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ON WEAKLY PERIODIC-LIKE RINGS AND COMMUTATIVITY THEOREMS

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Abstract. A ring R is called periodic if, for every x in R, there exist distinct positive integers m and n such that $x^m = x^n$. An element x of R is called potent if $x^k = x$ for some integer $k > 1$. A ring R is called weakly periodic if every x in R can be written in the form $x = a + b$ for some nilpotent element a and some potent element b in R. A ring R is called weakly periodic-like if every element x in R which is not in the center C of R can be written in the form $x = a + b$, with a nilpotent and b potent. Some structure and commutativity theorems are established for weakly periodic-like rings R satisfying certain torsion-freeness hypotheses along with conditions involving some elements being central.

Throughout, R is an associative ring, N is the set of nilpotent elements of R, C is the center of R, $C(R)$ is the commutator ideal of R, $[x, y]$ denotes the usual commutator $xy - yx$, and J denotes the Jacobson radical of R. We now state formally the definition of a weakly periodic-like ring.

Definition 1. A ring R is called *weakly periodic-like* if every x in $R\backslash C$ can be written in the form

$$x = a + b, \quad a \in N, \quad b \text{ potent } (b^k = b \text{ for some } k > 1), \quad x \in R \setminus C. \tag{1}$$

In preparation for the proofs of the main theorems, we state the following known lemmas.

Lemma 1.([3]) *If $[x, y]$ commutes with x, then $[x^k, y] = kx^{k-1}[x, y]$ for all positive integers k.*

Lemma 2.([10]) *Suppose R is a ring with identity 1. If $x^m[x, y] = 0$ and $(x + 1)^m[x, y] = 0$ for some x, y in R and some positive integer m, then $[x, y] = 0$. A similar statement holds if we assume $[x, y]x^m = 0$ and $[x, y](x + 1)^m = 0$ instead.*

Lemma 3.([4]) *Let R be an n-torsion-free ring with identity such that $[x^n, y^n] = 0$ for all x, y in R. Let N denote the set of nilpotent elements of R. Then*

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(i) *$a \in N$, $x \in R$ imply $[a, x^n] = 0$.*
(ii) *$a \in N$, $b \in N$ imply $[a, b] = 0$.*
(iii) *If N is contained in the center of R, then R is commutative.*

Lemma 4.([1]) *Suppose R is a ring with center C and N is the set of nilpotent elements of R. Suppose that (a) N is commutative; (b) for all a in N and all x in R, $ax - xa$ commutes with x; (c) for all $x \in R$, we have $x \in C$ or $x^n - x \in N$ for some $n = n(x) > 1$. Then R is commutative.*

Lemma 5.([6]) *Suppose R is a ring such that for all x, y in R, $[x^n, y^n] = 0$, where n is a fixed positive integer. Then the commutator ideal of R is nil, and hence N is an ideal of R.*

Lemma 6.([7]) *Suppose R is a ring such that for every x in R, there exists an integer $n = n(x) > 1$ such that $x - x^n$ is the center of R. Then R is commutative.*

Lemma 7.([9]) *Suppose R is a ring with identity and m and n are fixed positive integers which are relatively prime. Suppose that, for all x, y in R,*

$$[x^n, y^n] = 0 \quad \text{and} \quad [x^m, y^m] = 0.$$

Then R is commutative.

Lemma 8.([11]) *Let R be a weakly periodic-like ring, J the Jacobson radical of R, and $C(R)$ the commutator ideal of R. IF $C(R) \subseteq J$, then the set N of nilpotents is an ideal and R/N is commutative.*

We are now in a position to prove our main theorems. In the first theorem, we do not assume that the ground ring R is weakly periodic-like.

Theorem 1. *Let R be a ring with identity 1 and let $n > 1$ be a fixed integer. Suppose R is n-torsion-free and satisfies $[x^n, y^n] = 0$ for all x, y in R. Suppose, further, that for all x, y in R, either $(xy)^{n+1} - (yx)^{n+1} \in C$ or $(xy)^{n-1} - y^{n-1}x^{n-1} \in C$, where C is the center of R. Then R is commutative.*

Proof. By Lemma 3, the set N of nilpotent elements of R is commutative and

$$[a, b^n] = 0 \text{ for all } a \text{ in } N \text{ and all } b \text{ in } R. \tag{2}$$

Moreover, by Lemma 5, N is an ideal of R and hence (since N is commutative)

$$N^2 \subseteq C \quad (C = \text{ center of } R). \tag{3}$$

Let $a \in N$, $b \in R$. We now consider the two possibilities in the hypothesis of the theorem.

CASE 1. $(xy)^{n+1} - (yx)^{n+1} \in C$.

In this case,

$$\Big((a+1)b\Big)^{n+1} - \Big(b(a+1)\Big)^{n+1} \in C. \tag{4}$$

Since N is a commutative ideal of R and $N^2 \subseteq C$, a close look at (4) shows that

$$[a, b^{n+1}] \in C, \tag{5}$$

and hence

$$[a, b^{n+1}] = [a, b]b^n + b[a, b^n] \in C. \tag{6}$$

But , by (2), $[a, b^n] = 0$ and hence (6) implies that $[a, b]b^n \in C$, which, in turn, implies that $[[a, b]b^n, b] = 0$ and hence

$$\Big[[a, b]b\Big]b^n = 0, \quad (a \in N, b \in R). \tag{7}$$

CASE 2. $(xy)^{n-1} - y^{n-1}x^{n-1} \in C$.
As shown above, we have

$$N \text{ is a commutative ideal, } N^2 \subseteq C,\ [a, b^n] = 0 \text{ for all } a \text{ in } N, b \text{ in } R. \tag{8}$$

We may assume $((a+1)b)^{n+1} - (b(a+1))^{n+1}$ is not in C, and hence $(b(a+1))^{n+1} - ((a+1)b)^{n+1}$ is not in C. Therefore,

$$\Big((a+1)b\Big)^{n-1} - b^{n-1}(a+1)^{n-1} \in C, \tag{9}$$

and

$$\Big(b(a+1)\Big)^{n-1} - (a+1)^{n-1}b^{n-1} \in C. \tag{10}$$

By subtracting the two expressions in (9) and (10), and using the facts in (8), it can be seen that

$$[a, b^{n-1}] - \{-(n-1)[a, b^{n-1}]\} \in C,$$

and hence (since R is n-torsion-free)

$$[a, b^{n-1}] \in C \text{ for all } a \text{ in } N \text{ and all } b \text{ in } R. \tag{11}$$

Moreover, since $[a, b^n] = 0$ (see (8)), we have

$$0 = [a, b^n] = [a, b]b^{n-1} + b[a, b^{n-1}], \tag{12}$$

and hence

$$[a, b]b^{n-1} = -b[a, b^{n-1}]. \tag{13}$$

By (11), the right side of (13) commutes with b, and hence, $[a, b]b^{n-1}$ commutes with b. Thus,

$$\Big[[a, b]b^{n-1}, b\Big] = 0,$$

and hence

$$[a,b]b^n - b[a,b]b^{n-1} = 0,$$

which implies

$$\{[a,b]b - b[a,b]\}b^{n-1} = 0.$$

Therefore, $[[a,b],b]b^{n-1} = 0$, and hence

$$\Big[[a,b],b\Big]b^n = 0, \quad (a \in N, b \in R). \tag{14}$$

In view of (7) and (14), we see that

$$\Big[[a,b]b\Big]b^n = 0 \text{ in both cases}, \quad (a \in N, b \in R). \tag{15}$$

Replacing b by $b+1$ in the aove argument shows that (see (15))

$$\Big[[a,b],b\Big](b+1)^n = 0. \tag{16}$$

Combining (15), (16), and Lemma 2, we get

$$\Big[[a,b],b\Big] = 0, \quad (a \in N, b \in R). \tag{17}$$

By (2), $[a,b^n] = 0$, which when combined with (17) and Lemma 1 yields $nb^{n-1}[a,b] = 0$. Since R is n-torsion-free, it follows that

$$b^{n-1}[a,b] = 0, \quad (a \in N, b \in R). \tag{18}$$

Replacing b by $b+1$ in the above argument, we see that (see (18))

$$(b+1)^{n-1}[a,b] = 0, \quad (a \in N, b \in R). \tag{19}$$

Combining (18), (19), and Lemma 2, we otain

$$[a,b] = 0 \text{ for all } a \in N \text{ and all } b \in R. \tag{20}$$

The theorem now follows from (20) and Lemma 3(iii).

We now turn our attention to weakly periodic-like rings. We begin with the following main structure theorem.

Theorem 2. *Let R be a weakly periodic-like ring (not necessarily with identity), and let N be the set of nilpotent elements of R. Suppose P is a ring property which is satisfied by every subring S of R and also by every homomorphic image of the subring S. Suppose, further, that P is not satisfied by any complete matrix ring of $n \times n$ matrices over any division ring, where $n > 1$. Then*

(i) *N is an ideal of R.*

(ii) *R/N is commutative.*
(iii) *If x is in R, then either $x \in C$ or $x - x^m \in N$ for some integer $m > 1$.*

Proof. Let J be the Jacobson radical of R. Then

$$R/J \cong \text{ a subdirect sum of primitive rings } R_i, \quad i \in \Gamma. \tag{21}$$

In view of Jacobson's density theorem [8; p.33], the ring property P hypothesis guarantees that each primitive ring R_i in (21) nust be a division ring, and hence (21) yields

$$R/J \cong \text{ a subdirect sum of division rings } R_i, \quad i \in \Gamma. \tag{22}$$

Furthermore, since the homomorphic image of a weakly periodic-like ring is also weakly periodic-like, each division ring R_i in (22) is weakly periodic-like. Hence (see Definition 1), for any x_i in R_i, we have

$$x_i \in \text{ Center of } R_i \text{ or } x_i \text{ is potent } (x_i^k = x_i \text{ for some } k > 1). \tag{23}$$

Hence, for any x_i in R_i, $x_i - x_i^k \in$ Center of R_i, where $k > 1$, and thus by Lemma 6, R_i is commutative. Hence, by (22), R/J is commutative, and thus

$$C(R) \subseteq J \quad (C(R) \text{ denotes the commutator ideal of } R). \tag{24}$$

Parts (i) and (ii) of the theorem now follow from (24) and Lemma 8.

To prove part (iii), suppose $x \in R \setminus C$. Then, by Definition 1,

$$x = a + b;\ a \in N, b \text{ potent satisfying } b^m = b,\ m > 1.$$

Hence,

$$x - a = b = b^m = (x - a)^m,$$

which implies that $x - x^m \in N$, since N is an ideal (see part(i)). This proves part (iii), and the theorem is proved.

The following lemma will be needed for the proofs of the remaining theorems.

Lemma 9. *Let R be a weakly periodic-like ring with the set of nilpotents commutative and with idempotents central. If R has a property which implies commutativity in weakly periodic-like rings with identity and which is inherited by ideals, then R is commutative.*

Proof. First, we prove that the set P_0 of potent elements is central. Suppose

$$a \in P_0 \text{ with } a^n = a, \quad n > 1. \tag{25}$$

Let $e = a^{n-1}$. Then, since e is central idempotent, eR is a ring with identity. Indeed, eR is weakly periodic-like which, in fact, is an ideal of R. The hypothesis of the lemma (referring to the property inheritance) guarantees that

$$eR \text{ is commutative.} \tag{26}$$

Therefore,

$$(ea)(ex) = (ex)(ea) \text{ for all } x \text{ in } R. \tag{27}$$

Recall that $e = a^{n-1}$ is a *central* idempotent element of R, and hence (27) implies that

$$eax = exa = xae; \text{ that is, } a^{n-1}ax = xaa^{n-1}, \text{ or } a^n x = xa^n,$$

and thus, by (25), $ax = xa$ for all x in R. This proves that

$$\text{The set } P_0 \text{ of potent elements of } R \text{ is central.} \tag{28}$$

To complete the proof, suppose $x,\ y \in R$. If $x \in C$ or $y \in C$, then clearly $[x, y] = 0$. So suppose $x \notin C$ and $y \notin C$. Then, by Definition 1,

$$x = a + b,\ y = a' + b';\ a,\ a' \text{ nilpotent and } b,\ b' \text{ potent.} \tag{29}$$

By (28), b and b' are central, and hence

$$[x, y] = [a + b, a' + b'] = [a, a'] = 0, \text{ (since } N \text{ is commutative).}$$

This proves the lemma.

Next, we prove

Theorem 3. *Let R be a weakly periodic-like ring, and let n be a fixed positive integer. Suppose R is n-torsion-free and, for all $x,\ y$ in R, $(xy)^n - (yx)^n \in C$ (the center of R). Suppose, further, that the set N of nilpotents is commutative. Then, R is commutative.*

Proof. Let P be the ring property "$(xy)^n - (yx)^n$ is always central." Clearly, this property is satisfied by all subrings and all homomorphic images of any subring of R. Moreover, this property P is not satisfied by any complete matrix ring D_n of $n \times n$ matrices over any division ring D, where $n > 1$, as can be seen by taking x and y (in D_n) to be

$$x = E_{11}, \quad y = E_{11} + E_{12}, \quad (E_{11}, E_{12} \in D_n).$$

Hence, by Theorem 2(iii), we have

$$\text{If } x \in R \setminus C, \text{ then } x - x^m \in N \text{ for some integer } m > 1. \tag{30}$$

Moreover, by Theorem 2(i), N is an ideal which is commutative (by hypothesis). The net result is:

$$N \text{ is a commutative ideal and hence } N^2 \subseteq C. \tag{31}$$

We now distinguish two cases

CASE 1. $1 \in R$. Suppose $a \in N$, $b \in R$. Then, by hypothesis,

$$\Big((a + 1)b\Big)^n - \Big(b(a + 1)\Big)^n \in C. \tag{32}$$

In view of (31), an easy argument shows that (32) implies

$$[a, b^n] \in C \quad (a \in N, b \in R). \tag{33}$$

Let S be the subring of R generated by the set of all elements x^n, $x \in R$. By Theorem 2(ii), R/N is commutative, and hence

$$(x_1 \cdots x_k)^n - x_1^n \cdots x_k^n \in N \text{ for all } x_1, \ldots, x_k \text{ in } R.$$

Since N is commutative (see (31)), therefore

$$\Big[a, (x_1 \ldots x_k)^n\Big] = [a, x_1^n \ldots x_k^n], \quad (a \in N). \tag{34}$$

Moreover, by (33), the commutator on the left side of (34) is central, and hence

$$[a, x_1^n \ldots x_k^n] \in C \text{ for all } x_1, \ldots, x_k \text{ in } R. \tag{35}$$

Since the commutator is linear with respect to the second argument (in particular), (35) readily implies that

$$[a, x] \in C(S) \text{ for all } a \in N(S) \text{ and all } x \in S, \tag{36}$$

where $N(S)$ and $C(S)$ denote the set of nilpotents of S and the center of S, respectively. Also, by Theorem 2(iii), we have:

$$\text{If } x \in S, \text{ then } x \in C(S) \text{ or } x - x^m \in N(S) \text{ for some } m > 1. \tag{37}$$

Moreover, by (31),

$$N(S) \text{ is commutative.} \tag{38}$$

Combining (36), (37), (38), and Lemma 4, we conclude that S is commutative, and hence

$$[x^n, y^n] = 0 \text{ for all } x, y \text{ in } R. \tag{39}$$

Since R is n-torsion-free and satisfies (39), therefore by Lemma 3(i),

$$[a, x^n] = 0 \text{ for all } a \in N, x \in R. \tag{40}$$

Let $a \in N$, $u = 1 + a$. Then, by (40)

$$[u, x^n] = 0, \quad (u = 1 + a,\ a \in N). \tag{41}$$

By hypothesis,

$$\Big((x^{n-1}u)x\Big)^n - \Big(x(x^{n-1}u)\Big)^n \in C,$$

and hence

$$x^{n-1}ux^n ux^n \cdots ux^n(ux) - (x^n u)^n \in C. \tag{42}$$

Combining (41) and (42), we obtain

$$x^{n^2-1}u^n x - x^{n^2}u^n \in C,$$

and hence

$$x^{n^2-1}[u^n, x] \in C, \quad (u = 1 + a,\ a \in N). \tag{43}$$

By (31), $N^2 \subseteq C$, and hence

$$[u^n, x] = \Big[(1+a)^n, x\Big] = [na, x] = n[a, x]. \tag{44}$$

Combining (43), (44), we obtain

$$nx^{n^2-1}[a, x] \in C, \quad (a \in N,\ x \in R).$$

Since R is n-torsion-free, this implies that $x^{n^2-1}[a, x] \in C$, and hence

$$\Big[x^{n^2-1}[a, x], x\Big] = 0,$$

which implies

$$x^{n^2-1}\Big[[a, x], x\Big] = 0, \quad (a \in N,\ x \in R). \tag{45}$$

Replacing x by $x + 1$ in the above argument, we get

$$(x+1)^{n^2-1}\Big[[a, x], x\Big] = 0, \quad (a \in N,\ x \in R). \tag{46}$$

Combining (45), (46), and Lemma 2, we conclude that

$$\Big[[a, x], x\Big] = 0, \quad (a \in N,\ x \in R). \tag{47}$$

The theorem now follows from (30), (47), and the hypothesis that N is commutative (see Lemma 4).

CASE 2. R does not have an identity. In this case, we first prove the following:

CLAIM. The idempotents of R are central. Let $e^2 = e \in R$. By hypothesis,

$$\Big(e(e + er - ere)\Big)^n - \Big((e + er - ere)e\Big)^n \in C,$$

and hence $er - ere \in C$. Therefore,

$$er - ere = e(er - ere) = (er - ere)e = 0,$$

and thus $er = ere$. Similarly, $re = ere$, and the claim follows.

Since the properties of torsion-freeness and "$(xy)^n - (yx)^n$ is always central" are inherited by any ideal of R, the theorem now follows from Case 1 and Lemma 9. This completes the proof.

The following result was proved by the authors in [2], and is a corollary of Theorem 3.

Corollary 1. *Let n be a fixed positive integer and let R be an n-torsion-free periodic ring such that $(xy)^n - (yx)^n$ is central for all $x,\ y$ in R. If the set N of nilpotents of R is commutative, then R is commutative.*

This follows from Theorem 3, since a periodic ring is weakly periodic [5], and hence is also weakly periodic-like.

Theorem 4. *Suppose R is a weakly periodic-like ring, and suppose m and n are positive integers which are relatively prime. Suppose that*

$$(xy)^n - (yx)^n \in C \text{ and } (xy)^m - (yx)^m \in C \quad (C \text{ is center of } R).$$

Suppose, further, that the set N of nilpotents is commutative, Then R is commutative.

Proof. A careful examination of the proof of Theorem 3 shows that the hypothesis that R is n-torsion$-$free was *not* used in the proof of (31), and thus

$$N \text{ is a commutative ideal of } R \text{ and } N^2 \subseteq C. \tag{48}$$

CASE 1. $1 \in R$. Again, the proof of (39) did *not* use the hypopthesis that R is n-torsion-free, and hence

$$[x^n, y^n] = 0 \text{ for all } x, y \text{ in } R. \tag{49}$$

Similarly, by making use of the hypothesis "$(xy)^m - (yx)^m$ is central", we see that (see (49))

$$[x^m, y^m] = 0 \text{ for all } x, y \text{ in } R. \tag{50}$$

Hence, by (49), (50), and Lemma 7, we conclude that R is commutative (recall that $1 \in R$).

CASE 2. $1 \notin R$. Using the argument in the proof of the Claim of Theorem 3, we see that

$$\text{The idempotents of } R \text{ are central.} \tag{51}$$

As indicated in the proof of Case 2 of Theorem 3, the theorem now follows from Case 1 and Lemma 9. This completes the proof.

Recalling that a periodic ring is weakly periodic-like [5], and taking $m = 1$, $n = 1$ in Theorem 4, we obtain the following

Corollary 2. *A periodic ring with commuting nilpotents and central commutators is commutative.*

Theorem 5. *Suppose R is a weakly periodic-like ring, and suppose m and n are positive integers which are relatively prime. Suppose that R is mn-torsion-free, and*

$$[x^n, y^n] \in C \text{ and } [x^m, y^m] \in C \text{ for all } x, y \text{ in } R.$$

Suppose, further, that the set N of nilpotents is commutative. Then R is commutative.

Proof. Let P be the ring property "$[x^n, y^n] \in C$ for all x, y in R." Clearly, this property is satisfied by all subrings and all homomorphic images of any subring of R. Moreover, this property P is not satisfied by any complete matrix ring D_n of $n \times n$ matrices over any division ring D, where $n > 1$, as can be seen by taking

$$x = E_{11}, \quad y = E_{11} + E_{12}, \quad (E_{11}, E_{12} \in D_n).$$

Hence, by Theorem 2(i),

$$N \text{ is an ideal of } R, \quad (N \text{ is the set of nilpotents}). \tag{52}$$

Moreover,

$$N \text{ is commutative, and hence } N^2 \subseteq C \text{ (see (52))}. \tag{53}$$

We now distinguish two cases.

CASE 1. $1 \in R$. Let $a \in N$, $b \in R$. Then, $[(1+a)^n, b^n] \in C$, which when combined with (52) and (53) yields $n[a, b^n] \in C$. Since R is mn-torsion-free,

$$[a, b^n] \in C \text{ for all } a \in N, b \in R. \tag{54}$$

This is precisely (33) above. By repeating the argument used in (33) through (39) above, we conclude that

$$[x^n, y^n] = 0 \text{ for all } x, y \text{ in } R. \tag{55}$$

Similarly, using the hypothesis $[x^m, y^m] \in C$, the above argument yields

$$[x^m, y^m] = 0 \text{ for all } x, y \text{ in } R. \tag{56}$$

Hence, by (55), (56), and Lemma 7, we conclude that R is commutative (recall that $1 \in R$).

CASE 2. $1 \notin R$. First, note that the idempotents are central. To see this, let $e^2 = e \in R$, $r \in R$. By hypothesis,

$$\Big[e^n, (e + er - ere)^n\Big] \in C,$$

and hence $er - ere \in C$. Thus,

$$er - ere = e(er - ere) = (er - ere)e = 0,$$

and hence $er = ere$. Similarly, $re = ere$, and hence

$$\text{All idempotents of } R \text{ are central.} \tag{57}$$

Arguing as we did in the proof of Case 2 of Theorem 3, the theorem now follows from Case 1 and Lemma 9. This completes the proof.

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