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THE DEGREE OF APPROXIMATION OF FUNCTIONS, AND THEIR CONJUGATES, BELONGING TO SEVERAL GENERAL LIPSCHITZ CLASSES BY HAUSDORFF MATRIX MEANS OF THE FOURIER SERIES AND CONJUGATE SERIES OF A FOURIER SERIES

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Abstract. In this paper Hausdorff matrix approximations are obtained for a function and its conjugate belonging to any one of several generalized Lipschitz classes.

1. Introduction

A number of papers have been written dealing with the degree of approximation of the Fourier series representation of a function, or a conjugate function, by means of the matrix product of $(C,1)$, the Cesáro matrix of order one, with (E,q), the Euler matrix of order q. (See, e.g. [1] - [5], [7], and [8].)

In [3] three degree of approximation results were obtained for the product of an Euler matrix (E,q) with a Hausdorff matrix and the trigonometric approximation of the conjugate of a function belonging to certain Lipschitz classes. Since the product of two Hausdorff matrices is again a Hausdorff matrix, these theorems suggest that it is possible to prove these results using a single Hausdorff matrix. The main purpose of this paper is to show that this conjecture is true.

A function $f \in \text{Lip}(\alpha)$ if $|f(x+t)-f(x))| = O(t^\alpha)$ for $0<\alpha\le 1$. A function $f \in Lip(\alpha,r)$ if $\{\int_0^{2\pi}|f(x+t)-f(x)|^r dx\}^{1/r} = O(\xi(t))(r\ge 1)$, where ξ is a modulus of continuity; i.e., ξ is a nonnegative nondecreasing continuous function with the properties $\xi(0)=0$ and $\xi(t_1+t_2)\le \xi(t_1)+\xi(t_2)$. A function $f \in\text{Lip}(\xi,r)$ if $\{\int_0^{2\pi}|f(x+t)-f(x)|^r dx\}^{1/r} = O(\xi(t))(r\ge 1)$, where ξ is a modulus of continuity; i.e., ξ is nonnegative, nondecreasing, and continuous with the properties $\xi(0)=0, \xi(t_1+t_2)\le\xi(t_1)+\xi(t_2)$.

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Let $f(x)$ be a 2π-periodic function, Lebesgue integrable on $[0,2\pi]$ and belonging to any of the Lipschitz classes defined above. The Fourier series for $f(x)$ is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} A_n(x), \tag{1}$$

with nth partial sum $s_n(f;x)$.

The series

$$\sum_{n=1}^{\infty}(a_n \cos nx - b_n \sin nx) = -\sum_{n=1}^{\infty} B_n(x) \tag{2}$$

is called the conjugate series of series (1), with nth partial sum $\tilde{s}_n(f;x)$.

A Hausdorff matrix is a lower triangular matrix with nonzero entires

$$h_{nk} = \binom{n}{k}\Delta^{n-k}\mu_k,$$

where Δ is the forward difference operator defined by $\Delta\mu_k = \mu_k - \mu_{k+1}$, and $\Delta^{n+1}\mu_k = \Delta(\Delta^n\mu_k)$. A Hausdorff matrix is regular; i.e. is preserves the limit of each convergent sequence, if and only if

$$\int_0^1 |d\chi(u)| < \infty,$$

where the mass function $\chi \in BV[0,1], \chi(0+) = \chi(0)$ and $\chi(1) = 1$. In this case the μ_n have the representation

$$\mu_n = \int_0^1 u^n d\chi(u).$$

For any sequence $\{s_n\}$,

$$t_n := \sum_{k=0}^{n} h_{nk}s_k.$$

The norm $L^r = \|\,\|^r$ is defined by

$$\|f\|_r = \left\{\frac{1}{2\pi}\int_0^{2\pi}|f(x)|^r dr\right\}^{1/r}, \quad r \geq 1,$$

and the degree of trigonometric approximation $H_n(f)$ will be denoted by

$$H_n(f) = \min\|f - t_n\|_r.$$

The norm $\|\,\|_r^{\xi)}$ on the class of functions $L^r_{(\xi)}$ is defined by

$$\|f\|_r^{(\xi)} = \sup_{t\neq 0}\frac{\|f(\cdot + t) - f(\cdot)\|^r}{\xi(|t|)}.$$

For any regular Hausdorff transform, and $\psi(t,x)$ denoting one of the Lipschitz conditions,

$$(H)_n - f(x) = \frac{1}{\pi}\int_0^{\pi} \frac{\psi(x,t)}{\sin(t/2)} \sum_{k=0}^{n} h_{nk}\sin(k+\frac{1}{2})t\,dt$$
$$= I_1 + I_2, \quad \text{say.}$$

For $0 < t \le 1/(n+1), |\sin t| \le 1$, and $\sin(t/2) \ge (t/\pi)$. Therefore

$$\begin{aligned}|I_1| &= \Big|\frac{1}{2\pi}\int_0^{1/(n+1)} \psi(x,t)\sum_{k=0}^{n}\int_0^1 \binom{n}{k} u^k(1-u)^{n-k}d\chi(u)\frac{\sin(k+1/2)t}{\sin t/2}dt\Big| \\
&\le \frac{1}{2\pi}\int_0^{1/(n+1)} |\psi(x,t)|\sum_{k=0}^{n}\int_0^1 \binom{n}{k} u^k(1-u)^{n-k}|d\chi(u)|\Big|\frac{\sin(k+1/2)t}{\sin\frac{t}{2}}\Big|dt \\
&\le \frac{1}{2\pi}\int_0^{1/(n+1)} |\psi(x,t)|\sum_{k=0}^{n}\int_0^1 \binom{n}{k} u^k(1-u)^{n-k}|d\chi(u)|\frac{\pi}{t}dt \\
&= \frac{1}{2}\int_0^{1/(n+1)} \frac{|\psi(x,t)|}{t}\int_0^1 \sum_{k=0}^{n}\binom{n}{k} u^k(1-u)^{n-k}|d\chi(u)|dt \\
&\le \frac{\|H\|}{2}\int_0^{1/(n+1)} \frac{|\psi(x,t)|}{t}|dt = O\Big(\int_0^{1/(n+1)} \frac{|\psi(x,t)|}{t}|dt\Big) \qquad (3)\end{aligned}$$

$$\begin{aligned}I_2 &= \frac{1}{2\pi}\int_{1/(n+1)}^{\pi} \frac{\psi(x,t)}{\sin(t/2)}\sum_{k=0}^{n}\int_0^1 \binom{n}{k} u^k(1-u)^{n-k}d\chi(u)Im(e^{i(k+1/2)t}dt \\
&= \frac{1}{2\pi}\int_{1/(n+1)}^{\pi} \frac{\psi(x,t)}{\sin(t/2)}\int_0^1 Im\Big[\sum_{k=0}^{n}\binom{n}{k} u^k(1-u)^{n-k}e^{i(k+1/2)t}\Big]d\chi(u)dt.\end{aligned}$$

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k} u^k(1-u)^{n-k}e^{i(k+1/2)t} &= (1-u)^n e^{it/2}\sum_{k=0}^{n}\binom{n}{k}\Big(\frac{ue^{it}}{1-u}\Big)^n \\
&= (1-u)^n e^{it/2}\Big(1+\frac{ue^{it}}{1-u}\Big)^n \\
&= e^{it/2}(1-u+ue^{it})^n.\end{aligned}$$

Define

$$|g(u,t)| = |Im[e^{it/2}(1-u+ue^{it})^n| \le r^n(u,t), \qquad (4)$$

where

$$\begin{aligned}r^2(u,t) &= (1-u+u\cos t)^2 + (u\sin t)^2 \\
&= 1-2u+2u(1-u)\cos t + 2u^2 = 1-2u(1-u)+2u(1-u)\cos t \\
&= 1-2u(1-u)(1-\cos t) = 1-4u(1-u)/\sin^2(t/2)\end{aligned}$$

$$
\begin{aligned}
&= (1-4u(1-u))\sin^2(t/2)+\cos^2(t/2)\\
&= (1-2u)^2\sin^2(t/2)+\cos^2(t/2)\\
&\le (\sin(t/2))^2+\cos(t/2))^2 = 1. \qquad (5)
\end{aligned}
$$

Using (4) and (5),

$$|I_2| = O(1)\int_{1/(n+1)}^{\pi} \frac{|\psi(x,t)|h(t)}{t^2}dt,$$

where $h(t)=\sin(t/2)=O(1)$. Hence $h(t)=O(n+1)$, and it then follows that

$$
\begin{aligned}
|I_2| &= O\Big(\frac{1}{n+1}\Big)\int_{1/(n+1)}^{\pi} t^{\alpha-2}dt\\
&= O\Big(\frac{1}{n+1}\Big)\frac{1}{\alpha-1}t^{\alpha-1}\Big|_{1/(n+1)}^{\pi}\\
&= O\Big(\frac{1}{n+1}\Big)O\Big(\frac{1}{n+1}\Big)^{\alpha-1} = O((n+1)^{-\alpha}).
\end{aligned}
$$

Theorem 1. *The degree of approximation of a function f belonging to the class $Lip(\xi,r)$ by means of the Fourier series* (1) *satisfies*

$$\|(H)_n - f(x)\|_r^{(u)} = O\Big(\frac{1}{n+1}\int_{1/(n+1)}^{\pi}\frac{\xi(t)}{t^2u(t)}dt\Big),$$

where $\xi(t)$ and $u(t)$ are the modulus of continuity such that

$$\int_0^{v}\frac{\xi(t)}{tu(t)}dt = O\Big(\frac{\xi(v)}{u(v)}\Big), \quad 0<v<\pi.$$

Proof. From properties of the class $Lip(\xi,r)$ established in [3], (3) becomes

$$|I_1| = O\Big(u(|y|)\frac{\xi(1/(n+1)}{u(1/(n+1))}\Big),$$

and (6) becomes

$$|I_2| = O\Big(\frac{1}{n+1}\int_{1/(n+1)}^{\pi} u(|y|)\frac{\xi(t)}{t^2u(t)}dt\Big).$$

Thus

$$\|(H)_n - f(x)\| = O\Big(u(|y|)\frac{\xi(1/(n+1)}{u(1/(n+1))}\Big)+O\Big(\frac{1}{n+1}\int_{1/(n+1)}^{\pi} u(|y|)\frac{\xi(t)}{t^2u(t)}dt\Big),$$

and

$$\sup_{u\neq 0}\frac{\|(H)_n - f(x)\|}{u(|y|)} = O\Big(\frac{\xi(1/(n+1)}{u(1/(n+1))}\Big)+O\Big(\frac{1}{n+1}\int_{1/(n+1)}^{\pi}\frac{\xi(t)}{t^2u(t)}dt\Big).$$

Since ξ and u are moduli of continuity such that $\xi(t)/u(t)$ is positive and nondecreasing,

$$\frac{1}{n+1}\int_{1/(n+1)}^{\pi}\frac{\xi(t)}{t^2u(t)}dt \ge \frac{\xi(1/(n+1))}{u(1/(n+1))}\Big(\frac{1}{n+1}\Big)\int_{1/(n+1)}^{\pi}\frac{dt}{t^2} \ge \frac{\xi(1/(n+1))}{2u(1/(n+1))}.$$

Then

$$\frac{\xi(1/(n+1))}{u(1/(n+1))} = O\Big(\frac{1}{n+1}\int_{1/(n+1)}^{\pi}\frac{\xi(t)}{t^2u(t)}dt\Big),$$

and the conclusion follows. □

Theorem 2. *Let $\xi(t)$ be a modulus of continuity such that*

$$\int_0^{\nu} \frac{\xi(t)}{t}\,dt = O(\xi(\nu)), \quad 0 < \nu < \pi. \tag{6}$$

If $f : [0,2\pi] \to \mathbb{R}$ is 2π-periodic, Lebesgue integrable on $[0,2\pi]$ and belongs to the class $L(\xi,r)(r \geq 1)$, then the degree of approximation of f by the Hausdorff means of its Fourier series (1) *is given by*

$$\|(H)_n - f(x)\|_r = O\Big(\frac{1}{n+1}\int_{1/(n+1)}^{\pi} \frac{\xi(t)}{t^2}\,dt\Big) \quad \text{for} \quad n = 0,1,2,\ldots.$$

Proof. Applying condition (7) to (3), I_1 takes the form

$$I_1 = \Big(\int_0^{1/(n+1)} \frac{\xi(t)}{t}\,dt\Big) = O\Big(\xi\Big(\frac{1}{n+1}\Big)\Big),$$

and, using (6), I_2 becomes

$$I_2 = O\Big(\frac{1}{n+1}\int_0^{1/(n+1)} \frac{\xi(t)}{t^2}\,dt\Big).$$

Note that

$$\begin{aligned}\frac{1}{n+1}\int_{1/(n+1)}^{\pi} \frac{\xi(t)}{t^2}\,dt &\geq \frac{1}{(n+1)}\xi\Big(\frac{1}{(n+1)}\Big)\int_{1/(n+1)}^{\pi} \frac{1}{t^2}\,dt \\ &= \xi\Big(\frac{1}{n+1}\Big)\Big\{1 - \frac{1}{(n+1)\pi}\Big\} \geq \frac{1}{2}\xi\Big(\frac{1}{(n+1)}\Big).\end{aligned}$$

It then follows that

$$\xi\Big(\frac{1}{(n+1)}\Big) = O\Big(\frac{1}{(n+1)}\int_{1/(n+1)}^{\pi} \frac{\xi(t)}{t^2}\,dt\Big),$$

and the conclusion follows. □

Theorem 3. *If $f : [0,2\pi] \to \mathbb{R}$ is 2π-periodic, Lebesgue integrable on $[0,2\pi]$ and belongs to the class $Lip\,\alpha$, then the degree of approximation of f by a regular Hausdorff mean of its Fourier series* (1) *satisfies, for $n = 0,1,2,\ldots$,*

$$\|\tilde{t}_n - \tilde{f}\|_\infty = ess \sup_{0\leq x\leq 2\pi}\{\tilde{t}_n(x) - \tilde{f}(x)\} = \begin{cases} O((n+1)^{-\alpha}), 0 < \alpha < 1, \\ O\Big(\frac{\log(n+1)\pi}{(n+1)}\Big), \alpha = 1. \end{cases}$$

Proof.

In this case, $\psi(x,t) = t^{\alpha}$. Therefore, from (3),

$$I_1 = O\Big(\int_0^{1/(n+1)} t^{\alpha-1}\,dt\Big) = O((n+1)^{-\alpha}).$$

Using (6),

$$I_2 = O\Big(\frac{1}{n+1}\Big)\int_{1/(n+1)}^{\pi} t^{\alpha-2}dt = O(n+1)^{-\alpha}).$$

If $\alpha = 1$, then one obtains the result that

$$\|(H)_n - f(x)\|_\infty = O\Big(\frac{\log(n+1)\pi}{(n+1)}\Big). \qquad \square$$

Theorem 3 is Theorem 1 of [6]. However, the proof in [6] contains computational errors.

We now consider the corresponding results for the conjugate series (2).

$$|I_1| = \Big|\frac{1}{2\pi}\int_0^{1/(n+1)} \psi(x,t)\sum_{k=0}^{n}\binom{n}{k}u^k(1-u)^{n-k}\frac{\cos(k+1/2)t}{\sin(t/2)}\Big|d\chi(u)dt.$$

Since $|\cos(k+1/2)|\le 1$, and $\sin(t/2)\ge t/\pi$, (3) is unchanged.

For the conjugate series (2), I_2 takes the form

$$\begin{aligned}
I_2 &= \frac{1}{2\pi}\int_{1/(n+1)}^{\pi}\psi(x,t)\sum_{k=0}^{n}\int_0^1\binom{n}{k}u^k(1-u)^{n-k}d\chi(u)\frac{\cos(k+1/2)t}{\sin t/2}dt\\
&= \frac{1}{2\pi}\int_{1/(n+1)}^{\pi}\frac{\psi(x,t)}{\sin t/2}\int_{)}^{1}\sum_{k=0}^{n}\binom{n}{k}u^k(1-u)^{n-k}d\chi(u)Re(e^{(k+1/2)t)}dt\\
&= \frac{1}{2\pi}\int_{1/(n+1)}^{\pi}\frac{\psi(x,t)}{\sin t/2}\int_0^1 Re\{\sum_{k=0}^{n}\binom{n}{k}u^k(1-u)^{n-k}(e^{(k+1/2)t)}\}|d\chi(u)dt\\
&\le \frac{\|H\|}{2}\int_{1/(n+1)}^{\pi}\Big|\frac{\psi(x,t)}{t}\Big|\int_0^1 Re\Big[e^{it/2}(1-u)^n\sum_{k=0}^{n}\binom{n}{k}\Big(\frac{ue^{ikt}}{(1-u)}\Big)^k\Big]dt
\end{aligned}$$

and one obtains the same expressions as (4) and (5). Therefore $|I_2|$ takes the same form as (6). Consequently the analogues of Theorems 1 - 3 for the conjugate series take the following form.

Theorem 4. *The degree of approximation of a function* $(\tilde{f})$ *, belonging to the the* $Lip(\xi,r)$ *class by the Hausdorff means of the conjugate series (2) is given by*

$$\|\tilde{H}_n - \tilde{f}\| = O\Big(\frac{1}{n+1}\int_{1/(n+1)}^{\pi}\frac{\xi(t)}{t^2u(t)}dt\Big),$$

where $\xi(t)$ *and* $u(t)$ *are the moduli of continuity such that*

$$\int_0^{v}\frac{\xi(t)}{tu(t)}dt = O\Big(\frac{\xi(v)}{u(v)}\Big), \quad 0 < v < \pi.$$

Theorem 5. *Let $\xi(t)$ be a modulus of continuity such that*

$$\int_0^{\nu} \frac{\xi(t)}{t} = O(\xi(\nu)), \quad 0 < \nu < \pi.$$

If f belongs to the class $Lip(\xi, r)(r \geq 1)$, then the degree of approximation of $\tilde{f}$ by the Hausdorff means of the conjugate series (2) *is given by*

$$\|(\tilde{H})_n - \tilde{f}\|_r = O\Big(\frac{1}{n+1}\int_{1/(n+1)}^{\pi} \frac{\xi(t)}{t^2} dt\Big), \quad n = 0, 1, 2, \ldots.$$

Theorem 6. *If $f : [0, 2\pi] \to \mathbb{R}$ is 2π periodic, Lebesgue integrable on $[0, 2\pi]$ and belongs to the class $Lip\,\alpha$, then the degree of approximation of $\tilde{f}$ by the Hausdorff means of the conjugate series* (2) *satisfies, for $n = 0, 1, 2, \ldots$,*

$$\|(\tilde{H}_n) - \tilde{f}\|_\infty = ess \sup_{0 \leq x \leq 2\pi} |(\tilde{H}_n) - \tilde{f}| = \begin{cases} O(n+1)^{-\alpha}, & 0 < \alpha < 1, \\ O\Big(\frac{\log(n+1\alpha}{(n+1)}\Big), & \alpha = 1. \end{cases}$$

Theorems 4 - 6 are generalizations of Theorems 1 - 3 of [3].

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