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CLASSES WITH NEGATIVE COEFFICIENTS AND STARLIKE WITH RESPECT TO OTHER POINTS II

SUZEINI ABDUL HALIM, AINI JANTENG AND MASLINA DARUS

Abstract. A class $S_s^\star T(\alpha,\beta,\sigma,k)$ of functions f which are analytic and univalent in the open unit disk $D=\{z:|z|<1\}$ given by $f(z)=z-\sum_{n=2}^{\infty}a_nz^n$ and satisfying the condition

$$\left|\frac{zf'(z)}{f(z)-f(-z)}-k\right|<\beta\left|\frac{\alpha zf'(z)}{f(z)-f(-z)}-(2\sigma-k)\right|$$

for $0\le\alpha\le 1, 0<\beta\le 1, 0\le\sigma\le\frac{1}{2}<k\le 1, z\in D$, is introduced and studied. An analogous class $S_c^\star T(\alpha,\beta,\sigma,k)$ and $S_{sc}^\star T(\alpha,\beta,\sigma,k)$ are also examined.

1. Introduction

Let $\mathcal{S}$ be the class of functions f which are analytic and univalent in the open unit disk $D=\{z:|z|<1\}$ given by

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{1}$$

and a_n a complex number. Let $S_s^\star$ be the subclass of $\mathcal{S}$ consisting of functions given by (1) satisfying

$$Re\left\{\frac{zf'(z)}{f(z)-f(-z)}\right\}>0,\quad z\in D.$$

These functions are called starlike with respect to symmetric points and were introduced by Sakaguchi [4]. El-Ashwa and Thomas [2] obtained various results concerning functions in $S_s^\star$ and two other classes namely the class $S_c^\star$ of functions starlike with respect to conjugate points and the class $S_{sc}^\star$ of functions starlike with respect to symmetric conjugate points.

Now, we denote $\mathcal{T}$ the class consisting of functions f of the form

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n \tag{2}$$

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where a_n is a non negative real number.

For $f \in \mathcal{T}$, Halim et al. [3] studied the class $S_s^\star T(\alpha,\beta)$, $0 \le \alpha \le 1, \frac{1}{2} < \beta \le 1$, consisting of functions $f \in \mathcal{T}$ and starlike with respect to symmetric points. An analogous results are also obtained for the class $S_c^\star T(\alpha,\beta)$, $0 \le \alpha \le 1$, $\frac{1}{2} < \beta \le 1$, consisting of functions $f \in \mathcal{T}$ and starlike with respect to conjugate points and the class $S_{sc}^\star T(\alpha,\beta)$, $0 \le \alpha \le 1, \frac{1}{2} < \beta \le 1$, consisting of functions $f \in \mathcal{T}$ and starlike with respect to symmetric conjugate points.

For this paper, we consider a class $S_s^\star T(\alpha,\beta,\sigma,k)$, $0 \le \alpha \le 1$, $0 < \beta \le 1$, $0 \le \sigma \le \frac{1}{2} < k \le 1$, consisting of functions $f \in \mathcal{T}$ and starlike with respect to symmetric points as follows:

Definition 1. A function $f \in S_s^\star T(\alpha,\beta,\sigma,k)$ if it satisfies

$$\left|\frac{zf'(z)}{f(z)-f(-z)} - k\right| < \beta \left|\frac{\alpha z f'(z)}{f(z)-f(-z)} - (2\sigma - k)\right|$$

for some $0 \le \alpha \le 1$, $0 < \beta \le 1$, $0 \le \sigma \le \frac{1}{2} < k \le 1$ and $z \in D$.

An analogous results are also obtained for the class of functions $f \in \mathcal{T}$ and starlike with respect to conjugate points and functions starlike with respect to symmetric conjugate points. These classes are defined as below:

Definition 2. A function $f \in S_c^\star T(\alpha,\beta,\sigma,k)$ if it satisfies

$$\left|\frac{zf'(z)}{f(z)+\overline{f(\bar{z})}} - k\right| < \beta \left|\frac{\alpha z f'(z)}{f(z)+\overline{f(\bar{z})}} - (2\sigma - k)\right|$$

for some $0 \le \alpha \le 1, 0 < \beta \le 1, 0 \le \sigma \le \frac{1}{2} < k \le 1$ and $z \in D$.

Definition 3. A function $f \in S_{sc}^\star T(\alpha,\beta,\sigma,k)$ if it satisfies

$$\left|\frac{zf'(z)}{f(z)-\overline{f(-\bar{z})}} - k\right| < \beta \left|\frac{\alpha z f'(z)}{f(z)-\overline{f(-\bar{z})}} - (2\sigma - k)\right|$$

for some $0 \le \alpha \le 1, 0 < \beta \le 1, 0 \le \sigma \le \frac{1}{2} < k \le 1$ and $z \in D$.

2. Coefficient Inequalities

We first state some preliminary lemmas, required for proving our result.

Lemma 1.([5]) *If $f \in \mathcal{T}$ then $\sum_{n=2}^{\infty} n\,|a_n| < 1$.*

Lemma 2. *If $f \in \mathcal{T}$ then $\sum_{n=2}^{\infty} (n\alpha + (k-2\sigma)(1-(-1)^n))\,|a_n| < \alpha + 2(k-2\sigma)$.*

Proof. We note that

$$\begin{aligned}
&\sum_{n=2}^{\infty} (n\alpha + (k-2\sigma)(1-(-1)^n))\ |a_n| \\
&= \sum_{n=2}^{\infty} n\alpha\ |a_n| + \sum_{n=2}^{\infty}(k-2\sigma)(1-(-1)^n)\ |a_n| \\
&= \alpha \sum_{n=2}^{\infty} n\ |a_n| + (k-2\sigma) \sum_{n=2}^{\infty}(1-(-1)^n)|a_n| \\
&< \alpha(1) + (k-2\sigma)2(1) = \alpha + 2(k-2\sigma), \quad \text{by (Lemma 1)},
\end{aligned}$$

as required.

Lemma 3. *If $f \in \mathcal{T}$ then $\sum_{n=2}^{\infty} (n\alpha + 2(k-2\sigma))\ |a_n| < \alpha + 2(k-2\sigma)$.*

Proof. We note that

$$\begin{aligned}
&\sum_{n=2}^{\infty} (n\alpha + 2(k-2\sigma))\ |a_n| \\
&= \sum_{n=2}^{\infty} n\alpha\ |a_n| + \sum_{n=2}^{\infty} 2(k-2\sigma)\ |a_n| \\
&= \alpha \sum_{n=2}^{\infty} n\ |a_n| + 2(k-2\sigma) \sum_{n=2}^{\infty} |a_n| \\
&< \alpha(1) + 2(k-2\sigma)(1) = \alpha + 2(k-2\sigma), \quad \text{by (Lemma 1)},
\end{aligned}$$

as required.

For $S_s^{\star}T(\alpha,\beta,\sigma,k)$, we have the following:

Theorem 1. *Let $f \in \mathcal{T}$. A function $f \in S_s^{\star}T(\alpha,\beta,\sigma,k)$ if and only if*

$$\sum_{n=2}^{\infty} \left(\frac{(1+\beta\alpha)n}{\beta(2(k-2\sigma)+\alpha)-(2k-1)} + \frac{\beta(k-2\sigma)(1-(-1)^n) - k(1-(-1)^n)}{\beta(2(k-2\sigma)+\alpha)-(2k-1)} \right) a_n \le 1 \tag{3}$$

for $0 \le \alpha \le 1, \frac{1}{2} < \beta \le 1$ and $0 \le \sigma \le \frac{1}{2} < k \le 1$.

Proof. First we prove the 'if' part. We adopt the method used by Clunie and Keogh

[1]. We write

$$\begin{aligned}
&|zf'(z) - k(f(z) - f(-z))| - \beta|\alpha z f'(z) - (2\sigma - k)(f(z) - f(-z))| \\
&= |(1-2k)z - \sum_{n=2}^{\infty}(n - k(1-(-1)^n))a_n z^n| - \beta|(2(k-2\sigma)+\alpha)z \\
&\quad - \sum_{n=2}^{\infty}(n\alpha + (k-2\sigma)(1-(-1)^n))a_n z^n| \\
&\le \sum_{n=2}^{\infty}(n - k(1-(-1)^n))|a_n|r^n + (2k-1)r - \beta(2(k-2\sigma)+\alpha)r \\
&\quad + \sum_{n=2}^{\infty}\beta(n\alpha + (k-2\sigma)(1-(-1)^n))|a_n|r^n \\
&< \Big[\sum_{n=2}^{\infty}(n - k(1-(-1)^n))|a_n| + (2k-1) - \beta(2(k-2\sigma)+\alpha) \\
&\quad + \sum_{n=2}^{\infty}\beta(n\alpha + (k-2\sigma)(1-(-1)^n))|a_n|\Big]r \\
&= \Big[\sum_{n=2}^{\infty}((1+\beta\alpha)n + \beta(k-2\sigma)(1-(-1)^n) - k(1-(-1)^n))a_n \\
&\quad -(\beta(2(k-2\sigma)+\alpha) - (2k-1))\Big]r \\
&\le 0 \ \text{ by (3).}
\end{aligned}$$

Thus,

$$\left|\frac{\frac{zf'(z)}{f(z)-f(-z)} - k}{\frac{\alpha z f'(z)}{f(z)-f(-z)} - (2\sigma - k)}\right| < \beta$$

and hence $f \in S_s^{\star}T(\alpha, \beta, \sigma, k)$.

To prove the 'only if' part, we write

$$\begin{aligned}
&\left|\frac{\frac{zf'(z)}{f(z)-f(-z)} - k}{\frac{\alpha z f'(z)}{f(z)-f(-z)} - (2\sigma - k)}\right| \\
&= \left|\frac{(1-2k) - \sum_{n=2}^{\infty}(n - k(1-(-1)^n))a_n z^{n-1}}{2(k-2\sigma)+\alpha - \sum_{n=2}^{\infty}(n\alpha + (k-2\sigma)(1-(-1)^n))a_n z^{n-1}}\right| < \beta.
\end{aligned}$$

Since f is analytic, continuous and non constant in $\mathcal{D}$, the maximum modulus principle states that

$$\left|\frac{(1-2k)-\sum_{n=2}^{\infty}(n-k(1-(-1)^n))a_n z^{n-1}}{2(k-2\sigma)+\alpha-\sum_{n=2}^{\infty}(n\alpha+(k-2\sigma)(1-(-1)^n))a_n z^{n-1}}\right|$$
$$=\left|\frac{(2k-1)+\sum_{n=2}^{\infty}(n-k(1-(-1)^n))a_n z^{n-1}}{2(k-2\sigma)+\alpha-\sum_{n=2}^{\infty}(n\alpha+(k-2\sigma)(1-(-1)^n))a_n z^{n-1}}\right|$$
$$\leq \frac{(2k-1)+\sum_{n=2}^{\infty}(n-k(1-(-1)^n))|a_n||z|^{n-1}}{2(k-2\sigma)+\alpha-\sum_{n=2}^{\infty}(n\alpha+(k-2\sigma)(1-(-1)^n))|a_n||z|^{n-1}}, \quad \text{by (Lemma 2)}$$
$$< \frac{(2k-1)+\sum_{n=2}^{\infty}(n-k(1-(-1)^n))a_n r^{n-1}}{2(k-2\sigma)+\alpha-\sum_{n=2}^{\infty}(n\alpha+(k-2\sigma)(1-(-1)^n))a_n r^{n-1}}=f(r).$$

Since $f \in S_s^{\star}T(\alpha,\beta,\sigma,k)$ and $|z|<r<1$, we obtain

$$\left\{\frac{(2k-1)+\sum_{n=2}^{\infty}(n-k(1-(-1)^n))a_n r^{n-1}}{2(k-2\sigma)+\alpha-\sum_{n=2}^{\infty}(n\alpha+(k-2\sigma)(1-(-1)^n))a_n r^{n-1}}\right\}<\beta \tag{4}$$

for any $z \in \mathcal{D}$. Now letting $r \to 1$ in (4), we obtain

$$(2k-1)+\sum_{n=2}^{\infty}(n-k(1-(-1)^n))a_n$$
$$\leq \beta\left(2(k-2\sigma)+\alpha-\sum_{n=2}^{\infty}(n\alpha+(k-2\sigma)(1-(-1)^n))a_n\right)$$

and hence, we obtain

$$\sum_{n=2}^{\infty}\left(\frac{(1+\beta\alpha)n}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}+\frac{\beta(k-2\sigma)(1-(-1)^n)-k(1-(-1)^n)}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\right)a_n \leq 1$$

as required. This completes the proof of the theorem.

Remark 1. We note that the case $0<\beta\leq\frac{1}{2}$ remains open.

The result in Theorem 1 is sharp for functions given by

$$f_n(z)=z-\frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{(1+\beta\alpha)n+\beta(k-2\sigma)(1-(-1)^n)-k(1-(-1)^n)}\,z^n, \quad n\geq 2.$$

Corollary 1. *If* $f \in S_s^{\star}T(\alpha,\beta,\sigma,k)$ *then*

$$a_n \leq \frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{(1+\beta\alpha)n+\beta(k-2\sigma)(1-(-1)^n)-k(1-(-1)^n)}\,, \quad n\geq 2.$$

Next, similar coefficient properties for functions which belong to $S_c^{\star}T(\alpha,\beta,\sigma,k)$ and $S_{sc}^{\star}T(\alpha,\beta,\sigma,k)$ are obtained. Similar method of proving is used for Theorem 2 and Theorem 3.

Theorem 2. *Let $f \in \mathcal{T}$. A function $f \in S_c^{\star}T(\alpha,\beta,\sigma,k)$ if and only if*

$$\sum_{n=2}^{\infty}\left(\frac{(1+\beta\alpha)n}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}+\frac{2(\beta(k-2\sigma)-k)}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\right)a_n \leq 1 \qquad (5)$$

for $0 \leq \alpha \leq 1$, $\frac{1}{2} < \beta \leq 1$ and $0 \leq \sigma \leq \frac{1}{2} < k \leq 1$.

Proof. First we prove the 'if' part. We write

$$\begin{aligned}
&\left|zf'(z)-k(f(z)+\overline{f(\bar{z})})\right|-\beta\left|\alpha zf'(z)-(2\sigma-k)(f(z)+\overline{f(\bar{z})})\right| \\
&=|(1-2k)z-\sum_{n=2}^{\infty}(n-2k)a_nz^n|-\beta|(2(k-2\sigma)+\alpha)z-\sum_{n=2}^{\infty}(n\alpha+2(k-2\sigma))a_nz^n| \\
&\leq \sum_{n=2}^{\infty}(n-2k)|a_n|r^n+(2k-1)r-\beta(2(k-2\sigma)+\alpha)r+\sum_{n=2}^{\infty}\beta(n\alpha+2(k-2\sigma))|a_n|r^n \\
&< [\sum_{n=2}^{\infty}(n-2k)|a_n|+(2k-1)-\beta(2(k-2\sigma)+\alpha)+\sum_{n=2}^{\infty}\beta(n\alpha+2(k-2\sigma))|a_n|]r \\
&= [\sum_{n=2}^{\infty}((1+\beta\alpha)n+2(\beta(k-2\sigma)-k))a_n-(\beta(2(k-2\sigma)+\alpha)-(2k-1))]r \\
&\leq 0 \quad \text{by (5).}
\end{aligned}$$

Thus,

$$\left|\frac{\frac{zf'(z)}{f(z)+\overline{f(\bar{z})}}-k}{\frac{\alpha zf'(z)}{f(z)+\overline{f(\bar{z})}}-(2\sigma-k)}\right|<\beta$$

and hence $f \in S_s^{\star}T(\alpha,\beta,\sigma,k)$.

To prove the 'only if' part, as before we write

$$\left|\frac{\frac{zf'(z)}{f(z)+\overline{f(\bar{z})}}-k}{\frac{\alpha zf'(z)}{f(z)+\overline{f(\bar{z})}}-(2\sigma-k)}\right|=\left|\frac{(1-2k)-\sum_{n=2}^{\infty}(n-2k)a_nz^{n-1}}{2(k-2\sigma)+\alpha-\sum_{n=2}^{\infty}(n\alpha+2(k-2\sigma))a_nz^{n-1}}\right|<\beta.$$

Since f is analytic, continuous and non constant in $\mathcal{D}$, the maximum modulus principle states that

$$\begin{aligned}
&\left|\frac{(1-2k)-\sum_{n=2}^{\infty}(n-2k)a_nz^{n-1}}{2(k-2\sigma)+\alpha-\sum_{n=2}^{\infty}(n\alpha+2(k-2\sigma))a_nz^{n-1}}\right| \\
&=\left|\frac{(2k-1)+\sum_{n=2}^{\infty}(n-2k)a_nz^{n-1}}{2(k-2\sigma)+\alpha-\sum_{n=2}^{\infty}(n\alpha+2(k-2\sigma))a_nz^{n-1}}\right| \\
&\leq \frac{(2k-1)+\sum_{n=2}^{\infty}(n-2k)|a_n||z|^{n-1}}{2(k-2\sigma)+\alpha-\sum_{n=2}^{\infty}(n\alpha+2(k-2\sigma))|a_n||z|^{n-1}}, \quad \text{by (Lemma 3)} \\
&< \frac{(2k-1)+\sum_{n=2}^{\infty}(n-2k)a_nr^{n-1}}{2(k-2\sigma)+\alpha-\sum_{n=2}^{\infty}(n\alpha+2(k-2\sigma))a_nr^{n-1}}=f(r).
\end{aligned}$$

Since $f \in S_c^\star T(\alpha,\beta,\sigma,k)$ and $|z|<r<1$, we obtain

$$\left\{\frac{(2k-1)+\sum_{n=2}^{\infty}(n-2k)a_n r^{n-1}}{2(k-2\sigma)+\alpha-\sum_{n=2}^{\infty}(n\alpha+2(k-2\sigma))a_n r^{n-1}}\right\}<\beta \tag{6}$$

for any $z \in \mathcal{D}$. Now letting $r \to 1$ in (6), we obtain

$$(2k-1)+\sum_{n=2}^{\infty}(n-2k)a_n \le \beta\left(2(k-2\sigma)+\alpha-\sum_{n=2}^{\infty}(n\alpha+2(k-2\sigma))a_n\right)$$

and hence, we obtain

$$\sum_{n=2}^{\infty}\left(\frac{(1+\beta\alpha)n}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}+\frac{2(\beta(k-2\sigma)-k)}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\right)a_n \le 1$$

as required. This completes the proof of the theorem.

The result in Theorem 2 is sharp for function f_n given by

$$f_n(z)=z-\frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{(1+\beta\alpha)n+2(\beta(k-2\sigma)-k)}\,z^n.$$

Corollary 2. *If $f \in S_c^\star T(\alpha,\beta,\sigma,k)$ then*

$$a_n \le \frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{(1+\beta\alpha)n+2(\beta(k-2\sigma)-k)}\ , \quad n \ge 2.$$

For completeness, we state the following result with regards to the class $S_{sc}^\star T(\alpha,\beta,\sigma,k)$.

Theorem 3. *Let $f \in \mathcal{T}$. A function $f \in S_{sc}^\star T(\alpha,\beta,\sigma,k)$ if and only if*

$$\sum_{n=2}^{\infty}\left(\frac{(1+\beta\alpha)n}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}+\frac{\beta(k-2\sigma)(1-(-1)^n)-k(1-(-1)^n)}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\right)a_n \le 1 \tag{7}$$

for $0 \le \alpha \le 1$, $\frac{1}{2}<\beta\le 1$ and $0 \le \sigma \le \frac{1}{2} < k \le 1$.

The result in Theorem 3 is sharp for functions given by

$$f_n(z)=z-\frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{(1+\beta\alpha)n+\beta(k-2\sigma)(1-(-1)^n)-k(1-(-1)^n)}\,z^n, \quad n \ge 2.$$

Corollary 3. *If $f \in S_{sc}^\star T(\alpha,\beta,\sigma,k)$ then*

$$a_n \le \frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{(1+\beta\alpha)n+\beta(k-2\sigma)(1-(-1)^n)-k(1-(-1)^n)}\ , \quad n \ge 2.$$

3. Growth Theorem

A growth property for functions in the class $S_s^\star T(\alpha,\beta,\sigma,k)$ is given as follows.

Theorem 4. *Let the functions f be defined by* (2) *and belongs to the class $S_s^\star T(\alpha,\beta,\sigma,k)$. Then for $\{z : 0 < |z| = r < 1\}$,*

$$r - \frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{2(1+\beta\alpha)}\, r^2 \le |f(z)| \le r + \frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{2(1+\beta\alpha)}\, r^2.$$

The result is sharp.

Proof. First, it is obvious that

$$\frac{2(1+\beta\alpha)}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\sum_{n=2}^{\infty} a_n$$
$$\le \sum_{n=2}^{\infty}\left(\frac{(1+\beta\alpha)n}{\beta(2(k-2\sigma)+\alpha)-(2k-1)} + \frac{\beta(k-2\sigma)(1-(-1)^n)-k(1-(-1)^n)}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\right) a_n,$$

and as $f \in S_s^\star T(\alpha,\beta,\sigma,k)$, using the inequality in Theorem 1 yields

$$\sum_{n=2}^{\infty} a_n \le \frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{2(1+\beta\alpha)}. \tag{8}$$

From (2) with $|z| = r (r < 1)$, we have

$$|f(z)| \le r + \sum_{n=2}^{\infty} a_n r^n \le r + \sum_{n=2}^{\infty} a_n r^2$$

and

$$|f(z)| \ge r - \sum_{n=2}^{\infty} a_n r^n \ge r - \sum_{n=2}^{\infty} a_n r^2.$$

Finally, using (8) in the above inequalities, give the result in Theorem 4.

The result in Theorem 4 is sharp for functions given by

$$f_2(z) = z - \frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{2(1+\beta\alpha)}\, z^2$$

at $z = \pm r$.

Next, similar growth results for functions which belong to $S_c^\star T(\alpha,\beta,\sigma,k)$ and $S_{sc}^\star T(\alpha,\beta,\sigma,k)$ are obtained. Similar method of proving is used for Theorem 5 and Theorem 6.

Theorem 5. *Let the functions f be defined by* (2) *and belongs to the class $S_c^\star T(\alpha,\beta,\sigma,k)$. Then for $\{z : 0 < |z| = r < 1\}$,*

$$r - \frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{2(1+\beta\alpha)+2(\beta(k-2\sigma)-k)}\, r^2 \le |f(z)| \le r + \frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{2(1+\beta\alpha)+2(\beta(k-2\sigma)-k)}\, r^2.$$

The result is sharp.

Proof. First, it is obvious that

$$\frac{2(1+\beta\alpha)+2(\beta(k-2\sigma)-k)}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\sum_{n=2}^{\infty}a_n$$
$$\leq \sum_{n=2}^{\infty}\left(\frac{(1+\beta\alpha)n}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}+\frac{2(\beta(k-2\sigma)-k)}{\beta(2(k-2\sigma)+\alpha)-(2k-1)}\right)a_n,$$

and as $f \in S_c^{\star}T(\alpha,\beta,\sigma,k)$, using the inequality in Theorem 2 yields

$$\sum_{n=2}^{\infty} a_n \leq \frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{2(1+\beta\alpha)+2(\beta(k-2\sigma)-k)}. \tag{9}$$

From (2) with $|z| = r(r < 1)$, we have

$$|f(z)| \leq r + \sum_{n=2}^{\infty} a_n r^n \leq r + \sum_{n=2}^{\infty} a_n r^2$$

and

$$|f(z)| \geq r - \sum_{n=2}^{\infty} a_n r^n \geq r - \sum_{n=2}^{\infty} a_n r^2.$$

Finally, using (9) in the above inequalities, give the result in Theorem 5.

The result in Theorem 5 is sharp for functions given by

$$f_2(z) = z - \frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{2(1+\beta\alpha)+2(\beta(k-2\sigma)-k)}\, z^2$$

at $z = \pm r$.

For completeness, we state the following result with regards to the class $S_{sc}^{\star}T(\alpha,\beta,\sigma,k)$.

Theorem 6. *Let the functions f be defined by* (2) *and belongs to the class $S_{sc}^{\star}T(\alpha,\beta,\sigma,k)$. Then for $\{z : 0 < |z| = r < 1\}$,*

$$r - \frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{2(1+\beta\alpha)}\, r^2 \leq |f(z)| \leq r + \frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{2(1+\beta\alpha)}\, r^2.$$

The result is sharp.

The result in Theorem 6 is sharp for functions given by

$$f_2(z) = z - \frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{2(1+\beta\alpha)}\, z^2$$

at $z = \pm r$.

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Institute of Mathematical Sciences , Universiti Malaya, 50603 Kuala Lumpur, Malaysia.

E-mail: suzeini@um.edu.my

Institute of Mathematical Sciences , Universiti Malaya, 50603 Kuala Lumpur, Malaysia.

E-mail: aini_jg@ums.edu.my

School of Mathematical Sciences, Faculty of Sciences and Technology, Universiti Kebangsaan Malaysia, 43600 Bangi, Selangor, Malaysia.

E-mail: maslina@pkrisc.cc.ukm.my