda)+\mu\lambda\int_s^1 f_m(r,x(r),\dots,x^{(n-1)}(r))dr\Big]ds,$$
$$\lambda[\max\{x(t:t\in J)\}+\min\{x(t):t\in J\}]+(1-\lambda)x(1/2)+\mu\Big).$$

Then

$$\mathcal{H}(0,-x,-\mu)=\left(-(-1)^{n-p-1}\int_0^1 G(t,s)\Phi_{m'}(\mu)ds,-x(1/2)-\mu\right)=-\mathcal{H}(0,x,\mu)$$

for $(x,\mu)\in\overline{\Omega}$ and so $\mathcal{H}$ is an odd operator. Due to the fact that $f_m\in Car(J\times R^{n-1})$, $\mathcal{H}$ is a compact operator. Assume that $\mathcal{H}(\lambda_0,x_0,\mu_0)=(x_0,\mu_0)$ for some $\lambda_0\in[0,1]$ and $(x_0,\mu_0)\in\partial\Omega$. Then

$$x_0(t)=(-1)^{n-p-1}\int_0^1 G(t,s)\Phi_{m'}\left[\mu(1-\lambda_0)+\mu\lambda_0\int_s^1 f_m(r,x_0(r),\dots,x_0^{(n-1)}(r))dr\right]ds \tag{3.4}$$

for $t\in J$ and

$$\lambda_0[\max\{x_0(t):t\in J\}+\min\{x_0(t):t\in J\}]+(1-\lambda_0)x_0(1/2)=0. \tag{3.5}$$

Also from (2.3) it follows that

$$x_0(0)=0, x_0^{'}(t)\geq 0, t\in J.$$

So $x_0(t)>0$ for $t\in(0,1)$ and $\min\{x_0(t):t\in J\}=0$. Therefore

$$\begin{aligned}&\lambda_0[\max\{x_0(t):t\in J\}+\min\{x_0(t):t\in J\}]+(1-\lambda_0)x_0(1/2)\\&=\lambda_0\max\{x_0(t):t\in J\}+(1-\lambda_0)x_0(1/2)>0,\end{aligned}$$

contrary to (3.5). If $\mu_0<0$, then $x_0(t)<0$ for $t\in(0,1)$. By (2.3) $x_0(0)=0, x_0^{'}(0)\leq 0, t\in J$, so $\max\{x_0(t):t\in J\}=0$. Hence

$$\begin{aligned}&\lambda_0[\max\{x_0(t):t\in J\}+\min\{x_0(t):t\in J\}]+(1-\lambda_0)x_0(1/2)\\&=\lambda_0\min\{x_0(t):t\in J\}+(1-\lambda_0)x_0(1/2)<0,\end{aligned}$$

contrary to (3.5). If $\mu_0=0$ and then (3.4) gives $x_0=0$. Consequently, $(x_0,\mu_0)=(0,0)$, contrary to $(x_0,\mu_0)\in\partial\Omega$. The Borsuk antipodal theorem and the Leray-Schauder degree theory lead to $D(\mathcal{I}-\mathcal{H}(0,\cdot,\cdot),\Omega,0)\neq 0$ and

$$D(\mathcal{I}-\mathcal{F}_1,\Omega,0)=D(\mathcal{I}-\mathcal{H}(1,\cdot,\cdot),\Omega,0)=D(\mathcal{I}-\mathcal{H}(0,\cdot,\cdot),\Omega,0)$$

which implies (3.3).

Finally, let $\mathcal{F} : \overline{\Omega} \to C^n(J) \times R$ be defined by the formula

$$\mathcal{F}(x,\mu) = \Big((-1)^{n-p-1} \int_0^1 G(t,s)\Phi_{m'} \left[\mu \int_s^1 f_m(r, x(r), \dots, x^{(n-1)}(r))dr \right] ds, \\ \max\{x(t) : t \in J\} + \min\{x(t) : t \in J\} - A + \mu \Big).$$

We claim that to prove our theorem it is sufficient to verify:

$$D(\mathcal{I} - \mathcal{F}, \Omega, 0) \neq 0. \tag{3.6}$$

In fact, if (3.6) is true, then there exists a fixed point $(\widehat{x}, \widehat{\mu}) \in \Omega$ of the operator $\mathcal{F}$. Hence

$$\widehat{x}(t) = (-1)^{n-p-1} \int_0^1 G(t,s)\Phi_{m'} \left[\widehat{\mu} \int_s^1 f_m(r, \widehat{x}(r), \dots, x^{(n-1)}(r))dr \right] ds \tag{3.7}$$

for $t \in J$ and

$$\max\{\widehat{x}(t) : t \in J\} + \min\{\widehat{x}(t) : t \in J\} = A. \tag{3.8}$$

Moreover, $\widehat{\mu} > 0$ since in the case of $\widehat{\mu} \le 0$ (3.7) and Lemma 2.1 gives for $\widehat{x}(t) \le 0$ for $t \in J$, so $\max\{\widehat{x}(t) : t \in J\} = 0$, contrary to (3.8). Therefore (see (3.7)) $\widehat{x}$ is a solution of BVP (2.4), (1.6) with $\mu = \widehat{\mu}$ in (2.4), and for $t \in (0,1)$. So $\min\{\widehat{x}(t) : t \in J\} = 0$. Then, by (3.8), $\max\{\widehat{x}(t) : t \in J\} = A$, and we see that $\widehat{x}$ is a solution of BVP (2.4), (1.6), (1.7).

In order to prove (3.6) we consider the operator $\mathcal{H} : [0,1] \times \overline{\Omega} \to C^n(J) \times R$,

$$\mathcal{H}(\lambda, x,\mu) = \Big((-1)^{n-p-1} \int_0^1 G(t,s)\Phi_{m'} \left[\mu \int_s^1 f_m(r, x(r), \dots, x^{(n-1)}(r))dr \right] ds, \\ \max\{x(t) : t \in J\} + \min\{x(t) : t \in J\} - \lambda A + \mu \Big)$$

Then, $\mathcal{H}(1,\cdot,\cdot) = \mathcal{F}, \mathcal{H}(0,\cdot,\cdot) = \mathcal{F}_1$ and, by (3.3),

$$D(\mathcal{I} - \mathcal{H}(0,\cdot,\cdot), \Omega, 0) \neq 0. \tag{3.9}$$

Assume that $\mathcal{H}(\lambda_1, x_1, \mu_1) = (x_1, \mu_1)$ for some $\lambda_1 \in [0,1]$ and $(x_1, \mu_1) \in \partial\Omega$. If $\mu_1 = 0$ then from the equality

$$x_1(t) = (-1)^{n-p-1} \int_0^1 G(t,s)\Phi_{m'} \left[\mu_1 \int_s^1 f_m(r, x_1(r), \dots, x_1^{(n-1)}(r))dr \right] ds, \tag{3.10}$$

for $t \in J$, we get $x_1 = 0$, contrary to $(x_1, \mu_1) = (0,0) \in \partial\Omega$. let $\mu_1 < 0$. Then (see (3.10)) $x_1(t) < 0$ on $(0,1)$ and $\max\{x_1(t) : t \in J\} = 0$, contrary to $\max\{x_1(t) : t \in J\} + \min\{x_1(t) : t \in J\} = \lambda_1 A$. hence $\mu_1 > 0$ and then x_1 is a solution of BVP (2.5), (1.6) with $\max\{x(t) : t \in J\} = \lambda A$. Moreover, by Lemma 2.5, $\|x_1^{(j)}\| \le P$ for $0 \le j \le n-1$ and $0 < \mu \le \mu_*$. Consequently, $(x_1, \mu_1) \notin \partial\Omega$, a contradiction. we have proved $\mathcal{F}(\lambda, x, \mu) \neq (x,\mu)$ for each $\lambda \in [0,1]$ and $(x,\mu) \in \partial\Omega$, and since $\mathcal{H}$ is a compact homotopy,

$$D(\mathcal{I} - \mathcal{F}, \Omega, 0) = D(\mathcal{I} - \mathcal{H}(1,\cdot,\cdot), \Omega, 0) = D(\mathcal{I} - \mathcal{H}(0,\cdot,\cdot), \Omega, 0)$$

and then (3.9) gives (3.6), which finishes our proof.

Theorem 3.2. *Suppose the assumptions*(H_1) *be satisfied and* $A \in R_+$. *Then there exists a solution of BVP* (1.5), (1.6), (1.7) *for each* $A \in R_+$.

Proof. For each $m \in N$, there exists a solution x_m of BVP (2.5), (1.6), (1.7) with a $\mu = \mu_m$ by Theorem 3.1. Consider the sequence $\{x_m\}, \{\mu_m\}$. By Lemma 2.2, Lemma 2.5, $\{x_m^{(i)}\}, \{\mu_m\}$ are bounded for $i = 0, \dots, n-1$.

For $t_1, t_2 \in J, t_2 < t_1$,

$$\begin{aligned} & |x_m^{(n-1)}(t_1) - x_m^{(n-1)}(t_2)| \\ = {} & \Phi_{m'}\left(\Big|\int_0^{t_1} \mu_m f_m(t, x_m(t), \dots, x_m^{(n-1)}(t))dt\Big|\right) \\ & -\Phi_{m'}\left(\Big|\int_0^{t_2} \mu_m f_m(t, x_m(t), \dots, x_m^{(n-1)}(t))dt\Big|\right). \end{aligned}$$

We can use Lemma 2.6 and obtain that the sequence $\mu_m f_m(t, x_m(t), \dots, x_m^{(n-1)}(t))$ is uniformly absolutely continuous on J. Moreover by the continuity of $\Phi_{m'}$ we have $\{x_m^{(n-1)}\}_{n_0}^{\infty}$ is equi-continuous on J. The Arzalà-Ascoli theorem guarantees the existence of a subsequence, such that $\{x_{m_k}\}, \{\mu_{m_k}\}$ is convergent in $C^n(J)$ and R respectively. Let $\lim\limits_{k\to\infty} x_{m_k} = x$, $\lim\limits_{k\to\infty} \mu_{m_k} = \widehat{\mu}$, then $x \in C^{n-1}(J), x$ satisfies boundary condition (1.6), (1.7) and $0 \le \widehat{\mu} \le \mu_*$.

We now prove $(-1)^{n-p-1}x^{(n-1)}(t) > 0, t \in [0,1)$. If not, there exists $t_1 \in (0,1)$ such that

$$(-1)^{n-p-1}x^{(n-1)}(t) > 0, t \in [t_1, 1), \qquad (-1)^{n-p-1}x^{(n-1)}(t) = 0, t \in (0, t_1] \tag{3.11}$$

From (2.3) we obtain $x^{(i)}$ has at most one zero ξ_j on $[0, t_1]$ for $i = 0, \dots, p-1$. Now from the construction of $f_{m_k} \in Car(J \times R^{n-1})$ it follows that there exists a set $\mathcal{M} \in J, \mu(\mathcal{M}) = 0$ such that $f_{m_k}(t, \cdot, \dots, \cdot)$ are continuous on R^{n-1} for each $t \in J \setminus \mathcal{M}$ which implies that

$$\lim_{k\to\infty} \mu_{m_k} f_{m_k}(t, x_{m_k}(t), \dots, x_{m_k}^{(n-1)}(t)) = \widehat{\mu} f(t, x(t), \dots, x^{(n-1)}(t))$$

for $t \in [0, t_1] \backslash \mathcal{M}$. By Lemma 2.6 $\{\mu_{m_k} f_{m_k}(t, x_{m_k}(t), \dots, x_{m_k}^{(n-1)}(t))\}$ is uniformly absolutely continuous on $[0, t_1]$. Then $\widehat{\mu} f(t, x(t), \dots, x^{(n-1)}(t)) \in L^1[0, t_1]$ and

$$\lim_{k\to\infty} \mu_{m_k} \int_t^{t_1} f_{m_k}(s, x_{m_k}(s), \dots, x_{m_k}^{(n-1)}(s))ds = \widehat{\mu} \int_t^{t_1} f(s, x(s), \dots, x^{(n-1)}(s))ds$$

for $t \in [0, t_1]$ by Vitali's convergence theorem. Noticing $x_{m_k}^{(n-1)}(t_1)$ is bounded, we assume it is convergent, and let $\lim\limits_{k\to\infty} x_{m_k}^{(n-1)}(t_1) = d$. Taking the limit as $k \to \infty$ in the equality

$$x_{m_k}^{(n-1)}(t) = x_{m_k}^{(n-1)}(t_1) - (-1)^{n-k}\Phi_{m'}\left(\mu_{m_k}\int_t^{t_1} f_{m_k}(\tau, x_{m_k}(\tau), \dots, x_{m_k}^{(n-1)}(\tau))d\tau\right)$$

we get

$$x^{(n-1)}(t) = d - (-1)^{n-k}\Phi_{m'}\left(\widehat{\mu}\int_t^{t_1} f(\tau, x(\tau), \ldots, x^{(n-1)}(\tau))d\tau\right)$$

There are two cases to consider:

Case(i) If $\widehat{\mu} = 0.x^{(n-1)}(t) = 0$ for $t \in [0, t_1]$, and the equality $x^{(n-1)}(t_1) = 0$ yields $d = 0$. Hence $x^{(n-1)}(t) = 0$ for $t \in J$, contrary to (3.10).

Case(ii) If $\widehat{\mu} > 0$. By (2.4), we have

$$|x_{m_k}^{(n-1)}(t)| \geq \Phi_{m'}\left(\mu_{m_k}\int_t^1 \phi(\theta)d\theta\right), k \in N \tag{3.12}$$

Letting $k \to \infty$ in (3.11) we have

$$|x^{(n-1)}(t)| \geq \Phi_{m'}\left(\widehat{\mu}\int_t^1 \phi(\theta)d\theta\right), t \in J$$

Hence $|x^{(n-1)}(t)| > 0$ for $t \in [0, 1)$, contrary to (3.11). Thus $|x^{(n-1)}(t)| > 0$ for $t \in [0, 1)$. So $x^{(i)}(t) > 0, 0 \leq i \leq p-1$ on $(0, 1], (-1)^{2n-p-i}x^{(i)}(t) > 0, p \leq i \leq n-1$ on $[0, 1)$. Noticing $\{x_{m_k}^{(n-1)}(0)\}$ is convergent. Let $\lim_{k\to\infty} x_{m_k}^{(n-1)}(0) = \widehat{d}$. Since $\{\mu_{m_k} f_{m_k}(t, x_{m_k}(t), \ldots, x_{m_k}^{(n-1)})\}$ is uniformly absolutely continuous on J and

$$\lim_{k\to\infty} \mu_{m_k} f_{m_k}(t, x_{m_k}(t), \ldots, x_{m_k}^{(n-1)}(t)) = \mu f(t, x(t), \ldots, x^{(n-1)}(t)).$$

By the Vitali's Convergence theorem to get $\widehat{\mu} f(t, x(t), \ldots, x^{(n-1)}(t)) \in L_1(J)$ and letting $k \to \infty$ in the equality

$$x_{m_k}^{(n-1)}(t) = x_{m_k}^{(n-1)}(0) + (-1)^{n-k}\Phi_{m'}\left(\mu_{m_k}\int_0^t f_{m_k}(s, x_{m_k}(s), \ldots, x_{m_k}^{n-1}(s))ds\right),\ t \in J$$

we get

$$x^{(n-1)}(t) = \widehat{d} + (-1)^{n-k}\Phi_{m'}\left(\widehat{\mu}\int_0^t f(s, x(s), \ldots, x^{n-1}(s))ds\right),\ t \in J. \tag{3.13}$$

If $\widehat{\mu} = 0$. $x^{(n-1)}(t) = \widehat{d}$ for $t \in J$ and condition $x^{(n-1)}(1) = 0$ gives $\widehat{d} = 0$. So $x^{(n-1)}(t) = 0$ for $t \in J$. Without loss of generality, we suppose $\|x^{(i)}\| = x(\zeta_i)$.

$$\|x^{(i)}\| > \left|\frac{x^{(i-1)}(\zeta_{i-1}) - x^{(i-1)}(0)}{\zeta_{i-1}}\right| > \|x^{(i-1)}\|, \quad 0 \leq i \leq p-1$$

and

$$\|x^{(i)}\| > \left|\frac{x^{(i-1)}(1) - x^{(i-1)}(\zeta_{i-1})}{1-\zeta_{i-1}}\right| > \|x^{(i-1)}\|, \quad p \leq i \leq n-1$$

Thus $\|x^{(i)}\| > \|x^{(i-1)}\| > \cdots > \|x\| = A, 0 \leq i \leq n-1$. The fact $x^{(n-1)}(t) = 0$ for $t \in J$ contradicts $\|x^{(n-1)}\| > A$.

If $\widehat{\mu} > 0$ and from (3.13) we see that $x \in AC^{n-1}(J)$ and x satisfies (1.5) a.e. on J. We have proved that x is a solution of BVP (1.5)-(1.7) with $\mu = \widehat{\mu}$ in (1.5).

4. Example

Example 4.1. Let us consider the following fourth-order boundary value problem

$$\begin{cases} (\Phi_m(y^{(3)}(t)))' = \mu \left[1 - t + \sum_{i=0}^{3} q_i(t) y_i^{-\frac{1}{5}} + g_0(t) \sin(\Phi_3(y_0))^{\frac{1}{2}} + \sum_{i=1}^{3} g_i(t)\Phi_3(y_i)\right], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in (0,1), \\ y(0) = 0, \quad y'(1) = y''(1) = y^{(3)}(1) = 0, \end{cases} \tag{4.1}$$

with $\max\{y(t) : t \in [0,1]\} = A$, $q_i \in L_\infty([0,1]), g_i \in C[0,1], m = 3, p = 1$ for $i = 0,1,2,3$.

Corresponding to BVP (1.5)-(1.7) we have

$$f(t, y_0, y_1, y_2, y_3) = 1 - t + \sum_{i=0}^{3} q_i(t) y_i^{-\frac{1}{5}} + g_0(t)\sin(\Phi_3(y_0))^{\frac{1}{2}} + \sum_{i=1}^{3} g_i(t)\Phi_3(y_i)$$

where $\phi(t) = 1-t, \omega_i(|y_i|) = |y_i|^{-\frac{1}{5}}, i = 0,1,2,3, g_0(t,y_0) = g_0(t)\sin(\Phi_3(y_0))^{\frac{1}{2}}, g_i(t,y_i) = g_i(t)\Phi_3(y_i), i = 1,2,3$.

Then for any $A > 0$, there exists $\mu_A < \mu_* = \Phi_3\left(\frac{A}{\Gamma}\right)$ such that BVP (4.1) has a solution $y \in AC^3([0,1])$.

To see (4.1) has a solution $y \in AC^3([0,1])$, we apply theorem 3.2, It is easy to verify (H_1)

$$\lim_{|y_0| \to \infty} \sup_{t \in [0,1]} \frac{g_0(t,y_0)}{(\Phi_3(|y_0|))^{1/2}} = 0,$$

$$\lim_{|y_i| \to \infty} \sup_{t \in [0,1]} \frac{g_i(t,y_i)}{\Phi_3(|y_i|)} = \sup_{t \in [0,1]} g_i(t) \geq 0,$$

$$\int_0^1 \omega_0(s) ds < \infty,$$

$$\int_0^1 \omega_i(P_i(s)) ds < \infty, \quad i = 1,2,3$$

hold. So applying Theorem 3.2, for any $A > 0$, there exists μ_A such that BVP (4.1) has a solution $y \in AC^3([0,1])$.

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