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GENERALIZATION OF SOME INEQUALITIES VIA RIEMANN-LIOUVILLE FRACTIONAL CALCULUS

MARCELA V. MIHAI AND DANIEL ALEXANDRU ION

Abstract. Some Hermite-Hadamard type inequalities are provided. We deal with functions whose derivatives in absolute value are convex or concave. By defining two cumulative gaps which enable us to generalize known results in the framework of Riemann-Liouville fractional calculus, we open a new perspective on the classic statement of the inequality.

1. Introduction

The Hermite-Hadamard inequality states that if a function $f:[a,b]\longrightarrow\mathbb{R}$ is convex then

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\leq\frac{f(a)+f(b)}{2}, \tag{HH}$$

Both inequalities hold in the reversed direction if f is concave.

This inequality received a great deal of attention in the last decade instance, many generalizations and applications being obtained. See [1], [2], [4], [6], [8], [10] and the references therein. Of special interest to us is the following improvements of the Hermite-Hadamard inequality, that can be found in the monograph [7], p. 52:

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\leq\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x, \tag{LHH}$$

and

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\leq\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\leq\frac{f(a)+f(b)}{2}, \tag{RHH}$$

The purpose of the present paper is to establish new Hermite-Hadamard type inequalities within Riemann-Liouville fractional calculus. Unlike the classical case, the functions under attention are not assumed convex or concave, but this fact is asked for the absolute value

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of their derivatives. Under these circumstances we will prove the existence of two strings of inequalities refining the inequalities (LHH) and (RHH).

In what follows we will consider only real-valued functions defined on intervals $[a,b]$ with $0 \leq a < b$, and n is an odd number.

Let $f \in L^1[a,b]$ be an integrable function. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$, of order $\alpha > 0$, attached to f are defined respectively by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\mathrm{d}t, \quad \text{for } x > a,$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)\mathrm{d}t, \quad \text{for } x < b.$$

Here, $\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}\mathrm{d}t$ is the Gamma function. We make the convention

$$J^0_{a+}f(x) = J^0_{b-}f(x) = f(x).$$

The theory of Riemann-Liouville fractional integrals can be found in the book [3].

2. Main results

As above, we assume that $[a,b]$ is a compact subinterval of $[0,\infty)$ and $f : [a,b] \to \mathbb{R}$ is an integrable function.

We define the *cumulative to the left* $(\alpha,n)-$*gap* of f by the formula

$$\begin{aligned}\mathscr{L}_{\alpha,n}(a,b) = &\sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\\ &-\Gamma(\alpha+1)\left(\frac{n+1}{b-a}\right)^{\alpha}\sum_{k=0}^{(n-1)/2}\left[J^{\alpha}_{\frac{a(n-2k)+b(2k+1)}{n+1}-}f\left(\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)\right.\\ &\left.+J^{\alpha}_{\frac{a(n-2k-1)+b(2k+2)}{n+1}-}f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right].\end{aligned}$$

In the particular case where $\alpha = 1$ and $n = 3$ we have

$$\frac{\mathscr{L}_{1,3}(a,b)}{4} = \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t,$$

so the cumulative to the left gap $\mathscr{L}_{1,3}(a,b)$ estimates the precision of the right hand side inequality in (LHH).

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \geq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right].$$

The cumulative to the left gaps $\mathscr{L}_{\alpha,n}(a,b)$ have the same meaning, relative to higher order refinements of (LHH).

The following technical lemma provides a suitable formula for estimating $\mathscr{L}_{\alpha,n}(a,b)$ in absolute value:

Lemma 1. *We have*

$$\mathscr{L}_{\alpha,n}(a,b) = \frac{b-a}{n+1} \times \sum_{k=0}^{(n-1)/2} \left[\int_0^1 t^\alpha f'\left(t\frac{a(n-2k)+b(2k+1)}{n+1} + (1-t)\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right) \mathrm{d}t + \int_0^1 (t^\alpha - 1) f'\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1} + (1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\right) \mathrm{d}t \right].$$

Proof. Put

$$I_{1k} = \int_0^1 t^\alpha f'\left(t\frac{a(n-2k)+b(2k+1)}{n+1} + (1-t)\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right) \mathrm{d}t,$$

and

$$I_{2k} = \int_0^1 (t^\alpha - 1) f'\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1} + (1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\right) \mathrm{d}t.$$

By using the integration by parts and the substitutions

$$u = t\frac{a(n-2k)+b(2k+1)}{n+1} + (1-t)\frac{a(n-2k+1)+b\cdot 2k}{n+1},$$

and

$$v = t\frac{a(n-2k-1)+b(2k+2)}{n+1} + (1-t)\frac{a(n-2k)+b(2k+1)}{n+1},$$

we infer that

$$I_{1k} + I_{2k} = \frac{2(n+1)}{b-a} f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right) - \Gamma(\alpha+1)\cdot\left(\frac{n+1}{b-a}\right)^{\alpha+1} \times \left[J^\alpha_{\frac{a(n-2k)+b(2k+1)}{n+1}-} f\left(\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right) + J^\alpha_{\frac{a(n-2k-1)+b(2k+2)}{n+1}-} f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right]$$

The proof is completed. □

We are now in a position to state and prove the following result:

Theorem 1. *Assume that $f : [a,b] \to \mathbb{R}$ is a differentiable function such that $|f'|$ is convex on $[a,b]$. Then*

$$\left|\mathscr{L}_{\alpha,n}(a,b)\right| \le \frac{b-a}{n+1} \sum_{k=0}^{(n-1)/2} \left[\frac{\alpha^2+5\alpha+2}{2(\alpha+1)(\alpha+2)} \left| f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|\right.$$

$$+\frac{1}{(\alpha+1)(\alpha+2)}\left|f'\left(\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)\right| +\frac{\alpha}{2(\alpha+2)}\left|f'\left(\frac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right|\Bigg].$$

Proof. Using Lemma 1 and the convexity of $|f'|$ we obtain

$$\begin{aligned}\left|\mathscr{L}_{\alpha,n}(a,b)\right| \le \frac{b-a}{n+1}\sum_{k=0}^{(n-1)/2}\Bigg[&\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|\int_0^1 t^{\alpha+1}\mathrm{d}t\\ &+\left|f'\left(\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)\right|\int_0^1 t^{\alpha}(1-t)\mathrm{d}t\\ &+\left|f'\left(\frac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right|\int_0^1 t(1-t^{\alpha})\mathrm{d}t\\ &+\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|\int_0^1 (1-t^{\alpha})(1-t)\mathrm{d}t\Bigg].\end{aligned}$$

The proof ends after a straightforward computation in the right hand side term. □

The Beta function is defined by the formula

$$\mathrm{B}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t \quad \text{for } x,y>0.$$

Our next result is as follows:

Theorem 2. *Assume that $f:[a,b]\to\mathbb{R}$ is a differentiable function such that $|f'|^q$ is convex on $[a,b]$ for some exponent $q>1$. Then*

$$\begin{aligned}\left|\mathscr{L}_{\alpha,n}(a,b)\right| \le \frac{b-a}{2^{\frac{1}{q}}(n+1)}\sum_{k=0}^{(n-1)/2}\Bigg\{&\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\\ &\times\left[\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q+\left|f'\left(\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)\right|^q\right]^{\frac{1}{q}}\\ &+\left[\frac{1}{\alpha}\mathrm{B}\left(p+1,\frac{1}{\alpha}\right)\right]^{\frac{1}{p}}\\ &\times\left[\left|f'\left(\frac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right|^q+\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q\right]^{\frac{1}{q}}\Bigg\}.\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. According to Lemma 1 and Hölder's inequality, we have

$$\left|\mathscr{L}_{\alpha,n}(a,b)\right| \le \frac{b-a}{n+1}\sum_{k=0}^{(n-1)/2}\left[\left(\int_0^1 (t^{\alpha})^p\,\mathrm{d}t\right)^{\frac{1}{p}}\right.$$

$$\times\left(\int_0^1\left|f'\left(t\frac{a(n-2k)+b(2k+1)}{n+1}+(1-t)\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)\right|^q \mathrm{d}t\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 (1-t^{\alpha})^p\,\mathrm{d}t\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\left|f'\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1}+(1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q \mathrm{d}t\right)^{\frac{1}{q}}\Bigg].$$

Since $\left|f\right|^q$ is convex on $[a,b]$, we have:

$$\int_0^1\left|f'\left(t\frac{a(n-2k)+b(2k+1)}{n+1}+(1-t)\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)\right|^q \mathrm{d}t$$
$$\le\frac{1}{2}\left[\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q+\left|f'\left(\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)\right|^q\right]$$

and

$$\int_0^1\left|f'\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1}+(1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q \mathrm{d}t$$
$$\le\frac{1}{2}\left[\left|f'\left(\frac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right|^q+\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q\right].$$

A simple computation shows that

$$\int_0^1 t^{\alpha p}\mathrm{d}t=\frac{1}{\alpha p+1},\int_0^1 (1-t^{\alpha})^p\,\mathrm{d}t=\frac{1}{\alpha}\mathrm{B}\left(p+1,\frac{1}{\alpha}\right)$$

and the proof is completed. ☐

Theorem 3. *Assume that $f:[a,b]\to\mathbb{R}$ is a differentiable function such that $|f'|^q$ is convex on $[a,b]$ for some exponent $q\ge 1$. Then the following inequality holds:*

$$\left|\mathscr{L}_{\alpha,n}(a,b)\right|\le\frac{b-a}{(n+1)(\alpha+1)^{\frac{1}{p}}(\alpha+2)^{\frac{1}{q}}}$$
$$\times\sum_{k=0}^{(n-1)/2}\Bigg\{\left[\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q+\frac{1}{\alpha+1}\left|f'\left(\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)\right|^q\right]^{\frac{1}{q}}$$
$$+\frac{\alpha}{2^{\frac{1}{q}}}\left[\left|f'\left(\frac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right|^q+\frac{\alpha+3}{\alpha+1}\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q\right]^{\frac{1}{q}}\Bigg\},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 1 and the power mean inequality, we have

$$\left|\mathscr{L}_{\alpha,n}(a,b)\right|\le\frac{b-a}{n+1}\sum_{k=0}^{(n-1)/2}\left[\left(\int_0^1 t^{\alpha}\mathrm{d}t\right)^{\frac{1}{p}}\right.$$
$$\times\left(\int_0^1 t^{\alpha}\left|f'\left(t\frac{a(n-2k)+b(2k+1)}{n+1}+(1-t)\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)\right|^q \mathrm{d}t\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 \left(1-t^{\alpha}\right) \mathrm{d}t\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1 \left(1-t^{\alpha}\right) \left|f'\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1}+(1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q \mathrm{d}t\right)^{\frac{1}{q}}\Bigg].$$

Since $|f|^q$ is convex on $[a,b]$, we have:

$$\int_0^1 t^{\alpha}\left|f'\left(t\frac{a(n-2k)+b(2k+1)}{n+1}+(1-t)\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)\right|^q \mathrm{d}t$$
$$\leq \frac{1}{\alpha+2}\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q$$
$$+\frac{1}{(\alpha+1)(\alpha+2)}\left|f'\left(\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)\right|^q$$

and

$$\int_0^1 \left(1-t^{\alpha}\right)\left|f'\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1}+(1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q \mathrm{d}t$$
$$\leq \frac{\alpha}{2(\alpha+2)}\left|f'\left(\frac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right|^q$$
$$+\frac{\alpha^2+3\alpha}{2(\alpha+1)(\alpha+2)}\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q.$$

This completes the proof of the theorem. □

Theorem 4. *Assume that $f:[a,b]\to\mathbb{R}$ is a differentiable function such that $|f'|^q$ is concave on $[a,b]$ for some exponent $q>1$. Then*

$$\left|\mathcal{L}_{\alpha,n}(a,b)\right| \leq \frac{b-a}{n+1}\sum_{k=0}^{(n-1)/2}\left[\left(\frac{1}{\alpha p+1}\right)^{1/p}\left|f'\left(\frac{a(2n-4k+1)+b(4k+1)}{2(n+1)}\right)\right|\right.$$
$$\left.+\left(\frac{1}{\alpha}\mathrm{B}\left(p+1,\frac{1}{\alpha}\right)\right)^{1/p}\left|f'\left(\frac{a(2n-4k-1)+b(4k+3)}{2(n+1)}\right)\right|\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1 and Hölder's integral inequality for $q>1$ and $p=\frac{q}{q-1}$, we have

$$\left|\mathcal{L}_{\alpha,n}(a,b)\right| \leq \frac{b-a}{n+1}\sum_{k=0}^{(n-1)/2}\left[\left(\int_0^1 \left(t^{\alpha}\right)^p \mathrm{d}t\right)^{\frac{1}{p}}\right.$$
$$\times\left(\int_0^1 \left|f'\left(t\frac{a(n-2k)+b(2k+1)}{n+1}+(1-t)\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)\right|^q \mathrm{d}t\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 \left(1-t^{\alpha}\right)^p \mathrm{d}t\right)^{p}$$
$$\left.\times\left(\int_0^1 \left|f'\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1}+(1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q \mathrm{d}t\right)^{\frac{1}{q}}\right].$$

Since $|f'|^q$ is concave on $[a,b]$, we infer from Jensen's inequality for concave functions that

$$\begin{aligned}
&\int_0^1 \left|f'\left(t\frac{a(n-2k)+b(2k+1)}{n+1}+(1-t)\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)\right|^q \mathrm{d}t\\
&\le \left|f'\left(\frac{\frac{a(n-2k)+b(2k+1)}{n+1}+\frac{a(n-2k+1)+b\cdot 2k}{n+1}}{2}\right)\right|^q\\
&= \left|f'\left(\frac{a(2n-4k+1)+b(4k+1)}{2(n+1)}\right)\right|^q.
\end{aligned}$$

In the same manner,

$$\begin{aligned}
&\int_0^1 \left|f'\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1}+(1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q \mathrm{d}t\\
&\le \left|f'\left(\frac{a(2n-4k-1)+b(4k+3)}{2(n+1)}\right)\right|^q.
\end{aligned}$$

Using

$$\int_0^1 t^{\alpha p}\mathrm{d}t=\frac{1}{\alpha p+1},\int_0^1\left(1-t^\alpha\right)^p\mathrm{d}t=\frac{1}{\alpha}\mathrm{B}\left(p+1,\frac{1}{\alpha}\right),$$

we complete the proof. □

Our next result is as follows:

Theorem 5. *Assume that $f:[a,b]\to\mathbb{R}$ is a differentiable function such that $|f'|$ is concave on $[a,b]$ for some exponent $q>1$. Then*

$$\begin{aligned}
\left|\mathscr{L}_{\alpha,n}(a,b)\right| &\le \frac{b-a}{(\alpha+1)(n+1)}\\
&\times\sum_{k=0}^{(n-1)/2}\left[\left|f'\left(\frac{\alpha+1}{\alpha+2}\cdot\frac{a(n-2k)+b(2k+1)}{(n+1)}+\frac{1}{\alpha+2}\cdot\frac{a(n-2k+1)+b\cdot 2k}{(n+1)}\right)\right|\right.\\
&\left.+\alpha\left|f'\left(\frac{\alpha+1}{2(\alpha+2)}\cdot\frac{a(n-2k-1)+b(2k+2)}{(n+1)}+\frac{\alpha+3}{2(\alpha+2)}\cdot\frac{a(n-2k)+b(2k+1)}{(n+1)}\right)\right|\right].
\end{aligned}$$

Proof. From Lemma 1 we have

$$\begin{aligned}
\left|\mathscr{L}_{\alpha,n}(a,b)\right| &\le \frac{b-a}{n+1}\\
&\times\sum_{k=0}^{(n-1)/2}\left[\int_0^1 t^\alpha\left|f'\left(t\frac{a(n-2k)+b(2k+1)}{n+1}+(1-t)\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)\right|\mathrm{d}t\right.\\
&\left.+\int_0^1\left(1-t^\alpha\right)\left|f'\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1}+(1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|\mathrm{d}t\right].
\end{aligned}$$

Since $|f'|$ is concave, by Jensen's inequality we obtain

$$\left|\mathscr{L}_{\alpha,n}(a,b)\right|\le\frac{b-a}{n+1}\sum_{k=0}^{(n-1)/2}\left[\left(\int_0^1 t^\alpha\mathrm{d}t\right)I_1+\left(\int_0^1\left(1-t^\alpha\right)\mathrm{d}t\right)I_2\right],$$

where

$$I_1=\left|f'\left(\frac{\int_0^1 t^\alpha\left(t\frac{a(n-2k)+b(2k+1)}{n+1}+(1-t)\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)dt}{\int_0^1 t^\alpha dt}\right)\right|$$
$$=\left|f'\left(\frac{\alpha+1}{\alpha+2}\cdot\frac{a(n-2k)+b(2k+1)}{n+1}+\frac{1}{\alpha+2}\cdot\frac{a(n-2k+1)+b\cdot 2k)}{n+1}\right)\right|$$

and

$$I_2=\left|f'\left(\frac{\int_0^1 (1-t^\alpha)\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1}+(1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\right)dt}{\int_0^1 (1-t^\alpha)dt}\right)\right|$$
$$=\left|f'\left(\frac{\alpha+1}{2(\alpha+2)}\cdot\frac{a(n-2k-1)+b(2k+2)}{n+1}+\frac{\alpha+3}{2(\alpha+2)}\cdot\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|$$

and the proof is completed. □

Remark 1. For $\alpha=1$ in the Theorems 1, 2, 3, 4 respectively 5, we recover the results stated in ([9, Theorems 6-10]). Also, for $\alpha=1$ in Lemma 1, we get ([9, Lemma 1]).

Remark 2. For $n=3$ in the Theorems 1, 2, 3, respectively 4 , we recover the results stated in ([5, Theorems 1-4]). Also for $\alpha=1$ in Lemma 1, we get ([5, Lemma 1]).

We end our paper by considering the *cumulative to the right* $(\alpha,n)-$*gap* defined by the formula

$$\mathscr{R}_{\alpha,n}(a,b)=-\sum_{k=0}^{(n-1)/2}\left[f\left(\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)+f\left(\frac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right]$$
$$+\Gamma(\alpha+1)\left(\frac{n+1}{b-a}\right)^{\alpha}\sum_{k=0}^{(n-1)/2}\left[J^\alpha_{\frac{a(n-2k+1)+b\cdot 2k}{n+1}+}f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right.$$
$$\left.+J^\alpha_{\frac{a(n-2k)+b(2k+1)}{n+1}+}f\left(\frac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right],$$

In the particular case where $\alpha=1$ and $n=3$ we have

$$\frac{\mathscr{R}_{1,3}(a,b)}{4}=-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]+\frac{1}{b-a}\int_a^b f(t)dt,$$

so the cumulative to the right gap $\mathscr{R}_{1,3}(a,b)$ estimates the precision of the left hand side inequality in (RHH),

$$\frac{1}{b-a}\int_a^b f(x)\,dx\le\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right].$$

Using the above techniques, one can prove companions of all the results we proved for the cumulative to the left $(\alpha,n)-$gap. The starting point is the following formula for computing $\mathscr{R}_{\alpha,n}(a,b)$.

Lemma 2. *We have*

$$\begin{aligned}\mathscr{R}_{\alpha,n}(a,b) &= \frac{b-a}{n+1}\\ &\times \sum_{k=0}^{(n-1)/2}\left[\int_0^1 t^\alpha f'\left(t\frac{a(n-2k+1)+b\cdot 2k}{n+1}+(1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\right)\mathrm{d}t\right.\\ &\left.+\int_0^1 (t^\alpha-1)f'\left(t\frac{a(n-2k)+b(2k+1)}{n+1}+(1-t)\frac{a(n-2k-1)+b(2k+2)}{n+1}\right)\mathrm{d}t\right].\end{aligned}$$

Using Lemma 2, one can prove various estimates of $\mathscr{R}_{\alpha,n}(a,b)$ such as

$$\begin{aligned}\left|\mathscr{R}_{\alpha,n}(a,b)\right| \le \frac{b-a}{n+1}\sum_{k=0}^{(n-1)/2}&\left[\frac{1}{\alpha+2}\left|f'\left(\frac{a(n-2k+1)+b\cdot 2k}{n+1}\right)\right|\right.\\ &+\frac{\alpha^2+\alpha+2}{2(\alpha+1)(\alpha+2)}\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|\\ &\left.+\frac{\alpha^2+3\alpha}{2(\alpha+1)(\alpha+2)}\left|f'\left(\frac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right|\right].\end{aligned}$$

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Department of Mathematics, University of Craiova, Street A. I. Cuza 13, Craiova, RO-200585, Romania.

E-mail: marcelamihai58@yahoo.com

Department of Mathematics, University of Craiova, Street A. I. Cuza 13, Craiova, RO-200585, Romania.

E-mail: dan_alexion@yahoo.com