TAMKANG JOURNAL OF MATHEMATICS
Volume 37, Number 2, 117-122, Summer 2006

SOME SUFFICIENT CONDITIONS FOR UNIVALENCE AND CLOSE-TO-CONVEXITY

S. R. SWAMY

Abstract. The author uses the method of differential subordinations to obtain some new criteria for a normalised regular function, in unit disc $E = \{z : |z| < 1\}$ to be close-to-convex (univalent) in E.

1. Introduction

Let f and g be regular in the unit disc $E = \{z : |z| < 1\}$. We say that f is subordinate to g, written $f(z) \prec g(z)$ or $f \prec g$, if there exists a function w regular in E which satisfies $w(0) = 0$, $|w(z)| < 1$ and $f(z) = g(w(z))$. If g is univalent in E then $f \prec g$ if and only if $f(0) = g(0)$ and $f(E) \subset g(E)$.

Let V denote the class of all functions f regular in the unit disc E, with $f(0) = f'(0) - 1 = 0$. Suppose that the function f is regular in E. The function f, with $f'(0) \neq 0$ is convex (univalent) in E if and only if $Re[1 + zf''(z)/f'(z))] > 0$, $z \in E$. The function f is close-to-convex (univalent) in E if and only if there is a convex function g such that $Re[f'(z)/g'(z)] > 0$, $z \in E$[2].

Let $D^n f(z) = (z/(1-z)^{n+1}) * f(z)$, where $*$ denotes the Hadamard product (convolution) of two regular functions in E and $n \in N_0 = \{0, 1, 2, \ldots\}$ [9].

The aim of this paper is to give some sufficient conditions for a function $f \in V$ to be close-to-convex in E.

2. Preliminary Lemmas

For the proof of our results we need the following lemmas.

Lemma 2.1. *Let Ω be a set in the complex plane C. Suppose that the function $\psi : C^2 \times E \to C$ satisfies the condition $\psi(ir_2, s_1; z) \notin \Omega$, for all real r_2, $s_1 \leq -2^{-1}(1+r_2^2)$ and all $z \in E$.*

If $p(z)$ is regular in E, with $p(0) = 1$ and $\psi(p(z), zp'(z); z) \in \Omega$, when $z \in E$, then $Re(p(z)) > 0$ in E.

Received October 26, 2004; revised July 20, 2005.
2000 *Mathematics Subject Classification*. 30C45, 30C55.
Key words and phrases. Regular, convex and close-to-convex function, Hadamard product, Ruscheweyh derivatives, differential subordination.

More general form of the above lemma may be found in [3, 4].

Lemma 2.2.([5]) *Let h be a convex function in E and $u(z)$ be regular in E with $Re(u(z)) > 0$. If p is regular in E and $p(0) = h(0)$, then*

$$p(z) + zp'(z)(u(z)) \prec h(z) \Rightarrow p(z) \prec h(z), \qquad z \in E.$$

Lemma 2.3.([1, 4]) *If $\alpha \neq 0$, $Re(\alpha) \geq 0$, h be convex in E and p is regular in E with $p(0) = h(0)$, then*

$$p(z) + \frac{zp'(z)}{\alpha} \prec h(z), \qquad z \in E,$$

implies

$$p(z) \prec \alpha z^{-\alpha} \int_0^z h(t)t^{\alpha-1}dt \prec h(z), \qquad z \in E.$$

The following lemma is a special case of Lemma 1 of Ponnusamy [6] and is also due to S. Ponnusamy and V. Karunakaran [8], proved by them with the aid of Lemma 2.1:

Lemma 2.4. *Let u be a regular function in E with $Re(u(z)) > \delta > 0$ for $z \in E$. If p is regular in E with $p(0) = 1$, $\beta < 1$, $\alpha > 0$ and*

$$Re(p(z) + \frac{u(z)}{\alpha} zp'(z)) > \beta, \qquad z \in E,$$

then

$$Re(p(z)) > \frac{2\beta\alpha + \delta}{2\alpha + \delta}, \qquad z \in E.$$

3.. Main Results

Theorem 3.1. *Let $f \in V$, $n \in N_0$ and $\beta < 1$. If α, λ be complex numbers with $Re(\alpha) > 0$ and $|\lambda| \leq Re(\alpha)/|\alpha|$, then*

$$Re(1 + \lambda z) \left[\left(1 - \alpha - \frac{\alpha\lambda nz}{n+1} \right) (D^n f(z))' + \alpha(1 + \lambda z)(D^{n+1} f(z))' \right] > \beta, \qquad z \in E, \tag{3.1}$$

implies

$$Re((1 + \lambda z)(D^n f(z))') > \frac{2\beta(n+1) + Re(\alpha) - |\alpha\lambda|}{2(n+1) + Re(\alpha) - |\alpha\lambda|}, \qquad z \in E.$$

Proof. Let

$$p(z) = (1 + \lambda z)(D^n f(z))'$$

and

$$u(z) = \alpha(1 + \lambda z).$$

This $p(z)$ is regular in E and $p(0) = 1$. One can easily verify the identity

$$z(D^n f(z))'' = (n+1)[(D^{n+1} f(z))' - (D^n f(z))']. \tag{3.2}$$

Differentiating $p(z)$ and using (3.2) we obtain

$$\begin{aligned}&(1+\lambda z)\left[\left(1-\alpha-\frac{\alpha\lambda n z}{n+1}\right)(D^n f(z))' + \alpha(1+\lambda z)(D^{n+1} f(z))'\right]\\ &= p(z) + \frac{u(z)}{n+1} z p'(z).\end{aligned}$$

So by Lemma 2.4 and (3.1), we get

$$Re(1+\lambda z)(D^n f(z))' > \frac{2\beta(n+1)+\delta}{2(n+1)+\delta}, \qquad z \in E,$$

whenever $\delta < Re(\alpha + \alpha\lambda z)$. But δ can be chosen as near $Re(\alpha) - |\alpha\lambda|$ as we please and so by allowing $\delta \to Re(\alpha) - |\alpha\lambda|$ from below, we establish our claim.

Theorem 3.2. *Let $f \in V$, $n \in N_0$ and $\beta < 1$. If $\alpha > 0$ and λ be complex number such that $|\lambda| \leq 1$, then*

$$Re(e^{-\lambda z})\left[\left(1-\frac{\alpha(n+1+\lambda z)}{(n+1)(1+\lambda z)}\right)(D^n f(z))' + \frac{\alpha}{1+\lambda z}(D^{n+1} f(z))'\right] > \beta, \qquad z \in E \tag{3.3}$$

implies

$$Re(e^{-\lambda z}(D^n f(z))') > \frac{2\beta(n+1)(1+|\lambda|)+\alpha}{2(n+1)(1+|\lambda|)+\alpha}, \qquad z \in E.$$

Proof. If we let $p(z) = e^{-\lambda z}(D^n f(z))'$ and $u(z) = \alpha/(1+\lambda z)$, then using (3.2), it can be seen that (3.3) is equivalent to

$$Re\left(p(z) + \frac{u(z)}{n+1} z p'(z)\right) > \beta, \qquad z \in E.$$

and so by Lemma 2.4 we obtain that

$$Re(e^{-\lambda z}(D^n f(z))') > \frac{2\beta(n+1)+\delta}{2(n+1)+\delta}, \qquad z \in E,$$

whenever $\delta < \alpha Re(1/(1+\lambda z))$. Now Theorem 3.2 follows by allowing $\delta \to \alpha/(1+|\lambda|)$, from below.

If we set

$$v_1(z) = (1+\lambda z)\left[\left(\frac{1}{\alpha} - 1 - \frac{\lambda n z}{n+1}\right)(D^n f(z))' + (1+\lambda z)(D^{n+1} f(z))'\right]$$

and

$$v_2(z) = e^{-\lambda z}\left[\left(\frac{1}{\alpha} - \frac{n+1+\lambda z}{(n+1)(1+\lambda z)}\right)(D^n f(z))' + \frac{1}{1+\lambda z}(D^{n+1}f(z))'\right]$$

then for $\alpha > 0$ and $\beta = 0$, Theorem 3.1 and Theorem 3.2 reduces to

$$Re(v_1(z)) > 0, \qquad z \in E \tag{3.4}$$

implies

$$Re(1+\lambda z)(D^n f(z))' > \frac{\alpha(1-|\lambda|)}{2(n+1)+\alpha(1-|\lambda|)}, \qquad z \in E,$$

and

$$Re(v_2(z)) > 0, \qquad z \in E \tag{3.5}$$

implies

$$Re(e^{-\lambda z}(D^n f(z))') > \frac{\alpha}{2(n+1)(1+|\lambda|)+\alpha}, \qquad z \in E.$$

Let $\alpha \to \infty$. Then (3.4) and (3.5) are equivalent to

$$Re(\mathrm{v}_1(z)) \geq 0, \qquad z \in E \tag{3.6}$$

implies

$$Re(1+\lambda z)(D^n f(z))' \geq 1, \qquad z \in E,$$

and

$$Re(\mathrm{v}_2(z)) \geq 0, \qquad z \in E \tag{3.7}$$

implies

$$Re(e^{-\lambda z}(D^n f(z))') \geq 1, \qquad z \in E,$$

where

$$\mathrm{v}_1(z) = (1+\lambda z)\left[(1+\lambda z)(D^{n+1}f(z))' - \left(1+\frac{\lambda n z}{n+1}\right)(D^n f(z))'\right]$$

and

$$\mathrm{v}_2(z) = e^{-\lambda z}\left[\frac{1}{1+\lambda z}(D^{n+1}f(z))' - \frac{n+1+\lambda z}{(n+1)(1+\lambda z)}(D^n f(z))'\right].$$

In the following theorem we extend the results (3.6) and (3.7).

Theorem 3.3. *Let $f \in V$, $n \in N_0$ then for $\beta < 1$ and $|\lambda| \leq 1$*

$$Re(1+\lambda z)\left[(1+\lambda z)(D^{n+1}f(z))' - \left(1+\frac{\lambda n z}{n+1}\right)(D^n f(z))'\right]$$
$$> -\frac{(1-\beta)(1-|\lambda|)}{2(n+1)}, \qquad z \in E, \tag{3.8}$$

implies

$$Re(1+\lambda z)(D^n f(z))' > \beta \qquad z \in E,$$

and

$$Re(e^{-\lambda z})\left[\frac{(D^{n+1}f(z))'}{1+\lambda z}-\frac{(n+1+\lambda z)}{(1+\beta)(1+\lambda z)}(D^n f(z))'\right]$$
$$>-\frac{1-\beta}{2(n+1)(1+|\lambda|)},\qquad z\in E, \tag{3.9}$$

implies

$$Re(e^{-\lambda z}(D^n f(z))')>\beta \qquad z\in E.$$

Proof. It can be proved in a manner similar to that of Lemma 1 of [6] (using the identity (3.2)).

Remark. Theorem 1, Theorem 2 and Theorem 3 of Ponnusamy [6] are obtained for $n=0$ in our results.

Theorem 3.4. *Let α be a real number with $\alpha>0$, $n\in N_0$, h be convex in E with $h(0)=1$, and $g\in V$ satisfies*

$$Re\frac{(D^n g(z))'}{(D^{n+1}g(z))'}>0,\qquad z\in E.$$

If $f\in V$ satisfies

$$(1-\alpha)\frac{(D^n f(z))'}{(D^n g(z))'}+\alpha\frac{(D^{n+1}f(z))'}{(D^{n+1}g(z))'}\prec h(z),\qquad z\in E,$$

then we have

$$\frac{(D^n f(z))'}{(D^n g(z))'}\prec h(z),\qquad z\in E.$$

Proof. It can be proved in a manner similar to that of Theorem 1 of Ponnusamy and Juneja [7], ie using Lemma 2.2 and the identity (3.2) with

$$p(z)=\frac{(D^n f(z))'}{(D^n g(z))'}$$

and

$$u(z)=\frac{\alpha}{n+1}\frac{(D^n g(z))'}{(D^{n+1}g(z))'}.$$

Remark. Since the functions $D^n g_i(z)$, $(i=1,2)$ denoted by $(D^n g_1^n(z))'=\frac{1}{(1+\lambda z)}$ and $(D^n g_1^n(z))'=e^{\lambda z}$ satisfy $Re(D^n g_i(z)'/(D^{n+1}g_{i+1}(z))')>0$ in E $(i=1,2)$, it follows that (3.1) with $-\frac{Re(\alpha)-|\alpha\lambda|}{2(n+1)}\le\beta<1$, (3.3) with $-\frac{\alpha}{2(n+1)(1+|\lambda|)}\le\beta<1$, (3.8) and (3.9) with $0\le\beta<1$, are respectively sufficient conditions for a function $f\in V$ to be close-to-convex in E.

For $g(z) = z$, the Theorem 3.4 can be further sharpened in the following form and its proof follows in the similar lines of Theorem 3.4, using Lemma 2.3.

Theorem 3.5. *Let $f \in V$, $n \in N_0$ and h be convex function with $h(0) = 1$. Then for any complex number α with $Re(\alpha) \geq 0$ $(\alpha \neq 0)$*

$$(1-\alpha)(D^n f(z))' + \alpha(D^{n+1} f(z))' \prec h(z), \qquad z \in E$$

implies

$$(D^n f(z))' \prec \left(\frac{n+1}{\alpha}\right) z^{-(n+1)/\alpha} \int_0^z h(t) t^{\left(\frac{n+1}{\alpha}\right)-1} dt \prec h(z), \qquad z \in E.$$

The result is sharp.

Acknowledgement

The author thanks the referee for his valuable suggestions to improve the presentation.

References

[1] D. J. Hallenbeck and S. Ruscheweyh, *Subordination by conbvex functions*, Proc. Amer. Math. Soc. **52** (1975), 191-195.
[2] W. Kaplan, *Close-to-convex functions*, Michigan Math. J. **1**(1952), 169-185.
[3] S. S. Miller and T. Mocanu, *Second order differential inequalities in the complex plane*, J. Math Anal. Appl. **65** (1978), 289-305.
[4] S. S. Miller and P. T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math. J. **28**(1981), 157-171.
[5] S. S. Miller and P. T. Mocanu, *Differential subordination and inequalities in the complex plane*, J. Differentail Equations **67**(1987), 199-211.
[6] S. Ponnusamy, *On Bazilevic functions*, Annal. Univ. Marie-Curie Skla. Lub. Polon. (Sect. A) **42** (1988), 115-127.
[7] S. Ponnusamy and O. P. Juneja, *Some applications of first order differential subordinations*, Glansnik Math. **25** (1991), 48-58.
[8] S. Ponnusamy and V. Karunakaran, *Differential subordination and conformal mappings*, Complex Variables Theory Appl. **11** (1989), 79-86.
[9] St. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc. **49**(1975), 109-115.

P G Department of Computer Science, R. V. College of Engineering, Bangalore-560 059, India.

E-mail: mailtoswamy@rediffmail.com