THE DEGREE OF APPROXIMATION OF FUNCTIONS, AND THEIR CONJUGATES, BELONGING TO SEVERAL GENERAL LIPSCHITZ CLASSES BY HAUSDORFF MATRIX MEANS OF THE FOURIER SERIES AND CONJUGATE SERIES OF A FOURIER SERIES

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Abstract. In this paper Hausdorff matrix approximations are obtained for a function and its conjugate belonging to any one of several generalized Lipschitz classes.

1. Introduction

A number of papers have been written dealing with the degree of approximation of the Fourier series representation of a function, or a conjugate function, by means of the matrix product of \((C, 1)\), the Cesáro matrix of order one, with \((E, q)\), the Euler matrix of order \(q\). (See, e.g. [1] - [5], [7], and [8].)

In [3] three degree of approximation results were obtained for the product of an Euler matrix \((E, q)\) with a Hausdorff matrix and the trigonometric approximation of the conjugate of a function belonging to certain Lipschitz classes. Since the product of two Hausdorff matrices is again a Hausdorff matrix, these theorems suggest that it is possible to prove these results using a single Hausdorff matrix. The main purpose of this paper is to show that this conjecture is true.

A function \(f \in \text{Lip}(\alpha)\) if \(|f(x + t) - f(x)| = O(t^\alpha)\) for \(0 < \alpha \leq 1\). A function \(f \in \text{Lip}(\alpha, r)\) if \(\int_0^{2\pi} |f(x + t) - f(x)|^r \, dx \equiv O(\xi(t))(r \geq 1)\), where \(\xi\) is a modulus of continuity; i.e., \(\xi\) is a nonnegative nondecreasing continuous function with the properties \(\xi(0) = 0\) and \(\xi(t_1 + t_2) \leq \xi(t_1) + \xi(t_2)\). A function \(f \in \text{Lip}(\xi, r)\) if \(\int_0^{2\pi} |f(x + t) - f(x)|^r \, dx \equiv O(\xi(t))(r \geq 1)\), where \(\xi\) is a modulus of continuity; i.e., \(\xi\) is nonnegative, nondecreasing, and continuous with the properties \(\xi(0) = 0, \xi(t_1 + t_2) \leq \xi(t_1) + \xi(t_2)\).

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Let \( f(x) \) be a \( 2\pi \)-periodic function, Lebesgue integrable on \([0, 2\pi]\) and belonging to any of the Lipschitz classes defined above. The Fourier series for \( f(x) \) is given by

\[
f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} A_n(x),
\]

(1)

with \( n \)th partial sum \( s_n(f; x) \).

The series

\[
\sum_{n=1}^{\infty} (a_n \cos nx - b_n \sin nx) = -\sum_{n=1}^{\infty} B_n(x)
\]

(2)

is called the conjugate series of series (1), with \( n \)th partial sum \( \tilde{s}_n(f; x) \).

A Hausdorff matrix is a lower triangular matrix with nonzero entries

\[
h_{nk} = \binom{n}{k} \Delta^{n-k} \mu_k,
\]

where \( \Delta \) is the forward difference operator defined by \( \Delta \mu_k = \mu_k - \mu_{k+1} \), and \( \Delta^{n+1} \mu_k = \Delta(\Delta^n \mu_k) \). A Hausdorff matrix is regular; i.e. preserves the limit of each convergent sequence, if and only if

\[
\int_0^1 |d\chi(u)| < \infty,
\]

where the mass function \( \chi \in BV[0,1], \chi(0+) = \chi(0) \) and \( \chi(1) = 1 \). In this case the \( \mu_n \) have the representation

\[
\mu_n = \int_0^1 u^n d\chi(u).
\]

For any sequence \( \{s_n\} \),

\[
t_n := \sum_{k=0}^{n} h_{nk} s_k.
\]

The norm \( L_r = \|\|_r \) is defined by

\[
\|f\|_r = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^r \, dx \right\}^{1/r}, \quad r \geq 1,
\]

and the degree of trigonometric approximation \( H_n(f) \) will be denoted by

\[
H_n(f) = \min \|f - t_n\|_r.
\]

The norm \( \|\|_r^{(\xi)} \) on the class of functions \( L_r^{(\xi)} \) is defined by

\[
\|f\|_r^{(\xi)} = \sup_{t \neq 0} \frac{\|f(\cdot + t) - f(\cdot)\|^r}{\xi(|t|)}.
\]
For any regular Hausdorff transform, and \( \psi(t,x) \) denoting one of the Lipschitz conditions,

\[
(H)_n - f(x) = \frac{1}{\pi} \int_0^\pi \frac{\psi(x,t)}{\sin(t/2)} \sum_{k=0}^n h_{nk} \sin(k + \frac{1}{2}) t dt
\]

\[= I_1 + I_2, \text{ say.} \]

For \( 0 < t \leq 1/(n+1), |\sin t| \leq 1, \) and \( \sin(t/2) \geq (t/\pi) \). Therefore

\[
|I_1| = \left| \frac{1}{2\pi} \int_0^{1/(n+1)} \psi(x,t) \sum_{k=0}^n \int_0^1 \left( \begin{array}{c} n \\ k \end{array} \right) u^k (1-u)^{n-k} d\chi(u) \frac{\sin(k + \frac{1}{2}) t}{\sin t/2} dt \right|
\]

\[
\leq \frac{1}{2\pi} \int_0^{1/(n+1)} |\psi(x,t)| \left| \sum_{k=0}^n \int_0^1 \left( \begin{array}{c} n \\ k \end{array} \right) u^k (1-u)^{n-k} d\chi(u) \frac{\sin(k + \frac{1}{2}) t}{\sin t/2} dt \right|
\]

\[
\leq \frac{1}{2\pi} \int_0^{1/(n+1)} |\psi(x,t)| \int_0^1 \sum_{k=0}^n \left( \begin{array}{c} n \\ k \end{array} \right) u^k (1-u)^{n-k} d\chi(u) dt
\]

\[= \frac{\|H\|}{2} \int_0^{1/(n+1)} \frac{|\psi(x,t)|}{t} dt = O\left( \int_0^{1/(n+1)} \frac{|\psi(x,t)|}{t} dt \right)
\]

(3)

\[
I_2 = \frac{1}{2\pi} \int_0^\pi \frac{\psi(x,t)}{\sin(t/2)} \sum_{k=0}^n \int_0^1 \left( \begin{array}{c} n \\ k \end{array} \right) u^k (1-u)^{n-k} d\chi(u) Im(e^{i(k+1/2)t}) dt
\]

\[
= \frac{1}{2\pi} \int_0^\pi \frac{\psi(x,t)}{\sin(t/2)} \int_0^1 Im\left[ \sum_{k=0}^n \left( \begin{array}{c} n \\ k \end{array} \right) u^k (1-u)^{n-k} e^{i(k+1/2)t} \right] d\chi(u) dt.
\]

\[
\sum_{k=0}^n \left( \begin{array}{c} n \\ k \end{array} \right) u^k (1-u)^{n-k} e^{i(k+1/2)t} = (1-u)^n e^{it/2} \sum_{k=0}^n \left( \begin{array}{c} n \\ k \end{array} \right) \left( \frac{ue^{it}}{1-u} \right)^n
\]

\[
= (1-u)^n e^{it/2} \left( 1 + \frac{ue^{it}}{1-u} \right)^n
\]

\[
= e^{it/2} (1-u + ue^{it})^n.
\]

Define

\[
|g(u,t)| = |Im| e^{it/2} (1-u + ue^{it})^n | \leq r^n(u,t),
\]

(4)

where

\[
r^2(u,t) = (1-u + u \cos t)^2 + (u \sin t)^2
\]

\[= 1 - 2u + 2u(1-u) \cos t + 2u^2 = 1 - 2u(1-u) + 2u(1-u) \cos t
\]

\[= 1 - 2u(1-u)(1 - \cos t) = 1 - 4u(1-u)/\sin^2(t/2)
\]
Proof. From properties of the class $\text{Li p}(\xi, r)$ and the conclusion follows. □□□

and

where $\xi$ means of the Fourier series (1)

Then (3) becomes

where $h(t)$ and $u(t)$ are moduli of continuity such that

Thus

The degree of approximation of a function $f$ belonging to the class $\text{Li p}(\xi, r)$ by means of the Fourier series (1) satisfies

where $\xi(t)$ and $u(t)$ are the modulus of continuity such that

Proof. From properties of the class $\text{Li p}(\xi, r)$ established in [3], (3) becomes

and (6) becomes

Thus

and

Since $\xi$ and $u$ are moduli of continuity such that $\xi(t)/u(t)$ is positive and nondecreasing,

Then

and the conclusion follows. □
Theorem 2. Let $\xi(t)$ be a modulus of continuity such that

$$\int_0^\nu \frac{\xi(t)}{t} \, dt = O(\xi(\nu)), \quad 0 < \nu < \pi. \tag{6}$$

If $f : [0, 2\pi] \to \mathbb{R}$ is $2\pi$-periodic, Lebesgue integrable on $[0, 2\pi]$ and belongs to the class $L(\xi, r)(r \geq 1)$, then the degree of approximation of $f$ by the Hausdorff means of its Fourier series (1) is given by

$$\| (H)_n f(x) \|_r = O\left( \frac{1}{n+1} \int_{1/(n+1)}^{\pi} \frac{\xi(t)}{t^2} \, dt \right) \quad \text{for} \quad n = 0, 1, 2, \ldots.$$ 

Proof. Applying condition (7) to (3), $I_1$ takes the form

$$I_1 = \left( \int_0^{1/(n+1)} \frac{\xi(t)}{t} \, dt \right) = O\left( \xi\left( \frac{1}{n+1} \right) \right),$$

and, using (6), $I_2$ becomes

$$I_2 = O\left( \frac{1}{n+1} \int_{0}^{1/(n+1)} \frac{\xi(t)}{t^2} \, dt \right).$$

Note that

$$\frac{1}{n+1} \int_{1/(n+1)}^{\pi} \frac{\xi(t)}{t^2} \, dt \geq \frac{1}{n+1} \xi\left( \frac{1}{n+1} \right) \int_{1/(n+1)}^{\pi} \frac{1}{t^2} \, dt \quad \text{for} \quad n = 0, 1, 2, \ldots,$$

$$= \xi\left( \frac{1}{n+1} \right) \left\{ 1 - \frac{1}{(n+1)\pi} \right\} \geq \frac{1}{2} \xi\left( \frac{1}{n+1} \right).$$

It then follows that

$$\xi\left( \frac{1}{n+1} \right) = O\left( \frac{1}{n+1} \int_{1/(n+1)}^{\pi} \frac{\xi(t)}{t^2} \, dt \right),$$

and the conclusion follows. \qed

Theorem 3. If $f : [0, 2\pi] \to \mathbb{R}$ is $2\pi$-periodic, Lebesgue integrable on $[0, 2\pi]$ and belongs to the class $Lip \alpha$, then the degree of approximation of $f$ by a regular Hausdorff mean of its Fourier series (1) satisfies, for $n = 0, 1, 2, \ldots$,

$$\| \tilde{f}_n - f \|_\infty = ess \sup_{0 \leq x \leq 2\pi} \left| \tilde{f}_n(x) - \tilde{f}(x) \right| = \begin{cases} O((n+1)^{-a}), & 0 < \alpha < 1, \\ O\left( \frac{\log(n+1)}{(n+1)} \right), & \alpha = 1. \end{cases}$$

Proof. In this case, $\psi(x, t) = t^\alpha$. Therefore, from (3),

$$I_1 = O\left( \int_0^{1/(n+1)} t^{\alpha-1} \, dt \right) = O((n+1)^{-\alpha}).$$
Using (6),
\[ I_2 = O\left(\frac{1}{n+1}\right) \int_{1/(n+1)}^{\pi} t^{\alpha-2} \, dt = O(n+1)^{-\alpha}. \]

If \( \alpha = 1 \), then one obtains the result that
\[ \|(H)_n - f(x)\|_\infty = O\left(\frac{\log(n+1)\pi}{(n+1)}\right). \]

\[ \square \]

Theorem 3 is Theorem 1 of [6]. However, the proof in [6] contains computational errors.

We now consider the corresponding results for the conjugate series (2).

\[ |I_1| = \frac{1}{2\pi} \int_0^{1/(n+1)} \psi(x, t) \left( \sum_{k=0}^{n} \binom{n}{k} u^k(1-u)^{n-k} \frac{\cos(k+1/2)t}{\sin(t/2)} \right) d\chi(u) \, dt. \]

Since \( |\cos(k+1/2)| \leq 1 \), and \( \sin(t/2) \geq t/\pi \), (3) is unchanged.

For the conjugate series (2), \( I_2 \) takes the form
\[
I_2 = \frac{1}{2\pi} \int_{1/(n+1)}^{\pi} \psi(x, t) \sum_{k=0}^{n} \binom{n}{k} u^k(1-u)^{n-k} \frac{\cos(k+1/2)t}{\sin(t/2)} \, dt \\
= \frac{1}{2\pi} \int_{1/(n+1)}^{\pi} \psi(x, t) \sin(t/2) \int_{0}^{1} \sum_{k=0}^{n} \binom{n}{k} u^k(1-u)^{n-k} \cos(k+1/2)t \, d\chi(u) \, dt \\
= \frac{1}{2\pi} \int_{1/(n+1)}^{\pi} \frac{\psi(x, t)}{\sin(t/2)} \int_{0}^{1} \Re\left\{ e^{i(k+1/2)t} \sum_{k=0}^{n} \binom{n}{k} u^k(1-u)^{n-k} \right\} d\chi(u) \, dt \\
\leq \frac{\|H\|}{2} \int_{1/(n+1)}^{\pi} \frac{|\psi(x, t)|}{t} \int_{0}^{1} \Re\left\{ e^{i/2}(1-u)^n \sum_{k=0}^{n} \binom{n}{k} \left( \frac{ue^{ikt}}{(1-u)} \right)^k \right\} \, dt \\
\]
and one obtains the same expressions as (4) and (5). Therefore \( |I_2| \) takes the same form as (6). Consequently the analogues of Theorems 1 - 3 for the conjugate series take the following form.

**Theorem 4.** The degree of approximation of a function \((\tilde{f})\), belonging to the the \( \text{Li p}(\xi, r) \) class by the Hausdorff means of the conjugate series (2) is given by

\[ \|\tilde{H}_n - \tilde{f}\| = O\left(\frac{1}{n+1}\int_{1/(n+1)}^{\pi} \frac{\xi(t)}{t^2u(t)} \, dt\right), \]

where \( \xi(t) \) and \( u(t) \) are the moduli of continuity such that

\[ \int_{0}^{\nu} \frac{\xi(t)}{tu(t)} \, dt = O\left(\frac{\xi(v)}{u(v)}\right), \quad 0 < \nu < \pi. \]
Theorem 5. Let $\xi(t)$ be a modulus of continuity such that
\[
\int_0^\nu \frac{\xi(t)}{t} = O(\xi(\nu)), \quad 0 < \nu < \pi.
\]

If $f$ belongs to the class $Lip(\xi, r)(r \geq 1)$, then the degree of approximation of $\tilde{f}$ by the Hausdorff means of the conjugate series (2) is given by
\[
\| (\tilde{H}_n - \tilde{f}) \|_r = O\left(\frac{1}{n+1} \int_1^{\pi/(n+1)} \frac{\xi(t)}{t^2} dt\right), \quad n = 0, 1, 2, \ldots.
\]

Theorem 6. If $f : [0, 2\pi] \to \mathbb{R}$ is $2\pi$ periodic, Lebesgue integrable on $[0, 2\pi]$ and belongs to the class $Lip(\alpha)$, then the degree of approximation of $\tilde{f}$ by the Hausdorff means of the conjugate series (2) satisfies, for $n = 0, 1, 2, \ldots$,
\[
\| (\tilde{H}_n - \tilde{f}) \|_\infty = \text{ess sup}_{0 \leq x \leq 2\pi} |(\tilde{H}_n) - \tilde{f}| = \begin{cases} 
O(n+1)^{-\alpha}, & 0 < \alpha < 1, \\
O\left(\log\left(\frac{n+1}{n+1}\right)\right), & \alpha = 1.
\end{cases}
\]

Theorems 4 - 6 are generalizations of Theorems 1 - 3 of [3].

References


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