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THE (a, d)-ASCENDING SUBGRAPH DECOMPOSITION

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Abstract. Let G be a graph of size q and a, n, d be positive integers for which $\frac{n}{2}(2a+(n-1)d) \leq q < (\frac{n+1}{2})(2a+nd)$. Then G is said to have (a, d)- ascending subgraph decomposition into n parts $((a, d) - ASD)$ if the edge set of G can be partitioned into n-non-empty sets generating subgraphs G_1, G_2, G_3, ..., G_n without isolated vertices such that each G_i is isomorphic to a proper subgraph of G_{i+1} for $1 \leq i \leq n-1$ and $|E(G_i)| = a + (i-1)d$. In this paper, we find $(a, d) - ASD$ into n parts for W_m.

1. Introduction

By a graph we mean a finite undirected graph without loops or multiple edges. A wheel on p vertices is denoted by W_p. A path of length t is denoted by P_{t+1}. Terms not defined here are used in the sense of Harary [4]. Throughout this paper $G \subset H$ means G is a subgraph of H. Let $G = (V, E)$ be a simple graph of order p and size q. If G_1, G_2, ..., G_n are edge disjoint subgraphs of G such that $E(G) = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_n)$, then $\{G_1, G_2, \ldots, G_n\}$ is said to be a decomposition of G.

The concept of ASD was introduced by Alavi et al. [1]. The graph G of size q where $\binom{n+1}{2} \leq q < \binom{n+2}{2}$, is said to have an ascending subgraph decomposition (ASD) if G can be decomposed into n-subgraphs G_1, G_2, ..., G_n without isolated vertices such that each G_i is isomorphic to a proper subgraph of G_{i+1} for $1 \leq i \leq n-1$ and $|E(G_i)| = i$.

We generalize this concept into $(a, d) - ASD$ as follows:

G is a simple graph of size q and a, n, d are positive integers for which $\frac{n}{2}(2a + (n-1)d) \leq q < (\frac{n+1}{2})(2a+nd)$. Then (a, d)-ascending subgraph decomposition $((a, d)-ASD)$ of G is the edge disjoint decomposition of G into subgraphs G_1, G_2, ..., G_n without isolated vertices such that each G_i is isomorphic to a proper subgraph of G_{i+1} for $1 \leq i \leq n-1$ and $|E(G_i)| = a + (i-1)d$.

2. Main Results

Definition 2.1. Let G be a graph of size q and a, n, d be positive integers for which $\frac{n}{2}(2a + (n-1)d) \leq q < (\frac{n+1}{2})(2a + nd)$. Then G is said to have (a, d)- ascending subgraph decomposition into n parts $((a, d)-ASD)$ if the edge set of G can be partitioned into n non-empty sets generating subgraphs G_1, G_2, ..., G_n without isolated vertices

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such that each G_i is isomorphic to a proper subgraph of G_{i+1} for $1 \leq i \leq n-1$ and $|E(G_i)| = a + (i-1)d$.

Remark 2.2. From the above definition, the usual ASD of G coincides with $(1,1)-ASD$ of G.

Example 2.3. Consider the Graph G.

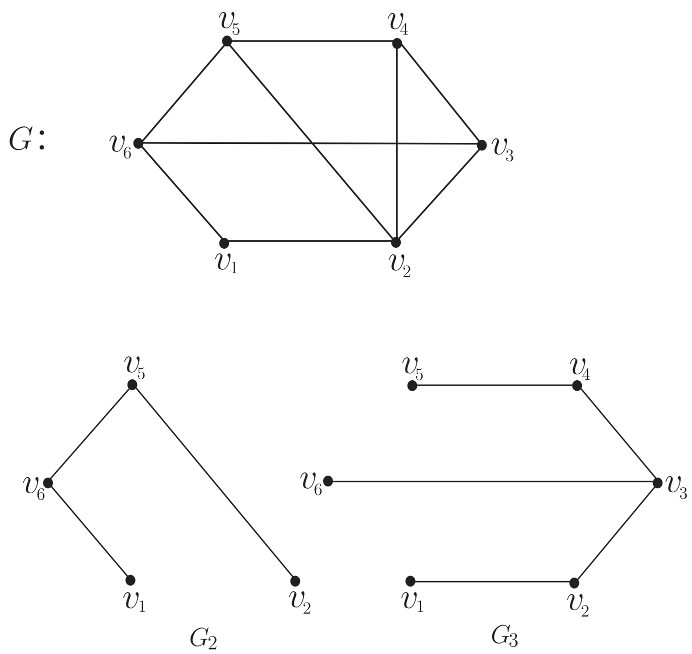


Figure 2.1.

Clearly $\{G_1, G_2, G_3\}$ is a $(1,2)-ASD$ of G.

Theorem 2.4. *Let G be a graph of size q, where $\frac{n}{2}(2a+(n-1)d) \leq q < \left(\frac{n+1}{2}\right)(2a+nd)$ for some positive integer n, such that G has $(a,d)-ASD$ into n parts, then G has an $(a,d)-ASD$ into n parts $G_1, G_2, \ldots, G_n$ such that each G_i has size $a+(i-1)d$ for $1 \leq i \leq n-1$ and G_n has size $q - \left(\frac{n-1}{2}\right)(2a+(n-2)d)$.*

Proof. If $q = \left(\frac{n}{2}\right)[2a+(n-1)d]$, then there is nothing to prove.

Now, suppse $(\frac{n}{2})[2a+(n-1)d] < q < (\frac{n+1}{2})[2a+nd]$. Suppose G has $H_1, H_2, \ldots, H_n$ as $(a,d)-ASD$. If the size of H_{n-1} is $a+(n-2)d$, then this decomposition has the desired properties. Therefore assume that the size of H_{n-1} exceeds $a+(n-1)d$. The size of H_1 must exceed a. Select the edges $e_{11}, e_{12}, \ldots, e_{1a}$ from H_1, inorder to define G_1, a subgraph of G induced by the set of edges $\{e_{11}, e_{12}, \ldots, e_{1a}\}$. Now let G_2 be a graph induced by the edges e_{21}, e_{22}, …, $e_{2(a+d)}$ from H_2 so that $G_1 \subset G_2$. Since H_2 is isomorphic to a subgraph H_3' of H_3, we can choose edges e_{31}, e_{32}, …, e_{3d} from $E(H_3) - E(H_3')$ so as to define G_3, a subgraph of G, induced by edges of $E(H_3')$ and the edges e_{31}, e_{32}, …, e_{3d}. Then it is clear that $|E(G_3)| = a + 2d$ and $G_2 \subset G_3$. Proceeding as before, we may define the graphs G_1, G_2, …, G_k $(3 \le k \le n-2)$ such that $|E(G_k)| = a + (k-1)d$ and $G_{k-1} \subset G_k$. From the above construction, we observe that each G_k $(1 \le k \le n-2)$ is a subgraph of H_k. Now we construct G_{k+1} as follows: Since G_k is isomorphic to a sbugraph H_k' of H_k, we choose the edges e_{k1}, e_{k2}, …, e_{kd} from $E(H_k) - E(H_k')$ such that the subgraph G_{k+1} is induced by the edges of $E(H_k')$ and $\{e_{k1}, e_{k2}, \ldots, e_{kd}\}$. Also note that $|E(G_{k+1})| = a + (k-1)d + d = a + kd$. Therefore there exist graphs G_1, G_2, …, G_{n-1} such that $|E(G_i)| = a + (i-1)d$ for $1 \le i \le n-1$ and $G_i \subset G_{i+1}$ for $1 \le i \le n-2$. Now define G_n, the subgraph of G induced by the edges of $E(G) - \bigcup_{i=1}^{n-1} E(G_i)$. Hence G has the required $(a, d) - ASD$ into n parts namely G_1, G_2, …, G_n. Clearly every graph does not posses $(a, d) - ASD$ into n parts. Now we wish to identify those graphs which admit $(a, d) - ASD$ into n parts.

The following number theoretical result will be useful for proving further results.

Lemma 2.5. *Given that the numbers a, $a + d$, $a + 2d$, …, $a + (n-1)d$ are in A.P $(a, d \in Z)$. Then their sum is*
i) $S_n = (a-d)n + d\binom{n+1}{2}$ *if* $d \le a$ *and*
ii) $S_n = a\binom{n+1}{2} + (d-a)\binom{n}{2}$ *if* $d \ge a$.

Theorem 2.6. *G admits $(a, d) - ASD$ into n parts. Then $a = q - k$, $2 \le k \le q-1$ if and only if $d = \frac{2(nk-(n-1)q)}{n(n-1)}$.*

Proof. Suppose $a = q - k$, $2 \le k \le q - 1$.
As G adimts $(a, d) - ASD$ into n-parts, we have

$$a + (a + d) + (a + 2d) + \cdots + a + (n-1)d = q$$

$$na + d\binom{n}{2} = q$$

$$n(n-1)d = 2(q - na)$$
$$n(n-1)d = 2(q - n(q-k)) \quad \text{as} \quad a = (q-k)$$
$$n(n-1)d = 2(nk - (n-1)q).$$

Hence $d = \dfrac{2(nk-(n-1)q)}{n(n-1)}$.

Conversely, suppose $d = \dfrac{2(nk-(n-1)q)}{n(n-1)}$ —— > (1).

As G admits $(a,d)-ASD$ into n parts, we have

$$\begin{aligned} a+(a+d)+(a+2d)+\cdots+a+(n-1)d &= q \\ na+d\binom{n}{2} &= q \\ na+[nk-(n-1)q] &= q \text{ by } (1) \\ n(q-a) &= nk. \\ \text{Hence } a &= q-k. \end{aligned}$$

Corollary 2.7. *If G admits $(a,d)-ASD$ into n even number of parts and let $a=q-k$, $2\le k\le q-1$, then $k\equiv 0 \pmod{n-1}$.*

Proof. Given $a=q-k$, $2\le k\le q-1$.

$$\begin{aligned} \text{By 2.6,}\quad n(n-1)d &= 2nk-2(n-1)q \\ (n-1)[nd+2q] &= 2nk \\ nd+2q &= \frac{2nk}{(n-1)} \quad (n>3). \end{aligned}$$

As $(n-1,n)=1$ and n is even, $n-1$ divides k. Therefore, $k\equiv 0 \pmod{n-1}$

Observation 2.8. If G admits $(a,d)-ASD$ into n parts, then $1\le a\le \frac{q-\binom{n}{2}}{n}$ and $1\le d\le \frac{q-n}{\binom{n}{2}}$.

Proof. Suppose G admits $(a,)-ASD$ into n parts. Then we have,

$$\begin{aligned} a+(a+d)+(a+2d)+\cdots+a+(n-1)d &= q \\ na+d\binom{n}{2} &= q \text{ ——} > (1) \\ na+\binom{n}{2} \le q \text{ as } d\ge 1, &\text{ therefore } a\le \frac{q-\binom{n}{2}}{n}. \end{aligned}$$

Alos from (1) and since $a\ge 1$, $n+\binom{n}{2}d\le q$, $d\le \frac{q-n}{\binom{n}{2}}$.

Hence we have $1\le a\le \frac{q-\binom{n}{2}}{n}$ and $1\le d\le \frac{q-n}{\binom{n}{2}}$.

Corollary 2.9. *If G admits $(a,d)-ASD$ into two parts, then $1\le a\le \frac{q-1}{2}$ and $1\le d\le q-2$.*

Corollary 2.10. *If G admits $(a,d)-ASD$ into two parts and if $a=\frac{q-1}{2}$, then $d=1$.*

Corollary 2.11. *If G admits $(a,d)-ASD$ into two parts and if $d=q-2$, then $a=1$.*

Corollary 2.12. *If G admits $(a,d)-ASD$ into two parts and let $d=q-k$ where $2 \le k \le q-1$, then k is even.*

Proof. Since G admits $(a,d)-ASD$ into 2 parts

$$\begin{aligned} & a+(a+d)=q \\ & 2a+d=q \\ & 2a+q-k=q, \text{ as } d=q-k. \\ & \text{Therefore,} \quad k=2a. \end{aligned}$$

Corollary 2.13. *If G admits $(a,d)-ASD$ into three parts, then $1 \le a \le \frac{q-3}{3}$ and $1 \le d \le \frac{q-3}{3}$.*

3. $(a,d)-ASD$ on Wheel

In this section for proving $W_m = K_1 + C_{m-1}$ $(m \ge 4)$ admits $(a,d)-ASD$ into n parts, we need the following results.

Theorem 3.1. *If W_m admits $(a,d)-ASD$ into n-parts, then*

a) *For $n \equiv 0 \pmod 4$,*
   i) *either $a \ge 1$ and $d \equiv 1 \pmod 2$ or $a \ge 1$ and $d \equiv 0 \pmod 2$*
   ii) *$m \equiv \frac{n}{4}+1 (\bmod \frac{n}{2})$ when $a \ge 1$ and $d \equiv 1 \pmod 2$ and*
   iii) *$m \equiv 1 (\bmod \frac{n}{2})$ when $a \ge 1$ and $d \equiv 0 \pmod 2$.*

b) *For $n \equiv 1 \pmod 4$,*
   i) *$m \equiv 1 \pmod n$ and* ii) *$a \equiv 0 \pmod 2$.*

c) *For $n \equiv 2 \pmod 4$,*
   i) *$m \equiv 1 \pmod{\frac{n}{2}}$ and* ii) *$d \equiv 0 \pmod 2$.*

d) *For $n \equiv 3 \pmod 4$,*
   i) *$m \equiv 1 \pmod n$ and* ii) *a is even (odd) if and only if d is even (odd).*

Proof. Suppose W_m admits $(a,d)-ASD$ into n-parts. Then we have,

$$\begin{aligned} a+(a+d)+(a+2d)+\cdots+a+(n-1)d &= q \\ \frac{n}{2}(2a+(n-1)d) &= 2(m-1) \quad \text{as } q=2(m-1) \\ n(2a+(n-1)d) &= 4(m-1) \text{ ——} > (1) \end{aligned}$$

Case (a): Suppose $n \equiv 0 \pmod 4$.
Let $n=4k$, $(k \in z^+)$.

Sub case (a)(i): Suppose k is odd, then by (i) $(m-1)$ is either odd or even.
Suppose $(m-1)$ is odd, then $a \ge 1$ and $d \equiv 1 \pmod 2$.
Suppose $(m-1)$ is even, then $a \ge 1$ and $d \equiv 0 \pmod 2$.

Sub case (a)(i)(a): Suppose k is even.
Then $(m-1)$ must be even. Therefore $d \equiv 0 \pmod 2$ or $d \equiv 1 \pmod 2$.
Hence either $a \geq 1$ and $d \equiv 1 \pmod 2$ or $a \geq 1$ and $d \equiv 0 \pmod 2$.

Sub case a(ii): Suppose $a \geq 1$ and $d \equiv 1 \pmod 2$.
Let $d = 2r+1$ ($r \in z^+ \cup \{0\}$), By using (1) we have,

$$\begin{aligned} n[2a+(n-1)d] &= 4(m-1) \\ k[2a+(4k-1)(2r+1)] &= (m-1) \text{ since } n = 4k \\ k[2a+(8kr-2r+4k-2)+1] &= m-1 \\ 2k[a+(4kr-r+2k-1)] &= m-(k+1). \end{aligned}$$

Therefore $m \equiv k+1 \pmod{2k}$.
Hence $m \equiv \frac{n}{4}+1 \pmod{\frac{n}{2}}$.

Sub case a(iii): Suppose $a > 1$ and $d \equiv 0 \pmod 2$.
Let $d = 2r$ ($r \in z^+$), By using (1) we have,

$$\begin{aligned} n(2a+(n-1)d) &= 4(m-1) \\ k(2a+(n-1)2r) &= (m-1) \text{ since } n = 4k \\ 2k(a+(n-1)r) &= m-1. \end{aligned}$$

Therefore $m \equiv 1 \pmod{2k}$.
Hence $m \equiv 1 \pmod{\frac{n}{2}}$.

Case (b): Suppose $n \equiv 1 \pmod 4$.
Let $n = 4k+1$ ($k \in z^+$), By using (1) we have,

$$\begin{aligned} n[2a+(n-1)d] &= 4(m-1) \\ n(2a+4kd) &= 4(m-1) \\ n(a+2kd) &= 2(m-1). \end{aligned}$$

As a, d are integers and n is odd, (b)(i) follows clearly.
As n is odd, (b)(ii) follows clearly.

Case (c): Suppose $n \equiv 2 \pmod 4$.
Let $n = 4k+2$ ($k \in z^+$), By using (1) we have,

$$\begin{aligned} n(2a+(n-1)d) &= 4(m-1) \\ (4k+2)(2a+(n-1)d) &= 4(m-1) \\ 2(m-1) &= (2k+1)\ell \text{ where } \ell = 2a+(n-1)d. \end{aligned}$$

The above equation is true only when ℓ is even. Then (c)(i) follows. Further, since ℓ is even and n is even, then (c)(ii) follows.

Case (d): Suppose $n \equiv 3 \pmod 4$.
Let $n = 4k + 3$ ($k \in z^+ \cup \{0\}$), By using (1) we have,

$$\begin{aligned} n(2a + (n-1)d) &= 4(m-1) \\ n(2a + (4k+2)d) &= 4(m-1) \\ n(a + (2k+1)d) &= 2(m-1). \end{aligned}$$

As a, d are integers and n is odd, then (d)(i) follows clearly.
As n is odd, (d)(ii) follows clearly.

Theorem 3.2. *If W_m admits $(a, d) - ASD$ into n-parts, then* $1 \le a \le \dfrac{q - \binom{n}{2}}{n}$ *and* $1 \le d \le \dfrac{q-n}{\binom{n}{2}}$.

Proof. Suppose W_m admits $(a, d) - ASD$ into n-parts. Then by 2.8, we have $1 \le a \le \dfrac{q - \binom{n}{2}}{n}$ and $1 \le d \le \dfrac{q-n}{\binom{n}{2}}$.

Theorem 3.3. *W_m adimts $(a, d) - ASD$ into n-parts if and only if*
a) *For* $n \equiv 0 \pmod 4$,
  i) *either* $a \ge 1$ *and* $d \equiv 1 \pmod 2$ *or* $a \ge 1$ *and* $d \equiv 0 \pmod 2$.
  ii) a) $m \equiv \frac{n}{4} + 1 (\bmod \frac{n}{2})$ *and* b) $m \ge \frac{n(n+1)}{4} + 1$ *when* $a \ge 1$ *and* $d \equiv 1 \pmod 2$.
  iii) a) $m \equiv 1 (\bmod \frac{n}{2})$ *and* b) $m \ge \frac{n^2}{2} + 1$ *when* $a \ge 1$ *and* $d \equiv 0 \pmod 2$.
b) *For* $n \equiv 1 \pmod 4$,
  i) $m \equiv 1 \pmod n$, ii) $a \equiv 0 \pmod 2$ *and* iii) $m \ge \frac{n(n+3)}{4} + 1$.
c) *For* $n \equiv 2 \pmod 4$,
  i) $m \equiv 1 (\bmod \frac{n}{2})$, ii) $d \equiv 0 \pmod 2$ *and* iii) $m \ge \frac{n^2}{2} + 1$.
d) *For* $n \equiv 3 \pmod 4$,
  i) $m \equiv 1 \pmod n$, ii) *a and d are both even or both odd and* iii) $m \ge \frac{n(n+1)}{4} + 1$.

Proof. The proof of the necessary part follows from 3.1. Conversely,

Let $V(W_m) = \{v_1, v_2, \ldots, v_m\}$ and
$E(W_m) = \{(v_i, v_{i+1}) | 1 \le i \le m-1\} \cup \{(v_m, v_i) | 1 \le i \le m-1\}$.
Define $L_i = (v_i, v_{i+1}) \cup (v_m, v_i), \quad 1 \le i \le m-1$.

Case (a): Let $n \equiv 0 \pmod 4$.
Subcase (a)(i): Suppose a and d are even.

$a = 4$, $d = 2$

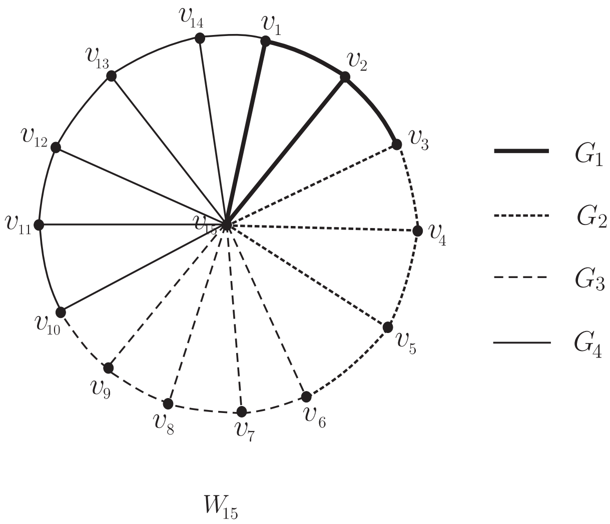


Figure 3.1.

Define $G_1 = \bigcup_{i=1}^{\frac{a}{2}} L_i$ and for $2 \leq j \leq n$, $G_j = \left\{ \bigcup_{i=\frac{1}{2}\sum_{k=0}^{j-2}(a+kd)+1}^{\frac{1}{2}\sum_{k=0}^{j-1}(a+kd)} L_i \right\}$.

Clearly $G_j \subset G_{j+1}$ for $1 \leq j \leq n-1$.
Therefore $G_1, G_2, \ldots, G_n$ is an $(a,d) - ASD$ into n-parts of W_m.

Subcase(a)(ii): Suppose a and d are odd.
Define when $a = 1$, $d = 1$, $G_1 = (v_m, v_1)$ and $G_2 = (v_1, v_2) \cup (v_m, v_2)$.
Define when $a = 1$, $d > 1$, $G_1 = (v_m, v_1)$

$G_2 = (v_{\ell+1}, v_{\ell+2}) \cup \bigcup_{i=\ell+2}^{p} L_i \cup (v_m, v_{p+1})$ where $\ell = \left\lfloor \frac{a}{2} \right\rfloor$ and $p = \left\lfloor \frac{a}{2} \right\rfloor + \left\lfloor \frac{a+d}{2} \right\rfloor$.

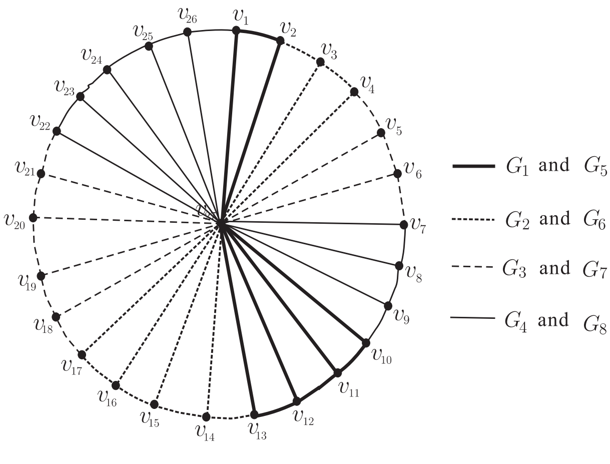
Define when $a > 1$ and $d > 1$

$$G_1 = \left\{ \bigcup_{i=1}^{\ell} L_i \right\} \cup (v_m, v_{\ell+1})$$

$$G_2 = (v_{\ell+1}, v_{\ell+2}) \cup \bigcup_{i=\ell+2}^{p} L_i \cup (v_m, v_{p+1}) \text{ where } \ell = \left\lfloor \frac{a}{2} \right\rfloor \text{ and } p = \left\lfloor \frac{a}{2} \right\rfloor + \left\lfloor \frac{a+d}{2} \right\rfloor.$$

Let $m_j = \left\lfloor \frac{a+kd}{2} \right\rfloor + \left\lfloor \frac{j}{4} \right\rfloor$

$a = 3$, $d = 1$



W_{27}

Figure 3.2.

When $j \equiv 3 \pmod 4$, define

$$G_j = (v_{\ell+1}, v_{\ell+2}) \cup \left\{ \bigcup_{i=\sum_{k=0}^{j-2} m_j + 2}^{\sum_{k=0}^{j-1} m_j + 1} L_i \right\} \text{ where } \quad \ell = \sum_{k=0}^{j-2} m_j.$$

When $j \equiv 0 \pmod 4$, define $G_j = \left\{ \bigcup_{i=\sum_{k=0}^{j-2} m_j + 1}^{\sum_{k=0}^{j-1} m_j} L_i \right\}$.

When $j \equiv 1 \pmod 4$, $(j > 1)$, define

$$G_j = \left\{ \bigcup_{i=\sum_{k=0}^{j-2} m_j + 1}^{\sum_{k=0}^{j-1} m_j} L_i \right\} \cup (v_m, v_{p+1}) \text{ where } p = \sum_{k=0}^{j-1} m_j.$$

When $j \equiv 2 \pmod 4$, $(j > 2)$, define

$$G_j = (v_{p+1}, v_{p+2}) \cup \left\{ \bigcup_{i=\sum_{k=0}^{j-2} m_j + 2}^{\sum_{k=0}^{j-1} m_j} L_i \right\} \cup (v_m, v_{\ell+1})$$

$$\text{where } p = \sum_{k=0}^{j-2} m_j \text{ and } \ell = \sum_{k=0}^{j-1} m_j.$$

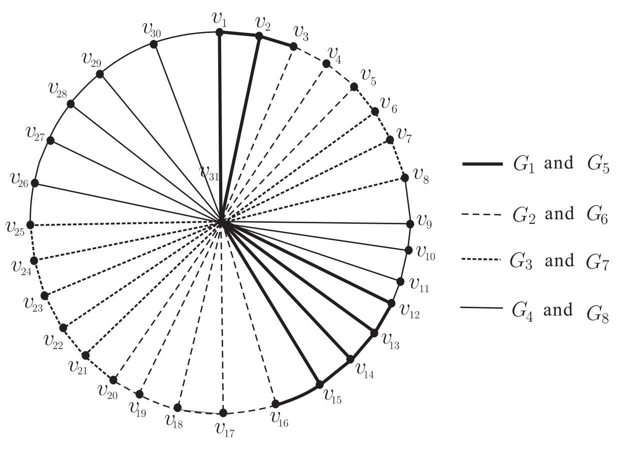
In the above construction addition of indices being taken modulo $(m-1)$ with residues $1, 2, \ldots, m-1$.
Clearly $G_j \subset G_{j+1}$ for $1 \leq j \leq n-1$.
Therefore, G_1, G_2, …, G_n is an $(a, d) - ASD$ of W_m.

Subcase (a)(iii) Suppose a is even and d is odd.

$$G_1 = \bigcup_{i=1}^{\ell} L_i$$

$$G_2 = \bigcup_{i=\ell+1}^{p} L_i \cup (v_m, v_{p+1}) \text{ where } \ell = \frac{a}{2} \text{ and } p = \frac{a}{2} + \left\lfloor \frac{a+d}{2} \right\rfloor$$

$a = 4$, $d = 1$



W_{31}

Figure 3.3.

Let $m_j = \left\lfloor \dfrac{a+kd}{2} \right\rfloor + \left\lfloor \dfrac{j}{4} \right\rfloor$.
When $j \equiv 3 \pmod 4$, define

$$G_j = (v_{\ell+1}, v_{\ell+2}) \cup \left\{ \bigcup_{i=\sum_{k=0}^{j-2} m_j + 2}^{\sum_{k=0}^{j-1} m_j} L_i \right\} \cup (v_m, v_{p+1})$$

$$\text{where } \ell = \sum_{k=0}^{j-2} m_j \text{ and } p = \sum_{k=0}^{j-1} m_j.$$

When $j \equiv 0 \pmod 4$, define

$$G_j = (v_\ell, v_{\ell+1}) \cup \left\{ \bigcup_{i=\sum_{k=0}^{j-2} m_j + 1}^{\sum_{k=0}^{j-1} m_j} L_i \right\} \text{ where } \ell = \sum_{k=0}^{j-2} m_j.$$

When $j \equiv 1 \pmod 4$, $(j > 1)$, define

$$G_j = \left\{ \bigcup_{i=\sum_{k=0}^{j-2} m_j + 1}^{\sum_{k=0}^{j-1} m_j} L_i \right\}.$$

When $j \equiv 2 \pmod 4$, $(j > 2)$, define

$$G_j = \left\{ \bigcup_{i=\sum_{k=0}^{j-2} m_j + 1}^{\sum_{k=0}^{j-1} m_j} L_i \right\} \cup (v_m, v_{\ell+1}) \text{ where } \ell = \sum_{k=0}^{j-1} m_j.$$

In the above construction addition of indices being taken modulo $(m-1)$ with residues $1, 2, \ldots, m-1$.
Clearly $G_j \subset G_{j+1}$ for $1 \leq j \leq n-1$.
Therefore, G_1, G_2, …, G_n is an $(a,d) - ASD$ into n parts of W_m.

Subcase (a)(iv): Suppose a is odd and d is even.
Define $G_1 = (v_m, v_1)$ when $a = 1$, $d \geq 2$

$$G_1 = \left\{ \bigcup_{i=1}^{\ell} L_i \right\} \cup (v_m, v_{\ell+1}) \text{ when } a > 1,\ d \geq 2$$

$$G_2 = (v_{\ell+1}, v_{\ell+2}) \cup \bigcup_{i=\ell+2}^{p} L_i \text{ where } \ell = \left\lfloor \frac{a}{2} \right\rfloor \text{ and } p = \left\lfloor \frac{a}{2} \right\rfloor + \left\lfloor \frac{a+d}{2} \right\rfloor + 1.$$

Let $m_j = \left\lfloor \frac{a+kd}{2} \right\rfloor + \left\lfloor \frac{j}{2} \right\rfloor$.
When $j \equiv 3 \pmod 4$, define

$$G_j = \left\{ \bigcup_{i=\sum_{k=0}^{j-2} m_j + 1}^{\sum_{k=0}^{j-1} m_j} L_i \right\} \cup (v_m, v_{\ell+1}) \text{ where } \ell = \sum_{k=0}^{j-1} m_j$$

$a = 1$, $d = 2$

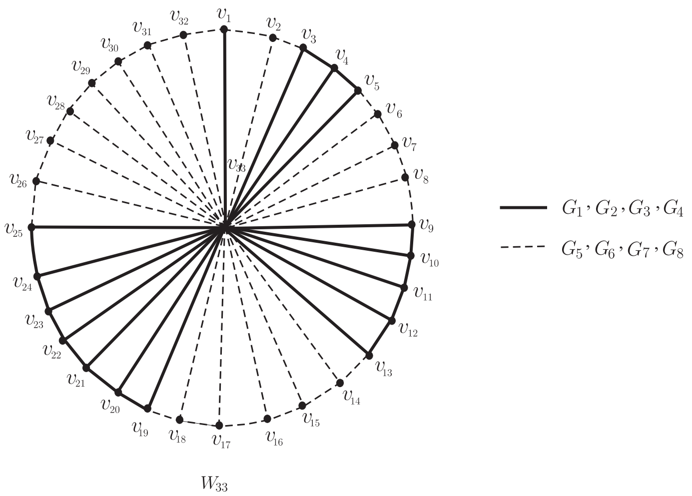


Figure 3.4.

When $j \equiv 0 \pmod 4$, define

$$G_j = (v_\ell, v_{\ell+1}) \cup \left\{ \bigcup_{i=\sum_{k=0}^{j-2} m_j + 1}^{\sum_{k=0}^{j-1} m_j} L_i \right\} \text{ where } \ell = \sum_{k=0}^{j-2} m_j.$$

When $j \equiv 1 \pmod 4$, $(j > 1)$, define

$$G_j = \left\{ \bigcup_{i=\sum_{k=0}^{j-2} m_j + 1}^{\sum_{k=0}^{j-1} m_j} L_i \right\} \cup (v_m, v_{p+1}) \text{ where } p = \sum_{k=0}^{j-1} m_j.$$

When $j \equiv 2 \pmod 4$, $(j > 2)$, define

$$G_j = (v_p, v_{p+1}) \cup \left\{ \bigcup_{i=\sum_{k=0}^{j-2} m_j + 1}^{\sum_{k=0}^{j-1} m_j} L_i \right\} \text{ where } p = \sum_{k=0}^{j-2} m_j.$$

In the above construction addition of indices beinig taken modulo $(m-1)$ with residues $1, 2, \ldots, m-1$.

Clearly $G_j \subset G_{j+1}$ for $1 \leq j \leq n-1$. Therefore, G_1, G_2, $\ldots$, G_n is an $(a,d) - ASD$ into n parts of W_m.

Case (b): Let $n \equiv 1 \pmod 4$.
The proof of this case is anologus to subcases a(i) and a(iii).

Case (c): Let $n \equiv 2 \pmod 4$.
The proof of this case is anologus to subcases a(i) and a(iv).

Case (d): Let $n \equiv 3 \pmod 4$.
The proof of this case is anologus to subcases a(i) and a(ii).

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