TAMKANG JOURNAL OF MATHEMATICS
Volume 45, Number 2, 195-206, June 2014
doi:10.5556/j.tkjm.45.2014.1492

Available online at http://journals.math.tku.edu.tw/

RECOVERING DIFFERENTIAL PENCILS ON GRAPHS WITH A CYCLE FROM SPECTRA

V. YURKO

Abstract. We study boundary value problems on compact graphs with a cycle for second-order ordinary differential equations with nonlinear dependence on the spectral parameter. We establish properties of the spectral characteristics and investigate inverse spectral problems of recovering coefficients of the differential equation from spectra. For these inverse problems we prove uniqueness theorems and provide procedures for constructing their solutions.

1. Introduction

In this paper we study inverse spectral problems for second-order differential pencils on compact graphs with a cycle. Inverse spectral problems consist in recovering coefficients of differential equations from their spectral characteristics. The main results on inverse spectral problems for ordinary differential operators on *an interval* are presented in the monographs [1]−[5]. Differential operators on graphs (spatial networks) often appear in mathematics, mechanics, physics, geophysics, physical chemistry, biology, electronics, nanoscale technology and other branches of natural sciences and engineering (see [6-7] and the references therein). Inverse spectral problems for *Sturm-Liouville operators* on compact graphs have been studied fairly completely in [8]−[14] and other works. Differential pencils (when differential equations depend nonlinearly on the spectral parameter) produce serious qualitative changes in the spectral theory. In particular, there are only a few works on inverse spectral problems for differential pencils on graphs. In [15] an inverse problem have been solved for differential pencils on trees (graphs without cycles). Inverse problems for differential pencils on graphs with cycles have not been studied yet.

In this paper we investigate inverse spectral problems for second-order differential pencils on compact graphs having a cycle under generalized matching conditions in interior vertices and boundary conditions in boundary vertices. For these inverse problems we prove uniqueness theorems and provide procedures for constructing their solutions.

Received September 2, 2013, accepted December 2, 2013.
2010 *Mathematics Subject Classification.* 34A55 34B45 34B07 47E05.
Key words and phrases. Geometrical graphs with a cycle, differential pencils, inverse spectral problems.

The paper is organized as follows: Properties of spectral characteristics are established in Section 2. In section 3 algorithms for the solutions of the inverse problems considered are provided and the corresponding uniqueness theorems are proved.

Consider a compact graph T in $\mathbf{R^m}$ with the set of vertices $V = \{v_0, \dots, v_r\}$, $r \ge 1$, and the set of edges $\mathscr{E} = \{e_0, \dots, e_r\}$, where $v_1, \dots, v_r$ are the boundary vertices, v_0 is the internal vertex, $e_j = [v_j, v_0]$, $j = \overline{1,r}$, $\bigcap_{j=0}^{r} e_j = \{v_0\}$, and e_0 is a cycle. Thus, the graph T has one cycle e_0 and one internal vertex v_0. Let T_j, $j = \overline{0,r}$, be the length of the edge e_j. Each edge $e_j \in \mathscr{E}$ is parameterized by the parameter $x_j \in [0, T_j]$. It is convenient for us to choose the following orientation: for $j = \overline{1,r}$, the vertex v_j corresponds to $x_j = 0$, and the vertex v_0 corresponds to $x_j = T_j$; for $j = 0$, both ends $x_0 = +0$ and $x_0 = T_0 - 0$ corespond to v_0.

An integrable function Y on T may be represented as $Y = \{y_j\}_{j=\overline{0,r}}$, where the function $y_j(x_j)$, $x_j \in [0, T_j]$, is defined on the edge e_j. Let $q = \{q_j\}_{j=\overline{0,r}}$ and $p = \{p_j\}_{j=\overline{0,r}}$ be complex-valued functions on T; the pair (q, p) is called the potential. Assume that $q_j(x_j) \in L(0, T_j)$, and $p_j(x_j)$ is absolutely continuous on $[0, T_j]$. Consider the following differential equation on T:

$$y_j''(x_j) + (\rho^2 + \rho p_j(x_j) + q_j(x_j))y_j(x_j) = 0, \quad x_j \in (0, T_j), \tag{1}$$

where ρ is the spectral parameter, $j = \overline{0,r}$, the functions $y_j(x_j)$, $y_j'(x_j)$ are absolutely continuous on $[0, T_j]$ and satisfy the following matching conditions in the internal vertex v_0:

$$y_0(0) = \alpha_j y_j(T_j),\ j = \overline{0,r}, \qquad y_0'(0) - (i\rho h_{01} + h_{00})y_0(0) = \sum_{j=0}^{r} \beta_j y_j'(T_j), \tag{2}$$

where α_j and β_j are complex numbers such that $\alpha_j\beta_j \neq 0$, $1 + \alpha_0\beta_0 \neq 0$. Matching conditions (2) are a generalization of Kichhoff's matching conditions [9]. Let us consider the boundary value problem $B_0 := B_0(q, p, h_1, h_0)$ on T for equation (1) with matching conditions (2) and with the following boundary conditions at the boundary vertices $v_1, \dots, v_r$:

$$U_j(Y) = 0, \quad j = \overline{1,r}. \tag{3}$$

where $U_j(Y) := y_j'(0) - (i\rho h_{j1} + h_{j0})y_j(0)$, h_{jk} are complex numbers, $h_k = \{h_{jk}\}_{j=\overline{0,r}}$, $k = 0, 1$, and $h_{j1} \neq \pm 1$ for $j = \overline{1,r}$. We also consider the boundary value problems $B_k := B_k(q, p, h_1, h_0)$, $k = \overline{1,r}$, for equation (1) with matching conditions (2) and with the boundary conditions

$$y_k(0) = 0, \quad U_j(Y) = 0, \quad j = \overline{1,r} \setminus k.$$

We denote by $\Lambda_k := \{\rho_{kn}\}_{n \in \mathbf{Z}}$ the eigenvalues (counting with multiplicities) of B_k, $k = \overline{0,r}$. In contrast to the case of trees (see [9, 15]), here the specification of the spectra Λ_k, $k = \overline{0,r}$ does not uniquely determine the potential, and we need an additional information. Let

$\Lambda_{-1} := \{\rho_{-1,n}\}_{n\in\mathbf{Z}}$ be the spectrum of the boundary value problem B_{-1} for equation (1) under boundary conditions (3) and matching conditions of the form (2), but with α_{-1} instead of α_0 ($\alpha_{-1} \neq \alpha_0$).

Inverse problem 1. Given Λ_k, $k = \overline{-1,r}$, construct the potential (q,p) on T and the coefficients h_1, h_0.

Let $\Lambda_{r+1} := \{\rho_{r+1,n}\}_{n\in\mathbf{Z}}$ be the spectrum of the boundary value problem B_{r+1} for equation (1) under boundary conditions (3) and matching conditions of the form (2), but with β_{-1} instead of β_0 ($\beta_{-1} \neq \beta_0$).

Inverse problem 2. Given Λ_k, $k = \overline{0,r+1}$, construct the potential (q,p) on T and the coefficients h_1, h_0.

For these inverse problems we provide constructive procedures for their solutions and prove their uniqueness. We note that the coefficients α_j, β_j from (2) are known a priori and fixed. Denote

$$z_0^{\pm} = \alpha_0(1 \mp h_{01}) + \beta_0, \quad z_{k+1}^{\pm} = \alpha_{k+1} z_k^{\pm} + \beta_{k+1} \prod_{j=0}^{k} \alpha_j, \quad k = \overline{0,r-1}.$$

We assume that $z_0^{\pm} z_r^{\pm} \neq 0$. This condition is called the regularity condition for matching. Differential operators on T which do not satisfy the regularity condition, possess qualitatively different properties for formulation and investigation of inverse problems, and are not considered in this paper; they require a separate investigation. We note that for classical Kirchhoff's matching conditions we have $\alpha_j = \beta_j = 1$, $h_{0k} = 0$, and the regularity condition is satisfied obviously.

Let us formulate uniqueness theorems for the solution of Inverse problems 1 and 2. For this purpose together with B_k we consider boundary value problems $\tilde{B}_k = B_k(\tilde{q}, \tilde{p}, \tilde{h}_1, \tilde{h}_0)$ of the same form but with different coefficients. Everywhere below if a symbol α denotes an object related to B_k, then $\tilde{\alpha}$ will denote the analogous object related to $\tilde{B}_k$.

Theorem 1.1. *If $\Lambda_k = \tilde{\Lambda}_k$, $k = \overline{-1,r}$, then $q = \tilde{q}$, $p = \tilde{p}$, $h_1 = \tilde{h}_1$, $h_0 = \tilde{h}_0$. Thus, the specification of the spectra Λ_k, $k = \overline{-1,r}$ uniquely determines the potential (q,p) on T and the coefficients h_1, h_0.*

Theorem 1.2. *If $\Lambda_k = \tilde{\Lambda}_k$, $k = \overline{0,r+1}$, then $q = \tilde{q}$, $p = \tilde{p}$, $h_1 = \tilde{h}_1$, $h_0 = \tilde{h}_0$. Thus, the specification of the spectra Λ_k, $k = \overline{0,r+1}$ uniquely determines the potential (q,p) on T and the coefficients h_1, h_0.*

These theorems will be proved below in Section 3. Moreover, we will give constructive procedures for the solutions of Inverse problems 1 and 2 (see Algorithms 3 and 5). In Section

2 properties of the spectra, characteristic functions and the Weyl functions are investigated for boundary value problems on the graph.

2. Auxiliary propositions

Denote $\mathscr{E}_k(x_k)=\frac{1}{2}\int_0^{x_k}p_k(t)\,dt$, $\omega_k=T_k^{-1}\mathscr{E}_k(T_k)$, $E^{\pm}(\rho)=\prod_{j=0}^r\exp(\mp i(\rho+\omega_j)T_j)$, $\Pi^{\pm}=\{\rho:\ \pm\mathrm{Im}\,\rho\ge 0\}$, $\Pi_\delta^{+}=\{\rho:\ \arg\rho\in[\delta,\pi-\delta]\}$, $\Pi_\delta^{-}=\{\rho:\ \arg\rho\in[\pi+\delta,2\pi-\delta]\}$. Fix $k=\overline{1,r}$. Let $\Phi_k=\{\Phi_{kj}\}_{j=\overline{0,r}}$, be the solution of equation (1) satisfying (2) and the boundary conditions

$$U_j(\Phi_k)=\delta_{jk},\quad j=\overline{1,r},\tag{4}$$

where δ_{jk} is the Kronecker symbol. Denote $M_k(\rho):=\Phi_{kk}(0,\rho)$, $k=\overline{1,r}$. The function $M_k(\lambda)$ is called the *Weyl function* with respect to the boundary vertex v_k. Clearly,

$$\Phi_{kk}(x_k,\rho)=S_k(x_k,\rho)+M_k(\rho)\varphi_k(x_k,\rho),\quad x_k\in[0,T_k],\quad k=\overline{1,r},\tag{5}$$

where $S_k(x_k,\rho)$, $\varphi_k(x_k,\rho)$, $k=\overline{0,r}$ are solutions of equation (1) on the edge e_k with the initial conditions $S_k(0,\rho)=0$, $S_k'(0,\rho)=\varphi_k(0,\rho)=1$, $\varphi_k'(0,\rho)=i\rho h_{k1}+h_{k0}$. For each fixed $x_k\in[0,T_k]$, the functions $S_k^{(\nu)}(x_k,\rho)$, $\varphi_k^{(\nu)}(x_k,\rho)$, $\nu=0,1$, are entire in ρ of exponential type, and $\langle\varphi_k(x_k,\rho),\Phi_{kk}(x_k,\rho)\rangle\equiv 1$, where $\langle y,z\rangle:=yz'-y'z$ is the Wronskian of y and z. For $k=\overline{0,r}$, $\nu=0,1$, $x_k\in[0,T_k]$, $|\rho|\to\infty$, one has (see [15]),

$$\begin{aligned}\varphi_k^{(\nu)}(x_k,\rho)&=(-i\rho)^\nu\frac{1-h_{k1}}{2}\exp(-i(\rho x_k+\mathscr{E}_k(x_k)))[1]\\&\quad+(i\rho)^\nu\frac{1+h_{k1}}{2}\exp(i(\rho x_k+\mathscr{E}_k(x_k)))[1].\end{aligned}\tag{6}$$

Similarly, for $k=\overline{1,r}$, $\nu=0,1$, $x_k\in[0,T_k)$, $\rho\in\Pi_\delta^{\pm}$, $|\rho|\to\infty$,

$$\Phi_{kk}^{(\nu)}(x_k,\rho)=\frac{1}{(\pm i\rho)^{1-\nu}(1\mp h_{k1})}\exp(\pm i(\rho x_k+\mathscr{E}_k(x_k)))[1],\tag{7}$$

$$M_k(\rho)=\frac{[1]}{(\pm i\rho)(1\mp h_{k1})},\quad k=\overline{1,r}.\tag{8}$$

Denote $M_{kj}^1(\rho):=\Phi_{kj}(0,\rho)$, $M_{kj}^0(\rho):=\Phi_{kj}'(0,\rho)-(i\rho h_{j1}+h_{j0})\Phi_{kj}(0,\rho)$. Then

$$\Phi_{kj}(x_j,\rho)=M_{kj}^0(\rho)S_j(x_j,\rho)+M_{kj}^1(\rho)\varphi_j(x_j,\rho),\quad x_j\in[0,T_j],\ j=\overline{0,r},\ k=\overline{1,r}.\tag{9}$$

In particular, $M_{kk}^0(\rho)=1$, $M_{kk}^1(\rho)=M_k(\rho)$, and $M_{kj}^0(\rho)=0$ for $j=\overline{1,r}\setminus k$. Substituting (9) into (2) and (4) we obtain a linear algebraic system s_k with respect to $M_{kj}^\nu(\rho)$, $\nu=0,1$, $j=\overline{0,r}$. The determinant $\Delta_0(\rho)$ of s_k does not depend on k and has the form

$$\Delta_0(\rho)=d(\rho)\prod_{j=1}^r(\alpha_j\varphi_j(T_j,\rho))+d_0(\rho)\sum_{i=1}^r(\beta_i\varphi_i'(T_i,\rho))\prod_{j=1,\,j\ne i}^r(\alpha_j\varphi_j(T_j,\rho)),\tag{10}$$

where

$$d(\rho) = \alpha_0\varphi_0(T_0,\rho) + \beta_0 S_0'(T_0,\rho) - (1+\alpha_0\beta_0), \quad d_0(\rho) = \alpha_0 S_0(T_0,\rho). \tag{11}$$

The function $\Delta_0(\rho)$ is entire in ρ of exponential type, and its zeros coincide with the eigenvalues of the boundary value problem B_0. Solving the algebraic system s_k we get by Cramer's rule: $M_{kj}^s(\rho) = \Delta_{kj}^s(\rho)/\Delta_0(\rho)$, $s = 0,1$, $j = \overline{0,r}$, where the determinant $\Delta_{kj}^s(\rho)$ is obtained from $\Delta_0(\rho)$ by the replacement of the column which corresponds to $M_{kj}^s(\rho)$ with the column of free terms. In particular,

$$M_k(\rho) = -\frac{\Delta_k(\rho)}{\Delta_0(\rho)}, \quad k = \overline{1,r}, \tag{12}$$

where

$$\Delta_k(\rho) = d(\rho)(\alpha_k S_k(T_k,\rho)) \prod_{j=1,\, j\neq k}^{r} (\alpha_j\varphi_j(T_j,\rho)) + d_0(\rho)\Big(\beta_k S_k'(T_k,\rho) \prod_{j=1,\, j\neq k}^{r} (\alpha_j\varphi_j(T_j,\rho))$$
$$+(\alpha_k S_k(T_k,\rho)) \sum_{i=1,\, i\neq k}^{r} (\beta_i\varphi_i'(T_i,\rho)) \prod_{j=1,\, j\neq i,k}^{r} (\alpha_j\varphi_j(T_j,\rho))\Big), \; k = 1,2. \tag{13}$$

We note that $\Delta_k(\rho)$ in (13) is obtained from $\Delta_0(\rho)$ by the replacement of $\varphi_k^{(\nu)}(T_k,\rho)$, $\nu = 0,1$, with $S_k^{(\nu)}(T_k,\rho)$, $\nu = 0,1$. The function $\Delta_k(\rho)$ is entire in ρ of exponential type, and its zeros coincide with the eigenvalues of the boundary value problem B_k. The functions $\Delta_k(\rho)$, $k = \overline{0,r}$, are called the characteristic functions for the boundary value problems B_k.

3. Solution of inverse problems 1-2

Fix $k = \overline{1,r}$, and consider the following auxiliary inverse problem on the edge e_k, which is called IP(k).

$\boldsymbol{I}P(k)$. Given the Weyl function $M_k(\rho)$, construct $q_k(x_k)$, $p_k(x_k)$, $x_k \in [0,T_k]$, h_{k1}, h_{k0}.

In $IP(k)$ we construct the potential only on the edge e_k, but the Weyl function brings a global information from the whole graph. In other words, $IP(k)$ is not a local inverse problem related to the edge e_k. Let us prove the uniqueness theorem for the solution of $IP(k)$.

Theorem 3.3. *Fix $k = \overline{1,r}$. If $M_k(\rho) = \tilde M_k(\rho)$, then $q_k(x_k) = \tilde q_k(x_k)$, $p_k(x_k) = \tilde p_k(x_k)$ a.e. on $[0,T_k]$, and $h_{k\nu} = \tilde h_{k\nu}$, $\nu = 0,1$. Thus, the specification of the Weyl function $M_k(\rho)$ uniquely determines the potential (q_k,p_k) on the edge e_k, and the coefficients h_{k1}, h_{k0}.*

Proof. We introduce the functions

$$P_{1s}^k(x_k,\rho) = (-1)^{s-1}\Big(\varphi_k(x_k,\rho)\tilde\Phi_{kk}^{(2-s)}(x_k,\rho) - \tilde\varphi_k^{(2-s)}(x_k,\rho)\Phi_{kk}(x_k,\rho)\Big), \quad s = 1,2. \tag{14}$$

By direct calculations we get

$$\varphi_k(x_k,\rho) = P^k_{11}(x_k,\rho)\tilde{\varphi}_k(x_k,\rho) + P^k_{12}(x_k,\rho)\tilde{\varphi}'_k(x_k,\rho). \tag{15}$$

Denote $\Omega_k(x_k) = \cos\hat{\mathscr{E}}_k(x_k)$, where $\hat{\mathscr{E}}_k(x_k) = \mathscr{E}_k(x_k) - \tilde{\mathscr{E}}_k(x_k)$. Since $M_k(\rho) = \tilde{M}_k(\rho)$, it follows from (8) that

$$h_{k1} = \tilde{h}_{k1}. \tag{16}$$

Taking (6), (7), (14) and (16) into account we obtain

$$P^k_{1s}(x_k,\rho) = \delta_{1s}\Omega_k(x_k) + O(\rho^{-1}), \quad \rho \in \Pi^{\pm}_{\delta},\ |\rho| \to \infty,\ x_k \in (0,T_k),\ s = 1,2. \tag{17}$$

According to (5) and (14),

$$P^k_{1s}(x_k,\rho) = (-1)^{s-1}\Big(\Big(\varphi_k(x_k,\rho)\tilde{S}^{(2-s)}_k(x_k,\rho) - S_k(x_k,\rho)\tilde{\varphi}^{(2-s)}_k(x_k,\rho)\Big) + (\tilde{M}_k(\rho) - M_k(\rho))\varphi_k(x_k,\rho)\tilde{\varphi}^{(2-s)}_k(x_k,\rho)\Big).$$

Since $M_k(\rho) = \tilde{M}_k(\rho)$, it follows that for each fixed x_k, the functions $P^k_{1s}(x_k,\rho)$ are entire in ρ of exponential type. Together with (17) this yields $P^k_{11}(x_k,\rho) \equiv \Omega_k(x_k)$, $P^k_{12}(x_k,\rho) \equiv 0$. Substituting these relations into (14) and (15) we get

$$\begin{aligned}(\tilde{\varphi}_k(x_k,\rho))^{-1}\varphi_k(x_k,\rho) &= (\tilde{\Phi}_{kk}(x_k,\rho))^{-1}\Phi_{kk}(x_k,\rho),\\ \varphi_k(x_k,\rho) &= \Omega_k(x_k)\tilde{\varphi}_k(x_k,\rho),\end{aligned} \tag{18}$$

for all x_k and ρ. Using the asymptotical formulae (6) and (7) we obtain for $|\rho| \to \infty$, $\rho \in \Pi^{\pm}_{\delta}$,

$$(\tilde{\varphi}_k(x_k,\rho))^{-1}\varphi_k(x_k,\rho) = \exp(\mp\hat{\mathscr{E}}_k(x_k))[1], \quad (\tilde{\Phi}_{kk}(x_k,\rho))^{-1}\Phi_{kk}(x_k,\rho) = \exp(\pm\hat{\mathscr{E}}_k(x_k))[1].$$

From this and from (18) we infer $\exp(2\hat{\mathscr{E}}_k(x_k)) \equiv 1$. Since $\hat{\mathscr{E}}_k(0) = 0$, it follows that $\hat{\mathscr{E}}_k(x_k) \equiv 0$, i.e. $P_{11}(x_k,\rho) \equiv 1$, $\varphi_k(x_k,\rho) \equiv \tilde{\varphi}_k(x_k,\rho)$, $\Phi_{kk}(x_k,\rho) \equiv \tilde{\Phi}_{kk}(x_k,\rho)$, and consequently, $q_k(x_k) = \tilde{q}_k(x_k)$, $p_k(x_k) = \tilde{p}_k(x_k)$ a.e. on $[0,T_k]$, and $h_{k0} = \tilde{h}_{k0}$. □

Using the method of spectral mappings [5] for equation (1) on the edge e_k one can get a constructive procedure for the solution of the local inverse problem $IP(k)$ of recovering the coefficients $q_k(x_k)$, $p_k(x_k)$ and h_{ks} (for details see [5], [15]).

Let the spectra Λ_k, $k = \overline{0,r}$, be given. The characteristic functions $\Delta_k(\rho)$, $k = \overline{0,r}$, are entire in ρ of exponential type. By Hadamard's factorization theorem,

$$\Delta_k(\rho) = B_k \exp(A_k\rho)\delta_k(\rho), \quad k = \overline{0,r}, \tag{19}$$

$$\delta_k(\rho) = \rho^{\xi_k}\prod_{n\in\Lambda'_k}\Big(1 - \frac{\rho}{\rho_{kn}}\Big)\exp(\rho/\rho_{kn}), \quad k = \overline{0,r}, \tag{20}$$

where $\Lambda'_k = \{n:\ \rho_{kn} \neq 0\}$, and $\xi_k \geq 0$ is the multiplicity of the zero eigenvalue. In view of (12), we deduce

$$M_k(\rho) = -b_k \exp(a_k\rho)\frac{\delta_k(\rho)}{\delta_0(\rho)}, \quad k = \overline{1,r}, \tag{21}$$

where $b_k = B_k/B_0$, $a_k = A_k - A_0$. Using (8) we calculate for $\rho \in \Pi_\delta^\pm$, $|\rho| \to \infty$:

$$b_k \exp(a_k\rho) = \frac{\delta_0(\rho)[1]}{\delta_k(\rho)(\mp i\rho)(1 \mp h_{k1})},$$

and consequently,

$$a_k = \lim_{|\rho|\to\infty} \frac{1}{\rho} \ln\left(\frac{\delta_0(\rho)}{\delta_k(\rho)}\right), \quad \rho \in \Pi_\delta^\pm, \quad k = \overline{1,r}. \tag{22}$$

Here and below we agree that if $z = |z|e^{i\xi}$, $\xi \in [0, 2\pi)$, then $\ln z = \ln|z| + i\xi$, and $\sqrt{z} := |z|^{1/2}e^{i\xi/2}$. Denote

$$\mu_k^\pm = \lim_{|\rho|\to\infty} \frac{\delta_0(\rho)\exp(-a_k\rho)}{\delta_k(\rho)(\mp i\rho)}, \quad \rho \in \Pi_\delta^\pm, \quad k = \overline{1,r}.$$

Then $b_k(1 \mp h_{k1}) = \mu_k^\pm$, hence

$$b_k = \frac{\mu_k^- + \mu_k^+}{2}, \quad h_{k1} = \frac{\mu_k^- - \mu_k^+}{\mu_k^- + \mu_k^+}, \quad k = \overline{1,r}. \tag{23}$$

Thus, we have uniquely constructed $M_k(\rho)$ and h_{k1} by (20)–(23). Solving auxiliary inverse problems $IP(k)$ for each $k = \overline{1,r}$, we find $q_k(x_k), p_k(x_k), h_{k1}$ and h_{k0} for $k = \overline{1,r}$. In particular, this means that the functions $\varphi_k^{(\nu)}(T_k,\rho)$ and $S_k^{(\nu)}(T_k,\rho)$, $k = \overline{1,r}$, $\nu = 0,1$, are known. Denote $\chi := \exp(2i\omega_0 T_0)$. Using (19)–(20), one can uniquely construct the functions $d_0(\rho)$, $d(\rho)$ and $\Delta_0(\rho)$ by the following algorithm (see [18] for details).

Algorithm 1.

(1) *Calculate* A_k, $k = 0,1$, *by*

$$A_k = -\kappa_k^\pm \mp i\sum_{j=0}^r T_j, \quad \kappa_k^\pm := \lim_{|\rho|\to\infty} \frac{\ln\delta_k(\rho)}{\rho}, \ \rho \in \Pi_\delta^\pm,$$

where $\delta_k(\rho)$ *is constructed by* (20).

(2) *Find* $\sigma_k^\pm$, $k = 0,1$, *via*

$$\sigma_0^\pm = \frac{1}{2^{r+1}}\prod_{j=1}^r (1 \mp h_{j1}) \prod_{j=1}^r \exp(\mp i\omega_j T_j) \lim_{|\rho|\to\infty} \frac{\exp(\kappa_0^\pm\rho)}{\delta_0(\rho)}, \quad \rho \in \Pi_\delta^\pm,$$

$$\sigma_1^\pm = \frac{1}{2^{r+1}}\prod_{j=2}^r (1 \mp h_{j1}) \prod_{j=1}^r \exp(\mp i\omega_j T_j) \lim_{|\rho|\to\infty} \frac{\exp(\kappa_1^\pm\rho)}{\delta_1(\rho)(\mp i\rho)}, \quad \rho \in \Pi_\delta^\pm.$$

(3) *Construct* $\Delta_k^\pm(\rho) = \sigma_k^\pm \exp(A_k\rho)\delta_k(\rho)$, $k = 0,1$.

(4) *Calculate*

$$d_0^{\pm}(\rho)=\frac{1}{\beta_1}\Big(\prod_{j=2}^{r}(\alpha_j\varphi_j(T_j,\rho))\Big)^{-1}\Big(\varphi_1(T_1,\rho)\Delta_1^{\pm}(\rho)-S_1(T_1,\rho)\Delta_0^{\pm}(\rho)\Big).$$

(5) *Find*

$$z_r^{\pm}=\frac{\alpha_0}{2}\lim_{|\rho|\to\infty}\frac{\exp(\mp i\rho T_0)}{d_0^{\pm}(\rho)(\mp i\rho)},\quad \rho\in\Pi_\delta^{\pm}.$$

(6) *Calculate* $\chi=(z_r^+\sigma_k^+)/(z_r^-\sigma_k^-),\ k=0,1.$

(7) *Construct* $\Delta_k^*(\rho)=z_r^{\pm}\chi^{\mp1/2}\Delta_k^{\pm}(\rho),\ k=0,1.$

(8) *Find* $F_s^*(\rho),\ s=1,2$*:*

$$F_1^*(\rho)=\frac{1}{\alpha_1}\Big(\Delta_0^*(\rho)S_1'(T_1,\rho)-\Delta_1^*(\rho)\varphi_1'(T_1,\rho)\Big),$$
$$F_2^*(\rho)=\frac{1}{\beta_1}\Big(\varphi_1(T_1,\rho)\Delta_1^*(\rho)-S_1(T_1,\rho)\Delta_0^*(\rho)\Big).$$

(9) *Calculate* $d_0^*(\rho)$ *and* $d^*(\rho)$ *by*

$$d_0^*(\rho)=\Big(\prod_{j=2}^{r}(\alpha_j\varphi_j(T_j,\rho))\Big)^{-1}F_2^*(\rho),$$
$$d^*(\rho)=\Big(\prod_{j=2}^{r}(\alpha_j\varphi_j(T_j,\rho))\Big)^{-1}$$
$$\times\Big(F_1^*(\rho)-d_0^*(\rho)\sum_{i=2}^{r}(\beta_i\varphi_i'(T_i,\rho))\prod_{j=2,\,j\neq i}^{r}(\alpha_j\varphi_j(T_j,\rho))\Big).$$

(10) *Find* ε*:*

$$\varepsilon=\frac{1}{1+\alpha_0\beta_0}\lim_{|\rho|\to\infty}\Big(\frac{z_0^-\sqrt{\chi}}{2}\exp(i\rho T_0)+\frac{z_0^+}{2\sqrt{\chi}}\exp(-i\rho T_0)-d^*(\rho)\Big).$$

(11) *Construct* $d(\rho)=\varepsilon d^*(\rho),\ d_0(\rho)=\varepsilon d_0^*(\rho),\ \Delta_0(\rho)=\varepsilon\Delta_0^*(\rho).$

Let $\Delta_{-1}(\rho)$ be the characteristic function of the boundary value problem B_{-1}. Similarly to (10) we calculate

$$\Delta_{-1}(\rho)=d_{-1}(\rho)\prod_{j=1}^{r}(\alpha_j\varphi_j(T_j,\rho))+d_0(\rho)\frac{\alpha_{-1}}{\alpha_0}\sum_{i=1}^{r}(\beta_i\varphi_i'(T_i,\rho))\prod_{j=1,\,j\neq i}^{r}(\alpha_j\varphi_j(T_j,\rho)),\tag{24}$$

where

$$d_{-1}(\rho)=\alpha_{-1}\varphi_0(T_0,\rho)+\beta_0S_0'(T_0,\rho)-(1+\alpha_{-1}\beta_0).\tag{25}$$

Since $\exp(i\omega_0T_0)$ and h_{01} are already found, it follows that the function $\Delta_{-1}(\rho)$ can be uniquely reconstructed from its zeros by the following algorithm.

Algorithm 2.

(1) *Calculate* A_{-1} *by*

$$A_{-1} = -\kappa_{-1}^{\pm} \mp i \sum_{j=0}^{r} T_j, \quad \kappa_{-1}^{\pm} := \lim_{|\rho|\to\infty} \frac{\ln \delta_{-1}(\rho)}{\rho}, \ \rho \in \Pi_{\delta}^{\pm},$$

$$\delta_{-1}(\rho) = \rho^{\xi_{-1}} \prod_{n\in\Lambda'_{-1}} \left(1 - \frac{\rho}{\rho_{-1,n}}\right) \exp(\rho/\rho_{-1,n}),$$

where $\Lambda'_{-1} = \{n : \rho_{-1,n} \neq 0\}$, *and* $\xi_{-1} \geq 0$ *is the multiplicity of the zero eigenvalue.*

(2) *Find* $\sigma_{-1}^{\pm}$ *via*

$$\sigma_{-1}^{\pm} = \frac{1}{2^{r+1}} \prod_{j=1}^{r} (1 \mp h_{j1}) \prod_{j=1}^{r} \exp(\mp i\omega_j T_j) \lim_{|\rho|\to\infty} \frac{\exp(\kappa_{-1}^{\pm}\rho)}{\delta_{-1}(\rho)}, \quad \rho \in \Pi_{\delta}^{\pm},$$

(3) *Calculate* $B_{-1} = \sigma_{-1}^{\pm} z_{r,-1}^{\pm} \exp(\mp i\omega_0 T_0)$, *where* $z_{r,-1}^{\pm}$ *is obtained from* $z_r^{\pm}$ *by the replacement of* α_0 *with* α_{-1}.

(4) *Construct* $\Delta_{-1}(\rho) = B_{-1} \exp(A_{-1}\rho)\delta_{-1}(\rho)$.

Using (24) we find the function $d_{-1}(\rho)$. Denote

$$\mu_0(\rho) := \varphi_0(T_0, \rho), \quad \mu_1(\rho) := S_0(T_0, \rho).$$

Taking (11) and (25) into account, one can calculate $\mu_0(\rho)$ and $\mu_1(\rho)$ by the formulae

$$\mu_0(\rho) = \frac{d(\rho) - d_{-1}(\rho)}{\alpha_0 - \alpha_{-1}} + \beta_0, \quad \mu_1(\rho) = \frac{d_0(\rho)}{\alpha_0}. \tag{26}$$

The functions $\mu_0(\rho)$ and $\mu_1(\rho)$ are the characteristic functions for the boundary value problems

$$-y_0''(x_0) + (\rho^2 + \rho p_0(x_0) + q_0(x_0)) y_0(x_0) = 0, \ x_0 \in (0, T_0), \quad y_0'(0) - (i\rho h_{01} + h_{00}) y_0(0) = y(T_0) = 0,$$

and

$$-y_0''(x_0) + (\rho^2 + \rho p_0(x_0) + q_0(x_0)) y_0(x_0) = 0, \ x_0 \in (0, T_0), \quad y_0(0) = y(T_0) = 0,$$

respectively. It was shown in [16] that the specification of $\mu_0(\rho)$ and $\mu_1(\rho)$ uniquely determines the potential (q_0, p_0) on the edge e_0, and the coefficients of boundary conditions. Moreover, a constructive procedure for this inverse problem is given in [16]. Thus, we have obtained a procedure for the solution of Inverse problem 1 and proved its uniqueness. In other words, Theorem 1 is proved, and the solution of Inverse problem 1 can be found by the following algorithm.

Algorithm 3. *Let Λ_k, $k=\overline{-1,r}$ be given.*

(1) *Construct $\delta_k(\rho)$, $k=\overline{-1,r}$ by (20).*

(2) *Calculate $M_k(\rho)$ and h_{k1}, $k=\overline{1,r}$ via (21)–(23).*

(3) *For each fixed $k=\overline{1,r}$, solve the inverse problem $IP(k)$ and find the functions $q_k(x_k)$, $p_k(x_k)$, $x_k\in(0,T_k)$ on the edge e_k, and the coefficient h_{k0}.*

(4) *For each fixed $k=\overline{1,r}$, calculate the functions $\varphi_k^{(\nu)}(T_k,\rho)$, $S_k^{(\nu)}(T_k,\rho)$, $\nu=0,1$.*

(5) *Construct the functions $d(\rho)$, $d_0(\rho)$ and $\Delta_0(\rho)$ by Algorithm 1.*

(6) *Find the function $\Delta_{-1}(\rho)$ by Algorithm 2.*

(7) *Calculate the function $d_{-1}(\rho)$ using (24).*

(8) *Construct the functions $\mu_0(\rho)$ and $\mu_1(\rho)$ via (26).*

(9) *Find $q_0(x_0)$, $p_0(x_0)$, $x_0\in(0,T_k)$ and h_{00}, h_{01} from $\mu_1(\rho)$ and $\mu_2(\rho)$, using results from [16].*

By similar arguments one can solve Inverse problem 2. Indeed, let $\Delta_{r+1}(\rho)$ be the characteristic function of the boundary value problem B_{r+1}. Then

$$\Delta_{r+1}(\rho)=d_{r+1}(\rho)\prod_{j=1}^{r}(\alpha_j\varphi_j(T_j,\rho))+d_0(\rho)\sum_{i=1}^{r}(\beta_i\varphi_i'(T_i,\rho))\prod_{j=1,\,j\neq i}^{r}(\alpha_j\varphi_j(T_j,\rho)),\tag{27}$$

where

$$d_{r+1}(\rho)=\alpha_0\varphi_0(T_0,\rho)+\beta_{-1}S_0'(T_0,\rho)-(1+\alpha_0\beta_{-1}).\tag{28}$$

The function $\Delta_{r+1}(\rho)$ can be uniquely reconstructed from its zeros by the following algorithm.

Algorithm 4.

(1) *Calculate A_{r+1} by*

$$A_{r+1}=-\kappa_{r+1}^{\pm}\mp i\sum_{j=0}^{r}T_j,\quad \kappa_{r+1}^{\pm}:=\lim_{|\rho|\to\infty}\frac{\ln\delta_{r+1}(\rho)}{\rho},\ \rho\in\Pi_\delta^{\pm},$$

$$\delta_{r+1}(\rho)=\rho^{\xi_{r+1}}\prod_{n\in\Lambda'_{r+1}}\Big(1-\frac{\rho}{\rho_{r+1,n}}\Big)\exp(\rho/\rho_{r+1,n}),$$

where $\Lambda'_{r+1}=\{n:\ \rho_{r+1,n}\neq 0\}$, and $\xi_{r+1}\geq 0$ is the multiplicity of the zero eigenvalue.

(2) *Find $\sigma_{r+1}^{\pm}$ via*

$$\sigma_{r+1}^{\pm}=\frac{1}{2^{r+1}}\prod_{j=1}^{r}(1\mp h_{j1})\prod_{j=1}^{r}\exp(\mp i\omega_jT_j)\lim_{|\rho|\to\infty}\frac{\exp(\kappa_{r+1}^{\pm}\rho)}{\delta_{r+1}(\rho)},\quad \rho\in\Pi_\delta^{\pm},$$

(3) *Calculate $B_{r+1}=\sigma_{r+1}^{\pm}z_{r,r+1}^{\pm}\exp(\mp i\omega_0T_0)$, where $z_{r,r+1}^{\pm}$ is obtained from $z_r^{\pm}$ by the replacement of β_0 with β_{-1}.*

(4) *Construct $\Delta_{r+1}(\rho)=B_{r+1}\exp(A_{r+1}\rho)\delta_{r+1}(\rho)$.*

Using (27) we find the function $d_{r+1}(\rho)$. Denote $\mu_2(\rho) := S_0'(T_0,\rho)$. Taking (11) and (28) into account, one can calculate $\mu_1(\rho)$ and $\mu_2(\rho)$ by the formulae

$$\mu_2(\rho) = \frac{d(\rho) - d_{r+1}(\rho)}{\beta_0 - \beta_{-1}} + \alpha_0, \quad \mu_1(\rho) = \frac{d_0(\rho)}{\alpha_0}. \tag{29}$$

The function $\mu_2(\rho)$ is the characteristic functions for the boundary value problems

$$-y_0''(x_0) + (\rho^2 + \rho p_0(x_0) + q_0(x_0))y_0(x_0) = 0,\ x_0 \in (0, T_0), \quad y_0(0) = y'(T_0) = 0.$$

It was shown in [16] that the specification of $\mu_1(\rho)$ and $\mu_2(\rho)$ uniquely determines the potential (q_0, p_0) on the edge e_0, and the coefficients of boundary conditions. Moreover, a constructive procedure for this inverse problem is given in [16]. Thus, we have obtained a procedure for the solution of Inverse problem 2 and proved its uniqueness. In other words, Theorem 2 is proved, and the solution of Inverse problem 2 can be found by the following algorithm.

Algorithm 5. *Let Λ_k, $k = \overline{0, r+1}$ be given.*

(1) *Construct $\delta_k(\rho)$, $k = \overline{0, r+1}$ by (20).*

(2) *Calculate $M_k(\rho)$ and h_{k1}, $k = \overline{1, r}$ via (21)-(23).*

(3) *For each fixed $k = \overline{1, r}$, solve the inverse problem $IP(k)$ and find the functions $q_k(x_k)$, $p_k(x_k)$, $x_k \in (0, T_k)$ on the edge e_k, and the coefficient h_{k0}.*

(4) *For each fixed $k = \overline{1, r}$, calculate the functions $\varphi_k^{(\nu)}(T_k,\rho)$, $S_k^{(\nu)}(T_k,\rho)$, $\nu = 0, 1$.*

(5) *Construct the functions $d(\rho)$, $d_0(\rho)$ and $\Delta_0(\rho)$ by Algorithm 1.*

(6) *Find the function $\Delta_{r+1}(\rho)$ by Algorithm 4.*

(7) *Calculate the function $d_{r+1}(\rho)$ using (27).*

(8) *Construct the functions $\mu_1(\rho)$ and $\mu_2(\rho)$ via (29).*

(9) *Find $q_0(x_0)$, $p_0(x_0)$, $x_0 \in (0, T_k)$ and h_{00}, h_{01} from $\mu_1(\rho)$ and $\mu_2(\rho)$, using results from [16].*

Denote by B the boundary value problem on the edge e_0 for equation (1) with $j = 0$, under the conditions $y_0(0) = \alpha_0 y_0(T_0)$, $U_0(y_0) = \beta_0 y_0'(T_0)$. Let $\Omega = \{\omega_n\}$ be the Ω– sequence for B (see [16]).

Inverse problem 3. Given Λ_k, $k = \overline{0, r}$, and Ω, construct the potential (q, p) on T and the coefficients h_1, h_0.

By similar arguments as above one can prove the uniqueness theorem for Inverse problem 3, and provide an algorithm for its solution (see [18] for details).

Acknowledgement

This research was supported in part by Grant 13-01-00134 of Russian Foundation for Basic Research.

References

[1] V. A. Marchenko, Sturm-Liouville operators and their applications, “Naukova Dumka", Kiev, 1977; English transl., Birkhäuser, 1986.

[2] B. M. Levitan, Inverse Sturm-Liouville problems, Nauka, Moscow, 1984; English transl., VNU Sci.Press, Utrecht, 1987.

[3] R. Beals, P. Deift and C. Tomei, Direct and Inverse Scattering on the Line, Math. Surveys and Monographs, v.28. Amer. Math. Soc. Providence: RI, 1988.

[4] G. Freiling and V. A. Yurko, Inverse Sturm-Liouville Problems and their Applications, NOVA Science Publishers, New York, 2001.

[5] V. A. Yurko, Method of Spectral Mappings in the Inverse Problem Theory, Inverse and Ill-posed Problems Series. VSP, Utrecht, 2002.

[6] Yu. V. Pokornyi and A. V. Borovskikh, *Differential equations on networks* (*geometric graphs*), J. Math. Sci. (N.Y.), **119**(2004), 691–718.

[7] Yu. Pokornyi and V. Pryadiev, *The qualitative Sturm-Liouville theory on spatial networks*, J. Math. Sci. (N.Y.), **119** (2004), 788–835.

[8] M. I. Belishev, *Boundary spectral inverse problem on a class of graphs* (*trees*) *by the BC method*, Inverse Problems, **20** (2004), 647–672.

[9] V. A. Yurko, *Inverse spectral problems for Sturm-Liouville operators on graphs*, Inverse Problems, **21**(2005), 1075–1086.

[10] B. M. Brown and R. Weikard, *A Borg-Levinson theorem for trees*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., **461**, no.2062 (2005), 3231–3243.

[11] V. A. Yurko, *Inverse problems for Sturm-Liouville operators on bush-type graphs*, Inverse Problems, **25**(2009), 105008, 14pp.

[12] V. A. Yurko, *Inverse spectral problems for differential operators on a graph with a rooted cycle*, Tamkang Journal of Mathematics, **40**(2009), 271–286.

[13] V. A. Yurko, *An inverse problem for Sturm-Liouville operators on A-graphs*, Applied Math. Letters, **23**(2010), 875–879.

[14] V. A. Yurko, *Inverse spectral problems for differential operators on arbitrary compact graphs*, Journal of Inverse and Ill-Posed Proplems, **18**(2010), 245–261.

[15] V. A. Yurko, *Recovering differential pencils on compact graphs*, J. Differ. Equations, **244** (2008), 431–443.

[16] V. A. Yurko, *Inverse problems for non-selfadjoint quasi-periodic differential pencils*, Analysis and Math. Physics, **2**(2012), 215–230.

[17] M. A. Naimark, Linear Differential Operators, 2nd ed., Nauka, Moscow, 1969; English transl. of 1st ed., Parts I,II, Ungar, New York, 1967, 1968.

[18] V. A. Yurko, *Differential pensils on graphs with a cycle*, Schriftenreiche des Fachbereichs Mathematik, SM-DUE-758, Universitaet Duisburg-Essen, 2013, 12pp.

Department of Mathematics, Saratov University, Astrakhanskaya 83, Saratov 410012, Russia.

E-mail: yurkova@info.sgu.ru