SOME SUFFICIENT CONDITIONS FOR UNIVALENCE AND CLOSE-TO-CONVEXITY

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Abstract. The author uses the method of differential subordinations to obtain some new criteria for a normalized regular function, in unit disc $E = \{ z : |z| < 1 \}$ to be close-to-convex (univalent) in $E$.

1. Introduction

Let $f$ and $g$ be regular in the unit disc $E = \{ z : |z| < 1 \}$. We say that $f$ is subordinate to $g$, written $f(z) \prec g(z)$ or $f \prec g$, if there exists a function $w$ regular in $E$ which satisfies $w(0) = 0$, $|w(z)| < 1$ and $f(z) = g(w(z))$. If $g$ is univalent in $E$ then $f \prec g$ if and only if $f(0) = g(0)$ and $f(E) \subset g(E)$.

Let $V$ denote the class of all functions $f$ regular in the unit disc $E$, with $f(0) = f'(0) - 1 = 0$. Suppose that the function $f$ is regular in $E$. The function $f$, with $f'(0) \neq 0$ is convex (univalent) in $E$ if and only if $\text{Re}[1 + zf''(z)/f'(z)] > 0$, $z \in E$. The function $f$ is close-to-convex (univalent) in $E$ if and only if there is a convex function $g$ such that $\text{Re}[f'(z)/g'(z)] > 0$, $z \in E$[2].

Let $D^n f(z) = (z/(1-z))^{n+1} * f(z)$, where $*$ denotes the Hadamard product (convolution) of two regular functions in $E$ and $n \in N_0 = \{0, 1, 2, \ldots \}$ [9].

The aim of this paper is to give some sufficient conditions for a function $f \in V$ to be close-to-convex in $E$.

2. Preliminary Lemmas

For the proof of our results we need the following lemmas.

Lemma 2.1. Let $\Omega$ be a set in the complex plane $C$. Suppose that the function $\psi : C^2 \times E \to C$ satisfies the condition $\psi(ir_2, s_1; z) \notin \Omega$, for all real $r_2$, $s_1 \leq -2^{-1}(1+r_2^2)$ and all $z \in E$.

If $p(z)$ is regular in $E$, with $p(0) = 1$ and $\psi(p(z), zp'(z); z) \in \Omega$, when $z \in E$, then $\text{Re}(p(z)) > 0$ in $E$.

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More general form of the above lemma may be found in [3, 4].

**Lemma 2.2.** ([5]) Let $h$ be a convex function in $E$ and $u(z)$ be regular in $E$ with $\Re(u(z)) > 0$. If $p$ is regular in $E$ and $p(0) = h(0)$, then
\[
p(z) + zp'(z)(u(z)) \prec h(z) \Rightarrow p(z) \prec h(z), \quad z \in E.
\]

**Lemma 2.3.** ([1, 4]) If $\alpha \neq 0$, $\Re(\alpha) \geq 0$, $h$ be convex in $E$ and $p$ is regular in $E$ with $p(0) = h(0)$, then
\[
p(z) + \frac{zp'(z)}{\alpha} \prec h(z), \quad z \in E,
\]
implies
\[
p(z) \prec \alpha z^{-\alpha} \int_0^z h(t)t^{\alpha-1}dt \prec h(z), \quad z \in E.
\]

The following lemma is a special case of Lemma 1 of Ponnusamy [6] and is also due to S. Ponnusamy and V. Karunakaran [8], proved by them with the aid of Lemma 2.1:

**Lemma 2.4.** Let $u$ be a regular function in $E$ with $\Re(u(z)) > \delta > 0$ for $z \in E$. If $p$ is regular in $E$ with $p(0) = 1$, $\beta < 1$, $\alpha > 0$ and
\[
\Re(p(z) + \frac{u(z)}{\alpha}zp'(z)) > \beta, \quad z \in E,
\]
then
\[
\Re(p(z)) > \frac{2\beta\alpha + \delta}{2\alpha + \delta}, \quad z \in E.
\]

3. Main Results

**Theorem 3.1.** Let $f \in V$, $n \in N_0$ and $\beta < 1$. If $\alpha$, $\lambda$ be complex numbers with $\Re(\alpha) > 0$ and $|\lambda| \leq \Re(\alpha)/|\alpha|$, then
\[
\Re((1 + \lambda z)(D^nf(z))') > 2\beta(n + 1) + \Re(\alpha) - |\alpha\lambda| \quad \frac{2(n + 1) + \Re(\alpha) - |\alpha\lambda|}{2(n + 1) + \Re(\alpha) - |\alpha\lambda|}, \quad z \in E.
\]
This \( p(z) \) is regular in \( E \) and \( p(0) = 1 \). One can easily verify the identity
\[
z(D^n f(z))'' = (n+1)[(D^{n+1} f(z))' - (D^n f(z))'].
\]
Differentiating \( p(z) \) and using (3.2) we obtain
\[
(1 + \lambda z) \left[ \left( 1 - \alpha - \frac{\alpha \lambda n z}{n+1} \right) (D^n f(z))' + \alpha (1 + \lambda z)(D^{n+1} f(z))' \right]
= p(z) + \frac{u(z)}{n+1} z p'(z).
\]
So by Lemma 2.4 and (3.1), we get
\[
\text{Re} \left( 1 + \lambda z \right) (D^n f(z))' > \frac{2\beta(n+1) + \delta}{2(n+1) + \delta}, \quad z \in E,
\]
whenever \( \delta < \text{Re}(\alpha + \alpha \lambda z) \). But \( \delta \) can be chosen as near \( \text{Re}(\alpha) - |\alpha \lambda| \) as we please and so by allowing \( \delta \to \text{Re}(\alpha) - |\alpha \lambda| \) from below, we establish our claim.

**Theorem 3.2.** Let \( f \in V \), \( n \in \mathbb{N}_0 \) and \( \beta < 1 \). If \( \alpha > 0 \) and \( \lambda \) be complex number such that \( |\lambda| \leq 1 \), then
\[
\text{Re}(e^{-\lambda z} (D^n f(z))') > \frac{2\beta(n+1) + \delta}{2(n+1) + \delta}, \quad z \in E,
\]
implies
\[
\text{Re}(e^{-\lambda z}(D^n f(z))') > \frac{2\beta(n+1)(1 + |\lambda|) + \alpha}{2(n+1)(1 + |\lambda|) + \alpha}, \quad z \in E.
\]

**Proof.** If we let \( p(z) = e^{-\lambda z} (D^n f(z))' \) and \( u(z) = \alpha/(1 + \lambda z) \), then using (3.2), it can be seen that (3.3) is equivalent to
\[
\text{Re} \left( p(z) + \frac{u(z)}{n+1} z p'(z) \right) > \beta, \quad z \in E.
\]
and so by Lemma 2.4 we obtain that
\[
\text{Re}(e^{-\lambda z}(D^n f(z))') > \frac{2\beta(n+1) + \delta}{2(n+1) + \delta}, \quad z \in E,
\]
whenever \( \delta < a \text{Re}(1/(1 + \lambda z)) \). Now Theorem 3.2 follows by allowing \( \delta \to a/(1 + |\lambda|) \), from below.

If we set
\[
u_1(z) = (1 + \lambda z) \left[ \left( \frac{1}{\alpha} - 1 - \frac{\lambda n z}{n+1} \right) (D^n f(z))' + (1 + \lambda z)(D^{n+1} f(z))' \right]
\]
and
\[ v_2(z) = e^{-\lambda z} \left[ \frac{1}{\lambda} - \frac{n + 1 + \lambda z}{(n + 1)(1 + \lambda z)} \right] (D^n f(z))' + \frac{1}{1 + \lambda z} (D^{n+1} f(z))' \]
then for \( \alpha > 0 \) and \( \beta = 0 \), Theorem 3.1 and Theorem 3.2 reduces to
\[ \text{Re}(v_1(z)) > 0, \quad z \in E \quad (3.4) \]
implies
\[ \text{Re}(1 + \lambda z)(D^n f(z))' > \frac{\alpha(1 - |\lambda|)}{2(n + 1) + \alpha(1 - |\lambda|)}, \quad z \in E, \]
and
\[ \text{Re}(v_2(z)) > 0, \quad z \in E \quad (3.5) \]
implies
\[ \text{Re}(e^{-\lambda z}(D^n f(z))') > \frac{\alpha}{2(n + 1)(1 + |\lambda|) + \alpha}, \quad z \in E. \]
Let \( \alpha \to \infty \). Then (3.4) and (3.5) are equivalent to
\[ \text{Re}(v_1(z)) \geq 0, \quad z \in E \quad (3.6) \]
implies
\[ \text{Re}(1 + \lambda z)(D^n f(z))' \geq 1, \quad z \in E, \]
and
\[ \text{Re}(v_2(z)) \geq 0, \quad z \in E \quad (3.7) \]
implies
\[ \text{Re}(e^{-\lambda z}(D^n f(z))') \geq 1, \quad z \in E, \]
where
\[ v_1(z) = (1 + \lambda z) \left[ (1 + \lambda z)(D^{n+1} f(z))' - \left( 1 + \frac{\lambda n z}{n + 1} \right) (D^n f(z))' \right] \]
and
\[ v_2(z) = e^{-\lambda z} \left[ \frac{1}{1 + \lambda z} (D^{n+1} f(z))' - \frac{n + 1 + \lambda z}{(n + 1)(1 + \lambda z)} (D^n f(z))' \right]. \]
In the following theorem we extend the results (3.6) and (3.7).

**Theorem 3.3.** Let \( f \in V, \quad n \in N_0 \) then for \( \beta < 1 \) and \( |\lambda| \leq 1 \)
\[ \text{Re}(1 + \lambda z) \left[ (1 + \lambda z)(D^{n+1} f(z))' - \left( 1 + \frac{\lambda n z}{n + 1} \right) (D^n f(z))' \right] > \frac{(1 - \beta)(1 - |\lambda|)}{2(n + 1)}, \quad z \in E, \quad (3.8) \]
implies
\[ \text{Re}(1 + \lambda z)(D^n f(z))' > \beta, \quad z \in E, \]
and
\[
Re(e^{-\lambda z}) \left[ \frac{(D^{n+1}f(z))'}{1+\lambda z} - \frac{(n+1+\lambda z)}{(1+\beta(1+\lambda z))} (D^n f(z))' \right] > -\frac{1 - \beta}{2(n+1)(1+|\lambda|)}, \quad z \in E, \quad (3.9)
\]
implies
\[
Re(e^{-\lambda z}(D^n f(z))') > \beta \quad z \in E.
\]

**Proof.** It can be proved in a manner similar to that of Lemma 1 of [6] (using the identity (3.2)).

**Remark.** Theorem 1, Theorem 2 and Theorem 3 of Ponnusamy [6] are obtained for \( n = 0 \) in our results.

**Theorem 3.4.** Let \( \alpha \) be a real number with \( \alpha > 0 \), \( n \in \mathbb{N}_0 \), \( h \) be convex in \( E \) with \( h(0) = 1 \), and \( g \in V \) satisfies
\[
Re \left( \frac{(D^n g(z))'}{(D^{n+1} g(z))'} \right) > 0, \quad z \in E.
\]
If \( f \in V \) satisfies
\[
(1 - \alpha) \frac{(D^n f(z))'}{(D^n g(z))'} + \alpha \frac{(D^{n+1} f(z))'}{(D^{n+1} g(z))'} \prec h(z), \quad z \in E,
\]
then we have
\[
\frac{(D^n f(z))'}{(D^n g(z))'} \prec h(z), \quad z \in E.
\]

**Proof.** It can be proved in a manner similar to that of Theorem 1 of Ponnusamy and Juneja [7], ie using Lemma 2.2 and the identity (3.2) with
\[
p(z) = \frac{(D^n f(z))'}{(D^n g(z))'}
\]
and
\[
u(z) = \frac{\alpha}{n+1} \frac{(D^n g(z))'}{(D^{n+1} g(z))'}.
\]

**Remark.** Since the functions \( D^n g_i(z), \quad (i = 1, 2) \) denoted by \( (D^n g_i(z))' = \frac{1}{(1+\lambda z)} \) and \( (D^n g_i^4(z))' = e^{\lambda z} \) satisfy \( Re(D^n g_i(z)'/(D^{n+1}g_{i+1}(z))') > 0 \) in \( E \) \( (i = 1, 2) \), it follows that (3.1) with \(-\frac{Re(\alpha - |\lambda|)}{2(n+1)} \leq \beta < 1 \), (3.3) with \(-\frac{\beta}{2(n+1)(1+|\lambda|)} \leq \beta < 1 \), (3.8) and (3.9) with \( 0 \leq \beta < 1 \), are respectively sufficient conditions for a function \( f \in V \) to be close-to-convex in \( E \).
For \( g(z) = z \), the Theorem 3.4 can be further sharpened in the following form and its proof follows in the similar lines of Theorem 3.4, using Lemma 2.3.

**Theorem 3.5.** Let \( f \in V, n \in N_0 \) and \( h \) be convex function with \( h(0) = 1 \). Then for any complex number \( \alpha \) with \( Re(\alpha) \geq 0 \) \((\alpha \neq 0)\)

\[
(1 - \alpha)(D^n f(z))' + \alpha(D^{n+1} f(z))' \prec h(z), \quad z \in E
\]

implies

\[
(D^n f(z))' \prec \left( \frac{n+1}{\alpha} \right) z^{-(n+1)/\alpha} \int_0^z h(t)t^{(n+1)/\alpha}dt \prec h(z), \quad z \in E.
\]

The result is sharp.

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**References**


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