TAMKANG JOURNAL OF MATHEMATICS
Volume 37, Number 4, 367-375, Winter 2006

DEDEKIND ZETA FUNCTIONS OF CERTAIN REAL QUADRATIC FIELDS

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Abstract. Using analytic and modular transformation methods, we represent the value of the product of two Dedekind zeta functions of certain real quadratic number fields at -3 by Dedekind sums of high rank in this paper.

1. Introduction and Results

The values of Dedekind zeta function of a number field $\mathbf{K}$ at rational integers are closely related with the algebraic character of the number field $\mathbf{K}$ itself. To represent these values as clearly as possible is one of the important tasks of algebraic number theory. In history many mathematicians had some work on this project. Hasse (see ref.[1]) expressed Dedekind zeta function of a number field as product of Riemann zeta function and usual Dirichlet L-functions. Siegel (see ref.[2]) got some properties of explicit values of Dekind zeta functions of quadratic number fields at negative integers, and a particular interesting case is at -1, using modular transformation method. Zagier (see ref.[3]) also obtained another expression of the values of Dedekind zeta functions of real quadratic fields at negative integers using Kronecker limit formula. Shintani (see ref.[4, 5]) using astonishing linear programming method expressed Dedekind zeta functions as a sum of Dirichlet series of some real cones.

In reference [6], we represented the value of the product of two Dedekind zeta functions of certain real quadratic number fields at -1 by Dedekind sums of high rank. Using the reciprocity law of Dedekind sums (see ref.[7]) and software of Mathematica 4.0, we got

Theorem 1. *If the class number of the real quadratic number field $\mathbf{Q}(\sqrt{q})$ is 1, with prime $q = 4n^2 + 1$. Then*

$$\zeta_{\mathbf{Q}\sqrt{5q}}(-1) = \frac{1}{45}(26n^3 - 41n \pm 9), \text{if}\, n \equiv \pm 2 \pmod 5.$$

Received August 31, 2005; revised March 16, 2006.
2000 *Mathematics Subject Classification.* 11R11, 11R42, 11R68.
Key words and phrases. Quadratic number field, ideal class group, Dedekind zeta function, Dedekind sum, modular group.
The first author would be happy to thank the financial support from the China Scholarship Council for his visiting to the University of Western Ontario as well as the hospitality of prof. Jan Minc and the Department of Mathematics of UWO. The first author is partly supported by NSFC project 10471121, and the second author is supported by NSFC project 10471104. Both authors will thank the referee for the valuable suggestions.

The main result of this paper is the following Theorem 2.

Theorem 2. *Let real quadratic fields* $\mathbf{K}_1 = \mathbf{Q}(\sqrt{p})$, $\mathbf{K}_2 = \mathbf{Q}(\sqrt{q})$, *where* $p \equiv q \equiv 1 \pmod 4$ *be different primes. Let the class number of* $\mathbf{K}_2$ *be 1, and write* $c = \frac{1-q}{4}$ *and* $\mathbf{K}_3 = \mathbf{Q}(\sqrt{pq})$, *then we have*

$$
\begin{aligned}
&14400 j \delta_{\mathbf{K}_2} \zeta_{\mathbf{K}_1}(-3) \zeta_{\mathbf{K}_3}(-3) \\
=& \frac{3q^2p^6}{35840}\left(\frac{64q^3U^3}{3T} - 16q^2\frac{U}{T}\right)\left(p - \frac{1}{p^7}\right)B_8 \\
&+ \frac{1}{8!}\sum_{m=0}^{4}(-1)^{m+1}\binom{8}{m} S_{m,8-m}\left(\frac{T-U}{2}, \frac{U}{p}\right)\cdot\left(9qU^2(m-1)(m-7) - 540 + 45T^2\right) \\
&\quad \cdot \frac{1}{T^{3-m}} \sum_{f=0}^{[\frac{4-m}{2}]} \binom{4-m}{2f} 2^{5-m-2f}(-U^2q)^f \\
&+ \frac{90}{8!}\sum_{m=0}^{3}(-1)^{m+1}\binom{8}{m} S_{m,8-m}\left(\frac{T-U}{2}, \frac{U}{p}\right)\frac{4-m}{T^{3-m}} \sum_{f=1}^{[\frac{5-m}{2}]} \binom{4-m}{2f-1} 2^{6-m-2f}(-U^2q)^f \\
&+ \frac{T}{576}\Big(S_{4,4}\left(\frac{T-U}{2}, \frac{U}{p}\right) - \sum_{l (\mathrm{mod}\, p)} \chi(l^2+l+c) S_{4,4}\left(\frac{T-U}{2} - lU, pU\right)\Big) \\
&\quad \cdot(-81qU^2 - 540 + 45T^2) \\
&+ \frac{p^6}{2\cdot 8!}\sum_{l (\mathrm{mod}\, p)} \chi(l^2+l+c) \sum_{m=0}^{3}(-1)^{m+1}\binom{8}{m} \\
&\quad \cdot\Big(S_{m,8-m}\left(\frac{T-U}{2-lU}, pU\right) + S_{m,8-m}\left(\frac{T+U}{2} + lU, pU\right)\Big) \\
&\quad \cdot \frac{1}{T^{3-m}} \sum_{f=0}^{[\frac{4-m}{2}]} \binom{4-m}{2f} 2^{5-m-2f}(-U^2q)^f \cdot \left(9qU^2(m-1)(m-7) - 540 + 45T^2\right) \\
&+ \frac{90p^6}{2\cdot 8!}\sum_{l (\mathrm{mod}\, p)} \chi(l^2+l+c) \sum_{l=0}^{3}(-1)^{m+1}\binom{8}{m}(4-m)\Big(S_{m,8-m}\left(\frac{T-U}{2} - lU, pU\right) \\
&+ S_{m,8-m}\left(\frac{T+U}{2} + lU, pU\right)\Big)\frac{1}{T^{3-m}} \sum_{f=0}^{[\frac{5-m}{2}]} \binom{4-m}{2f-1} 2^{6-m-2f}(-U^2q)^f
\end{aligned}
$$

where $S_{k,l}(u,m) = \sum_{v \pmod m} B_k(\{\frac{v}{m}\})B_l(\{\frac{uv}{m}\})$ *be Dedekind sum; and* $B_n(x)$ *be the usual Bernoulli polynomial, with* $[x]$ *and* $\{x\}$ *denote the integral part and fractional part of* x *respectively;* $\delta_{\mathbf{K}_2} = \frac{\log \epsilon_+}{\log \epsilon}$, *with* ϵ *and* ϵ_+ *denote the fundamental and totally positive fundamental unit of* $\mathbf{K}_2$ *respectively;* χ *be the Kronecker symbol mod* p; $\epsilon_+^{2j} = \frac{T+U\sqrt{q}}{2}$ *with positive integer* j *such that* $p \mid U$. *From the define equation of Dedekind sums one can see that* $S_{k,l}(u_1,m) = S_{k,l}(u_2,m)$, *if* $u_1 \equiv u_2 \pmod m$; $S_{k,l}(u,m) = S_{l,k}(\overline{u},m)$, *if* $u\overline{u} \equiv 1 \pmod m$.

Of course Theorem 2 is a effective computing formulae in the case of the the conditions in Theorem 1.

2. Main Lemma

Let $\mathbf{K} = \mathbf{Q}(\sqrt{\Delta})$ be a real quadratic number field with basic discriminant Δ, and let $A = [a, \frac{-b+\sqrt{\Delta}}{2}]$ be a integral ideal of $\mathbf{K}$. Set $A^* = \sqrt{\Delta}A$, $B = [a, \frac{b+\sqrt{\Delta}}{2}]$, where a, b, c be rational integers with $0 \leq\mid a \mid\leq b$, $\Delta = b^2 - 4ac$ and $g.c.d(a, b, c) = 1$. Let $F_A(Z) = aZ^2 - bZ + c$ and $F_B(Z) = aZ^2 + bZ + c$. Denote $\omega = \frac{b+\sqrt{\Delta}}{2a}$, $\omega' = \frac{b-\sqrt{\Delta}}{2a}$, $\overline{\omega} = \frac{-b+\sqrt{\Delta}}{2a}$, $\overline{\omega}' = \frac{-b-\sqrt{\Delta}}{2a}$. let ϵ_+ be a totally positive fundamental unit of the number field $\mathbf{K}$, For positive rational integer j, set

$$\epsilon_+^j = \frac{T_j + U_j\sqrt{\Delta}}{2}, \qquad \rho_j = \frac{\epsilon_+^{-j} + i\epsilon_+^j}{\epsilon_+^{-j} - i\epsilon_+^j},$$

$$Z_{A,j} = \frac{b}{2a} + \frac{\sqrt{\Delta}}{2a}\rho_j, \qquad Z_{B,j} = -\frac{b}{2a} + \frac{\sqrt{\Delta}}{2a}\rho_j,$$

$$\widetilde{Z}_{A,j} = \frac{b}{2a} + \frac{\sqrt{\Delta}}{2a}\overline{\rho}_j, \qquad \widetilde{Z}_{B,j} = -\frac{b}{2a} + \frac{\sqrt{\Delta}}{2a}\overline{\rho}_j.$$

Let Γ denote the upper half circle with center $\frac{b}{2a}$ and radius $\frac{\sqrt{\Delta}}{2a}$, and $\Gamma_{A,j}$ denote the arc of Γ located between $Z_{A,j}$ and $\widetilde{Z}_{A,j}$.

We write $Z = X + iY$, where X and Y denote real and imaginary part of Z respectively. Let χ be a real primitive Dirichlet character of mod k. Define

$$\mathcal{L}(s, \chi, A) = \sum_{\substack{\lambda >> 0 \\ \lambda \in A/\epsilon_+}} \frac{\chi(N((\lambda))/N(A))}{(N((\lambda))/N(A))^s}, \mathrm{Re}(s) > 1, \tag{1}$$

where N denote the norm map of $\mathbf{K}/\mathbf{Q}$. Obviously, such defined $\mathcal{L}(s, \chi, A)$ is a ideal class function of A.

We got the following Lemma 1 in ref. [8]:

Lemma 1. *With notations above, and let s be complex variable with $Re(s) > 1$, then we have*

$$j(\mathcal{L}(s, \chi, A) + \chi(-1)\mathcal{L}(s, \chi, A^*)) = -\frac{\Gamma(s)\Delta^{-\frac{s-1}{2}}}{2\Gamma(\frac{s}{2})^2} \int_{\Gamma_{A,j}} \frac{E(s, Z, \chi, A)}{F_A(Z)} \, \mathrm{d}Z, \tag{2}$$

where the Eisenstein series

$$E(s, Z, \chi, A) = \sum_{\substack{(m,n)\neq(0,0) \\ m,n=-\infty}}^{+\infty} \frac{\chi(am^2 + bmn + cn^2)Y^s}{\mid m + nZ \mid^{2s}}.$$

We got the Fourier expansion of $E(s,Z,A,\chi)$ in ref. [7], i.e.

$$E(s,Z,\chi,A)=2Y^s\chi(a)\zeta(2s)\prod_{p|k}(1-p^{-2s})+\frac{2\sqrt{\pi}\Gamma(s-\frac{1}{2})Y^{1-s}}{k\Gamma(s)}$$
$$\times\sum_{n=1}^{+\infty}n^{1-2s}\sum_{m \bmod k}\chi(am^2+bmn+cn^2)+\frac{8\pi^s k^{-s-\frac{1}{2}}Y^{\frac{1}{2}}}{\Gamma(s)}\sum_{u=1}^{+\infty}u^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(\frac{2\pi uY}{k})$$
$$\times\sum_{1\le n|u}n^{1-2s}\sum_{m \bmod k}\chi(am^2+bmn+cn^2)\cos\frac{2\pi u(\frac{m}{n}+X)}{k}, \tag{3}$$

where $K_s(z)$ be Bessel function, i.e.

$$K_s(z)=\frac{1}{2}\int_0^{+\infty}\exp^{-\frac{1}{2}z(t+\frac{1}{t})}t^{s-1}\,\mathrm{d}t,\ z>0.$$

It is easy to know that the Eisenstein series $E(s,Z,A,\chi)$ have the analytic continuations to the whole complex plane by (3). It is well known that

$\mathcal{L}(\int,\chi,\mathcal{A})$ and $\Gamma(s)$ could also have the analytic continuations to the whole complex plane.

Taking the limit of both sides of (2) when $s\to -3$, substituting (3) into (2), and then write the R.H.S of (2) as three summands, i.e. $I_1+I_2+I_3$, and let's compute each summand individually.

Firstly, by the well-known functional equation of $\zeta(s)$ and $\lim_{s\to-3}\Gamma(\frac{s}{2})^2=\frac{16\pi}{9}$ we get

$$\lim_{s\to-3}I_1=-\frac{135\chi(a)\Delta^2\zeta(7)}{(2\pi)^7}(1-p^6)\int_{\Gamma_{A,j}}\frac{Y^{-3}}{F_A(Z)}\,\mathrm{d}Z. \tag{4}$$

It is not difficult to get $Y\frac{\mathrm{d}F_A(Z)}{\mathrm{d}Z}=-iF_A(Z)$ for $Z\in\Gamma_{A,j}$. Hence substituting it in (4) we get

$$\lim_{s\to-3}I_1=i\frac{135\chi(a)\Delta^2\zeta(7)}{(2\pi)^7}\prod_{p|k}(p^6-1)(2aF_A^{-2}(Z)+\frac{\Delta}{3}F_A^{-3}(Z))\Big|_{Z_{A,j}}^{\widetilde{Z}_{A,j}} \tag{5}$$

Secondly, let's calculate $\lim_{s\to-3}I_2$. It is easy to get $\sum_{n=1}^{+\infty}n^{1-2s}\sum_{m \bmod k}\chi(am^2+bmn+cn^2)=\sum_{1\le m,n\le k}\chi(am^2+bmn+cn^2)\zeta(2s-1,n/k)k^{1-2s}$, where $\zeta(\star,\star)$ be the Hurwitz zeta function. We know that $\zeta(-7,n/k)=-\frac{1}{8}B_8(n/k)$ and $\Gamma(-7/2)=16\sqrt{\pi}/105$, so through a not difficult computation, we have

$$\lim_{s\to-3}I_2=\frac{3\Delta^2k^6}{35840a^4}\sum_{1\le m,n\le k}\chi(am^2+bmn+cn^2)B_8(\frac{n}{k})(\frac{64\Delta^3U_{2j}^3}{3T_{2j}}-16\Delta^2\frac{U_{2j}}{T_{2j}}) \tag{6}$$

Finally, we deal with $\lim_{s\to-3}I_3$. In ref.[9], we get $K_{n+\frac{1}{2}}(z)=\sqrt{\frac{pi}{2z}}\exp(-z)\sum_{l=0}^{n}\frac{(n+l)!}{l!(n-l)!}(2z)^{-l}$. So applying the similar integral techniques as in ref.[6] though, through a long but not tough calculation we have:

$$\lim_{s\to-3} I_3 = -\frac{9\Delta^2k^3}{16\pi^4}\sum_{u=1}^{+\infty}u^{-4}\sum_{1\le n|u}n^7\sum_{m \pmod k}\chi(am^2+bmn+cn^2)$$
$$\cdot\Big(\Big(\frac{k}{2\pi iuF_A(Z)}+\frac{5k^2F_A'(Z)}{4\pi^2u^2F_A^2(Z)}+\frac{30ak^2i}{8\pi^3u^3F_A^2(Z)}+\frac{5\Delta k^3 i}{8\pi^3u^3F_A^3(Z)}\Big)$$
$$\cdot\exp\Big(\frac{2\pi iu}{k}(\frac{m}{n}+Z)\Big)\Big|_{Z_{A,j}}^{\widetilde{Z}_{A,j}}$$
$$+\Big(\frac{k}{2\pi iuF_B(Z)}+\frac{5k^2F_B'(Z)}{4\pi^2u^2F_B^2(Z)}+\frac{30ak^2i}{8\pi^3u^3F_B^2(Z)}+\frac{5\Delta k^3 i}{8\pi^3u^3F_B^3(Z)}\Big)$$
$$\cdot\exp\Big(\frac{2\pi iu}{k}(-\frac{m}{n}+Z)\Big)\Big|_{Z_{B,j}}^{\widetilde{Z}_{B,j}}\Big) \tag{7}$$

By (2), (3), (5), (6), and (7) we have:

Main Lemma. Notations as explained above,

$$\lim_{s\to-3} j(\mathcal{L}(\int,\chi,\mathcal{A})+\chi(-\infty)\mathcal{L}(\int,\chi,\mathcal{A}^*))$$
$$= i\frac{135\chi(a)\Delta^2\zeta(7)}{(2\pi)^7}\prod_{p|k}(p^6-1)(2aF_A^{-2}(Z)+\frac{\Delta}{3}F_A^{-3}(Z))\Big|_{Z_{A,j}}^{\widetilde{Z}_{A,j}}$$
$$+\frac{3\Delta^2k^6}{35840a^4}\sum_{1\le m,n\le k}\chi(am^2+bmn+cn^2)B_8(\frac{n}{k})(\frac{64\Delta^3U_{2j}^3}{3T_{2j}}-16\Delta^2\frac{U_{2j}}{T_{2j}})$$
$$-\frac{9\Delta^2k^3}{16\pi^4}\sum_{u=1}^{+\infty}u^{-4}\sum_{1\le n|u}n^7\sum_{m \pmod k}\chi(am^2+bmn+cn^2)$$
$$\cdot\Big(\Big(\frac{k}{2\pi iuF_A(Z)}+\frac{5k^2F_A'(Z)}{4\pi^2u^2F_A^2(Z)}+\frac{30ak^2i}{8\pi^3u^3F_A^2(Z)}+\frac{5\Delta k^3 i}{8\pi^3u^3F_A^3(Z)}\Big)$$
$$\cdot\exp(\frac{2\pi iu}{k}(\frac{m}{n}+Z))\Big|_{Z_{A,j}}^{\widetilde{Z}_{A,j}}$$
$$+\Big(\frac{k}{2\pi iuF_B(Z)}+\frac{5k^2F_B'(Z)}{4\pi^2u^2F_B^2(Z)}+\frac{30ak^2i}{8\pi^3u^3F_B^2(Z)}+\frac{5\Delta k^3 i}{8\pi^3u^3F_B^3(Z)}\Big)$$
$$\cdot\exp(\frac{2\pi iu}{k}(-\frac{m}{n}+Z))\Big|_{Z_{B,j}}^{\widetilde{Z}_{B,j}}\Big) \tag{8}$$

3. Proof of Theorem 2

$\mathcal{L}(s,\chi,A)$ and χ as in the last section. We define $\mathcal{L}(s,\chi)=\sum_A\mathcal{L}(s,\chi,A)$, where A runs through representative set of the narrow ideal class group of the real quadratic number field $\mathbf{K} = \mathbf{Q}(\sqrt{\Delta})$ with basic discriminant Δ. Obviously, $\mathcal{L}(s,\chi)$ is analytic for $Re(s) > 1$, we also proved

Theorem 3. *Symbols as explained above, we have*

$$L(s,\chi)L(s,\chi\chi_\Delta) = \mathcal{L}(s,\chi), \tag{9}$$

where χ_Δ *is the Kronecker symbol* $(\frac{\Delta}{\star})$, *and* $L(s,\chi)$ *together with* $L(s,\chi\chi_\Delta)$ *is the usual Dirichlet L-function.*

To prove Theorem 2, we need to apply Theorem 3 to the Main Lemma.

In the Main Lemma, we may assume k to be a prime p with $p \equiv 1(mod 4)$ and $g.c.d(p,\Delta) = 1$, and take an integral ideal A described in the beginning of the second section satisfying $a = b = 1, c = \frac{1-\Delta}{4}$. If the class number of the real quadratic number field $\mathbf{K}$ is 1, and χ be a real primitive Dirichlet character of module p then using the well-known fact that $\zeta(s)L(s,\chi_\Delta) = \zeta_{\mathbf{Q}(\sqrt{\Delta})}(s)$ and $\zeta(s)L(s,\chi\chi_\Delta) = \zeta_{\mathbf{Q}(\sqrt{p\Delta})}(s)$ with $\zeta(-3) = 1/120$, we can easily get as $s \to -3$

$$\text{L.H.S of (2)} = 14400 j \delta_{\mathbf{K}} \zeta_{\mathbf{K}}(-3) \zeta_{\mathbf{Q}(\sqrt{p\Delta})}(-3) \tag{10}$$

To calculate the R.H.S of (2), we set

$$Q_m(z) = \frac{1}{2}\zeta(2m+1) + \sum_{n=1}^{+\infty} \frac{\sigma_{2m+1}(n)}{n^{2m+1}} \exp(2\pi i n z), Im(z) > 0, \tag{11}$$

Using standard summation techniques in the R.H.S of (2), we have

$$\begin{aligned}
\lim_{s\to -3} I_1 + I_3 = \frac{9\Delta^2 p^3}{(2\pi i)^7} \Bigg(& \frac{1}{F_A(Z)}[p^{-1}Q_3''(pZ) + p \sum_{m (\operatorname{mod} p)} \chi(m^2+m+c) Q_3''(\frac{m+Z}{p})]\Big|_{Z_{A,j}}^{\widetilde{Z}_{A,j}} \\
& - \frac{5F_A'(Z)}{F_A^2(Z)}[p^{-2}Q_3'(pZ) + p^2 \sum_{m (\operatorname{mod} p)} \chi(m^2+m+c) Q_3'(\frac{m+Z}{p})]\Big|_{Z_{A,j}}^{\widetilde{Z}_{A,j}} \\
& + \frac{30}{F_A^2(Z)}[p^{-3}Q_3(pZ) + p^3 \sum_{m (\operatorname{mod} p)} \chi(m^2+m+c) Q_3(\frac{m+Z}{p})]\Big|_{Z_{A,j}}^{\widetilde{Z}_{A,j}} \\
& + \frac{5\Delta}{F_A^3(Z)}[p^{-3}Q_3(pZ) + p^3 \sum_{m (\operatorname{mod} p)} \chi(m^2+m+c) Q_3(\frac{m+Z}{p})]\Big|_{Z_{A,j}}^{\widetilde{Z}_{A,j}}
\end{aligned}$$

$$+\frac{1}{F_B(Z)}[p^{-1}Q_3''(pZ)+p\sum_{m(\mathrm{mod}\,p)}\chi(m^2+m+c)Q_3''(\frac{m+Z}{p})]\Big|_{Z_{B,j}}^{\widetilde{Z}_{B,j}}$$
$$-\frac{5F_B'(Z)}{F_B^2(Z)}[p^{-2}Q_3'(pZ)+p^2\sum_{m(\mathrm{mod}\,p)}\chi(m^2+m+c)Q_3'(\frac{m+Z}{p})]\Big|_{Z_{B,j}}^{\widetilde{Z}_{B,j}}$$
$$+\frac{30}{F_B^2(Z)}[p^{-3}Q_3(pZ)+p^3\sum_{m(\mathrm{mod}\,p)}\chi(m^2+m+c)Q_3(\frac{m+Z}{p})]\Big|_{Z_{B,j}}^{\widetilde{Z}_{B,j}}$$
$$+\frac{5\Delta}{F_B^3(Z)}[p^{-3}Q_3(pZ)+p^3\sum_{m(\mathrm{mod}\,p)}\chi(m^2+m+c)Q_3(\frac{m+Z}{p})]\Big|_{Z_{B,j}}^{\widetilde{Z}_{B,j}}\Bigg) \qquad (12)$$

To calculate the last equation above, we need some modular transformation formulae for $Q_3(Z)$, $Q_3'(Z)$, and $Q_3''(Z)$. Thanks for the work of Apostol (ref.[10]) and Carlitz (ref.[11]).

Lemma 1. *$Q_k(Z)$ as in* (10)*, for $M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}$ be an element in modular group, with $C>0$, we have*

$$(CZ+D)^{2k}Q_k(M<Z>)=Q_k(Z)$$
$$+\frac{(-1)^{k+1}(2\pi)^{2k+1}i}{2(2k+1)!}\sum_{m=0}^{2k+2}\binom{2k+2}{m}(CZ+D)^{m-1}S_{m,2k+2-m}(D,C),\ Im(Z)>0. \qquad (13)$$

For M as in the condition of Lemma 1, Fixing $k=3$ and taking derivations of Z on both sides of (13) we get

$$Q_3'(M<Z>)=\frac{Q_3'(Z)}{(CZ+D)^4}-\frac{6CQ_3(Z)}{(CZ+D)^5}$$
$$+\frac{(2\pi)^7 i}{2\cdot 8!}\sum_{m=0}^{8}(-1)^m\binom{8}{m}(m-7)C(CZ+D)^{m-6}S_{m,8-m}(D,C), \qquad (14)$$

and

$$Q_3''(M<Z>)=\frac{Q_3''(Z)}{(CZ+D)^2}-\frac{10CQ_3'(Z)}{(CZ+D)^3}+\frac{30C^2Q_3(Z)}{(CZ+D)^4}$$
$$+\frac{(2\pi)^7 i}{2\cdot 8!}\sum_{m=0}^{8}(-1)^m\binom{8}{m}(m-7)(m-6)C^2(CZ+D)^{m-5}S_{m,8-m}(D,C), \qquad (15)$$

It is easy to prove the following two lemmas:

Lemma 2. *Let $F_A(Z)$, ω, and ω' as explained in the beginning of Section* 2*. Matrix M as in Lemma* 1 *above, in addition that ω and ω' are fixed by M, then*

$$(CZ+D)^2F_A(M<Z>)=F_A(Z),\quad Im(Z)>0. \qquad (16)$$

Lemma 3. *Let $F_B(Z)$, $\overline{\omega}$, and $\overline{\omega}'$ as explained in the beginning of Section* 2*. Matrix M as in Lemma* 1 *above, in addition that $\overline{\omega}$ and $\overline{\omega}'$ are fixed by M, then*

$$(CZ+D)^2F_B(M<Z>)=F_B(Z), Im(Z)>0. \tag{17}$$

Finally, we have already found the modular matrix translate the variables in (12); $\epsilon_+ = \frac{T_1+U_1\sqrt{\Delta}}{2}$ denote the fundamental and totally positive fundamental unit of $\mathbf{K}$, set $\epsilon_+^l = \frac{T_l+U_l\sqrt{\Delta}}{2}$ and matrix $M = \begin{pmatrix} \frac{T_1+bU_1}{2} & -cU_1 \\ aU_1 & \frac{T_1-bU_1}{2} \end{pmatrix}$. For any integer l, set matrix $M_l = M^l = \begin{pmatrix} \frac{T_l+bU_l}{2} & -cU_l \\ aU_l & \frac{T_l-bU_l}{2} \end{pmatrix}$.

Let's choose an positive integer j such that $p|U_j$, then all the following matrixes are in the modular group:

$$M_{2j}, M_{2j}^{(p)} = \begin{pmatrix} \frac{T_{2j}+bU_{2j}}{2} & -cpU_{2j} \\ a\frac{U_{2j}}{p} & \frac{T_{2j}-bU_{2j}}{2} \end{pmatrix}, M_{2j}^{(p,m)} \begin{pmatrix} \frac{T_{2j}+bU_{2j}}{2}+maU_{2j} & -(am^2+bm+c)\frac{U_{2j}}{p} \\ apU_{2j} & \frac{T_{2j}-bU_{2j}}{2}-maU_{2j} \end{pmatrix},$$

$$\overline{M}_{2j} = \begin{pmatrix} \frac{T_{2j}-bU_{2j}}{2} & -cU_{2j} \\ aU_{2j} & \frac{T_{2j}+bU_{2j}}{2} \end{pmatrix},$$

$$\overline{M}_{2j}^{(p)} = \begin{pmatrix} \frac{T_{2j}-bU_{2j}}{2} & -cpU_{2j} \\ a\frac{U_{2j}}{p} & \frac{T_{2j}+bU_{2j}}{2} \end{pmatrix}, \overline{M}_{2j}^{(p,m)} \begin{pmatrix} \frac{T_{2j}-bU_{2j}}{2}-maU_{2j} & -(am^2+bm+c)\frac{U_{2j}}{p} \\ apU_{2j} & \frac{T_{2j}+bU_{2j}}{2}+maU_{2j} \end{pmatrix}.$$

And it is not difficult to check that these matrixes transfer $Z_{A,j}$, $pZ_{A,j}$, $\frac{m+Z_{A,j}}{p}$, $Z_{B,j}$, $pZ_{B,j}$, $\frac{m+Z_{B,j}}{p}$ to $\widetilde{Z}_{A,j}$, $p\ \widetilde{Z}_{A,j}$, $\frac{m+\widetilde{Z}_{A,j}}{p}$, $\widetilde{Z}_{B,j}$, $p\ \widetilde{Z}_{B,j}$, $\frac{m+\widetilde{Z}_{B,j}}{p}$ respectively. And it is also easy to check that these matrixes satisfy the conditions in Lemma 1, and Lemma 2 or Lemma 3. So by a long and tedious calculation,using corresponding modular transformation (13)-(17) in (12), and we feel at ease at the end, for the irrational part disappears. Comparing (10) and (12), and the R.H.S of (2) equal $I_1+I_2+I_3$, further more let the fundamental discriminant Δ be a prime $q \equiv 1 \pmod 4$ we get the proof of Theorem 2.

At the end of this paper, we would add a remark, though the methods are a little similar to Siegel's [2], our start point is different from his, and the results obviously are different from his.

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