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CONTINUOUS ANALOGUE OF ALZER'S INEQUALITY

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Abstract. Let $b > a > 0$ and $\delta > 0$ be real numbers, then, for all real r,

$$\frac{b}{b+\delta} < \left(\frac{1}{b-a}\int_a^b x^r dx \Big/ \frac{1}{b+\delta-a}\int_a^{b+\delta} x^r dx\right)^{1/r} < 1.$$

Both bounds are best possible.

1. Introduction

It has been shown in [3, 12] that

$$\frac{n}{n+1} < \left(\frac{1}{n}\sum_{i=1}^{n} i^r \Big/ \frac{1}{n+1}\sum_{i=1}^{n+1} i^r\right)^{1/r} < 1 \tag{1}$$

for all natural numbers n, and all real r. Both bounds of (1) are best possible. This extends a result given by H. Alzer [1], who established this inequality for $r > 0$. For $r > 0$, several easy proofs of (1) have been published by different authors, see [2, 11, 15]. For convience, we call (1) Alzer's inequality.

In this paper, we present a continuous analogue of (1) as follows:

Theorem. *Let $b > a > 0$ and $\delta > 0$ be real numbers, then, for all real r,*

$$\frac{b}{b+\delta} < \left(\frac{1}{b-a}\int_a^b x^r dx \Big/ \frac{1}{b+\delta-a}\int_a^{b+\delta} x^r dx\right)^{1/r} < 1. \tag{2}$$

Both bounds are best possible.

Remark. Our theorem extends a result given by F. Qi [10], who established the inequality (2) for $r > 0$. In fact, (2) can be written as

$$\frac{b}{b+\delta} < \frac{L_r(a,b)}{L_r(a,b+\delta)} < 1, \tag{3}$$

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where $L_r(a,b)$ denotes the generalized logarithmic mean of two positive numbers a, b. For convience, let us recall that the generalized logarithmic mean $L_r(a,b)$ of two positive numbers a, b is defined in [5, 13, 14] for $a = b$ by $L_r(a,b) = a$ and for $a \neq b$ by

$$\begin{aligned} L_r(a,b) &= \left(\frac{b^{r+1} - a^{r+1}}{(r+1)(b-a)}\right)^{1/r}, \quad r \neq -1, 0; \\ L_{-1}(a,b) &= \frac{b-a}{\ln b - \ln a} = L(a,b); \\ L_0(a,b) &= \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)} = I(a,b). \end{aligned}$$

$L(a,b)$ and $I(a,b)$ are respectively called the logarithmic mean and exponential mean of two positive numbers a, b. When $a \neq b$, $L_r(a,b)$ is a strictly increasing function of r. In particular,

$$\lim_{r \to -\infty} L_r(a,b) = \min\{a,b\}, \quad \lim_{r \to +\infty} L_r(a,b) = \max\{a,b\}.$$

2. Proof of (3)

For $r = -1$. Then the left hand inequality of (3) is

$$\frac{b}{b+\delta} < \frac{L(a,b)}{L(a,b+\delta)},$$

which is equivalent to

$$\begin{gathered} \frac{b+\delta-a}{(b+\delta)[\ln(b+\delta) - \ln a]} < \frac{b-a}{b(\ln b - \ln a)}, \\ \frac{[a/(b+\delta)] - 1}{\ln[a/(b+\delta)]} < \frac{(a/b) - 1}{\ln(a/b)}. \end{gathered} \tag{4}$$

Define the function f by

$$f(t) = \frac{t-1}{\ln t} \quad (0 < t < 1).$$

Differentiation yields

$$\begin{aligned} f'(t) = \frac{1 - t + t\ln t}{t(\ln t)^2} &= \frac{1}{t(\ln t)^2}\left(1 - t - t\sum_{n=1}^{\infty}\frac{(1-t)^n}{n}\right) \\ &> \frac{1}{t(\ln t)^2}\left(1 - t - t\sum_{n=1}^{\infty}(1-t)^n\right) = 0, \end{aligned}$$

which implies (4).

For $r = 0$. Then the left hand inequality of (3) is

$$\frac{b}{b+\delta} < \frac{I(a,b)}{I(a,b+\delta)},$$

which is equivalent to

$$\frac{1}{b+\delta}\left(\frac{(b+\delta)^{b+\delta}}{a^a}\right)^{1/(b+\delta-a)} < \frac{1}{b}\left(\frac{b^b}{a^a}\right)^{1/(b-a)},$$

$$\left(\frac{b+\delta}{a}\right)^{1/(b+\delta-a)} < \left(\frac{b}{a}\right)^{1/(b-a)},$$

$$L(a,b) = \frac{b-a}{\ln b - \ln a} < \frac{b+\delta-a}{\ln(b+\delta)-\ln a} = L(a,b+\delta). \tag{5}$$

Since the logarithmic mean $L(a,b)$ is strictly increasing with respect to the two variables a and b, (5) holds obviously.

For $r(r+1) \neq 0$. Then the left hand inequality of (3) is equivalent to

$$\frac{b^{r+1}-a^{r+1}}{b-a}\Big/\frac{(b+\delta)^{r+1}-a^{r+1}}{b+\delta-a} \gtrless \frac{b^r}{(b+\delta)^r}, \text{ according as } r \gtrless 0.$$

i.e. (since $(b+\delta)^{r+1} - b^{r+1} \gtrless 0$ according as $r+1 \gtrless 0$),

$$\frac{b^{r+1}-a^{r+1}}{(b-a)b^r} \gtrless \frac{(b+\delta)^{r+1}-a^{r+1}}{(b+\delta-a)(b+\delta)^r}, \text{ according as } r(r+1) \gtrless 0,$$

i.e.

$$\frac{1-(a/b)^{r+1}}{1-(a/b)} \gtrless \frac{1-[a/(b+\delta)]^{r+1}}{1-[a/(b+\delta)]}, \text{ according as } r(r+1) \gtrless 0. \tag{6}$$

By the mean value theorem for derivatives,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1-x^{r+1}}{1-x}\right) &= \frac{1}{1-x}\left[\frac{1-x^{r+1}}{1-x} - (r+1)x^r\right] \\ &= \frac{(r+1)(\xi^r - x^r)}{1-x} \gtrless 0 \quad (0 < x < \xi < 1),\end{aligned}$$

according as $r(r+1) \gtrless 0$, which implies (6).

Since the generalized logarithmic mean $L_r(a,b)$ is strictly increasing with respect to the two variables a and b (see [4, 6, 7, 8, 9]), the right hand inequality of (3) holds obviously.

It is clear that

$$\lim_{r\to+\infty} \frac{L_r(a,b)}{L_r(a,b+\delta)} = \frac{b}{b+\delta}, \quad \lim_{r\to-\infty} \frac{L_r(a,b)}{L_r(a,b+\delta)} = 1.$$

Thus, the both bounds given in (3) are best possible. The proof of (3) is complete.

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