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DIFFERENTIATING-DOMINATING SETS IN GRAPHS UNDER BINARY OPERATIONS

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Abstract. In this paper we characterize the differentiating-dominating sets in the join, corona, and lexicographic product of graphs. We also determine bounds or the exact differentiating-domination numbers of these graphs.

1. Introduction

Let $G = (V(G), E(G))$ be a connected graph and $v \in V(G)$. The neighborhood of v is the set $N_G(v) = N(v) = \{u \in V(G) : uv \in E(G)\}$. The *degree* of a vertex $v \in V(G)$ is equal to the cardinality of $N_G(v)$ and the *maximum degree* of G is $\Delta(G) = \max\{deg(v) : v \in V(G)\}$.

If $X \subseteq V(G)$, then the *open neighborhood* of X is the set $N_G(X) = N(X) = \cup_{v\in X} N_G(v)$. The *closed neighborhood* of X is $N_G[X] = N[X] = X \cup N(X)$.

Now a connected graph G of order $n \geq 3$ is *point distinguishing* if for any two distinct vertices u and v of G, $N_G[u] \neq N_G[v]$. It is *totally point determining* if for any two distinct vertices u and v of G, $N_G(u) \neq N_G(v)$ and $N_G[u] \neq N_G[v]$. These concepts were parts of investigation in [2] and [7].

A subset X of $V(G)$ is a *dominating set* of G if for every $v \in V(G)\backslash X$, there exists $x \in X$ such that $xv \in E(G)$, i.e., $N[X] = V(G)$. The *domination number* $\gamma(G)$ of G is the smallest cardinality of a dominating set of G.

A subset S of $V(G)$ is a *locating set* in a connected graph G if for any two distinct vertices u and v in $V(G)\backslash S$, $N_G(u) \cap S \neq N_G(v) \cap S$. A subset S of $V(G)$ is a *differentiating set* in a connected graph G if for every two distinct vertices u and v of G, $N_G[u] \cap S \neq N_G[v] \cap S$. It is a *strictly differentiating set* if it is differentiating and $N_G[u] \cap S \neq S$ for all $u \in V(G)$. The

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minimum cardinality of a differentiating set in G, denoted by $dn(G)$, is called the *differentiating number* of G. The minimum cardinality of a strictly differentiating set in G, denoted by $sdn(G)$, is called the *strict differentiating number* of G. A differentiating (resp. strictly differentiating) subset S of $V(G)$ which is also dominating is called a *differentiating-dominating* (resp. *strictly differentiating-dominating*) set in a graph G. The minimum cardinality of a differentiating-dominating (resp. strictly differentiating-dominating) set in G, denoted by $\gamma_D(G)$ (resp. $\gamma_{SD}(G)$), is called the *differentiating-domination* (resp. *strict differentiating-domination*) number of G. Some of these concepts may be found in [4] and are investigated in [1], [3], [5], and [7].

In a given network or graph, a differentiating set can be viewed as a set of monitors which can actually determine the exact location of an intruder (e.g. a burglar, a fire, etc.). By requiring such a set to be dominating implies that every vertex where there is no monitor in it is connected to at least one monitoring device. Hence, determination of the differentiating-domination number of a graph is equivalent to finding the least number of monitors that can do the certain task in a given graph or network. In some contexts, differentiating dominating sets are called identifying codes (see [8]).

Now let G be a connected graph of order n and suppose that there exist (distinct) adjacent vertices u and v of G such that $N_G[u] = N_G[v]$. Then $N_G[u] \cap S = N_G[v] \cap S$ for any subset S of $V(G)$. This implies that G cannot have a differentiating set. Also, if $\Delta(G) = n-1$ and $v \in V(G)$ with $deg(v) = n-1$, then $N_G[v] \cap S = S$ for any subset S of $V(G)$. Consequently, G cannot have a strictly differentiating set. Thus, unless otherwise stated, throughout this paper, G is a point distinguishing graph of order $n \geq 3$. Moreover, whenever the concept of strictly differentiating set of a graph G is mentioned in this paper, it is always assumed that $\Delta(G) \leq n-2$.

2. Preliminary results and characterizations

The following two simple observations are worth mentioning.

Remark 2.1. Every differentiating set in a connected graph G is a locating set.

Remark 2.2. Let G be a connected graph of order $n \geq 3$. Then $2 \leq \gamma_D(G) \leq n-1$.

Theorem 2.3. *Let G be a connected graph. Then $\gamma_D(G) = 2$ if and only if $G = P_3$.*

Proof. Suppose $\gamma_D(G) = 2$, say $S = \{a, b\}$ is a differentiating dominating set in G. If $ab \in E(G)$, then $N_G[a] \cap S = \{a, b\} = N_G[b] \cap S$, contrary to our assumption of S. Therefore, $ab \notin E(G)$. Now, since S has only three different non-empty subsets, $|V(G)| = 3$. Therefore, since $G \neq K_3$, $G = P_3$.

For the converse, suppose that $G = [a, c, b] = P_3$. Let $S = \{a, b\}$. Then, clearly, S is a differentiating dominating set in G. Thus $\gamma_D(G) = 2$. □

Remark 2.4. Let G be a connected graph of order $n \geq 3$. Then $dn(G) \leq \gamma_D(G) \leq \gamma_{SD}(G)$ and $dn(G) \leq sdn(G) \leq \gamma_{SD}(G)$.

The following simple results give specific relationships between $dn(G)$, $sdn(G)$, $\gamma_D(G)$, and $\gamma_{SD}(G)$ for a connected graph G.

Lemma 2.5. *Let G be a connected graph of order $n \geq 3$ such that $dn(G) < \gamma_D(G)$. Then $1 + dn(G) = \gamma_D(G)$.*

Proof. Let S be a minimum differentiating set in G. Then S is not a dominating set in G. Hence, there exists a $y \in V(G)\setminus S$ such that $xy \notin E(G)$ for all $x \in S$. This implies that $N_G[y] \cap S = N_G(y) \cap S = \emptyset$. Set $S^* = S \cup \{y\}$ and let $z \in V(G)\setminus S^*$. Since S is a locating set (Remark 2.1), $N_G(z) \cap S \neq \emptyset$. This implies that there exists $w \in S$ such that $wz \in E(G)$. This shows that S^* is a dominating set in G. Next, let $a, b \in V(G)$. Then $N_G[a] \cap S \neq N_G[b] \cap S$ since S is a differentiating set in G. Therefore, $N_G[a] \cap S^* \neq N_G[b] \cap S^*$. This implies that S^* is a differentiating set in G. Therefore $\gamma_D(G) \leq 1 + dn(G)$. Since $dn(G) < \gamma_D(G)$, $1 + dn(G) \leq \gamma_D(G)$. This shows that $1 + dn(G) = \gamma_D(G)$. □

Lemma 2.6. *Let G be a connected graph of order $n \geq 3$ such that $dn(G) < sdn(G)$ and $\Delta(G) \leq n-2$. Then $1 + dn(G) = sdn(G)$.*

Proof. Let S be a minimum differentiating set in G. By assumption, S is not a strictly differentiating set in G. Hence, there exists a $y \in V(G)$ such that $N_G[y] \cap S = S$. Since $deg(y) \leq n-2$, there exists $z \in V(G)\setminus(S \cup \{y\})$ such that $z \notin N_G(y)$. Set $S^* = S \cup \{z\}$. If $a, b \in V(G)$ $(a \neq b)$, then $N_G[a] \cap S \neq N_G[b] \cap S$ since S is a differentiating set. Thus, $N_G[a] \cap S^* \neq N_G[b] \cap S^*$, showing that S^* is a differentiating set. Now let $x \in V(G)$. If $x = y$, then $z \notin N_G[x]$. This implies that $z \notin N_G[x] \cap S^*$. Hence $N_G[x] \cap S^* \neq S^*$. If $x \neq y$, then $N_G[x] \cap S \neq S$ since S is differentiating. This implies that there exists $w \in S$ such that $w \notin N_G[x]$. Hence, $N_G[x] \cap S^* \neq S^*$. Therefore S^* is a strictly differentiating set in G. Consequently, $sdn(G) \leq 1 + dn(G)$. Since $dn(G) < sdn(G)$, $1 + dn(G) \leq sdn(G)$. This establishes the desired equality. □

Lemma 2.7. *Let G be a connected graph of order $n \geq 3$ such that $sdn(G) < \gamma_{SD}(G)$. Then $1 + sdn(G) = \gamma_{SD}(G)$.*

Proof. Let S be a minimum strictly differentiating set in G. From the assumption, S is not a dominating set in G. Hence, there exists a $y \in V(G)\setminus S$ such that $xy \notin E(G)$ for all $x \in S$. This implies that $N_G[y] \cap S = N_G(y) \cap S = \emptyset$. Set $S^* = S \cup \{y\}$ and let $z \in V(G)\setminus S^*$. Since S is a differentiating set, $N_G[z] \cap S = N_G(z) \cap S \neq \emptyset$. This implies that there exists $q \in S \subseteq S^*$ such that $qz \in E(G)$. Hence S^* is a dominating set in G.

Next, let $a, b \in V(G)$. Then $N_G[a] \cap S \neq N_G[b] \cap S$ since S is a differentiating set in G. Therefore, $N_G[a] \cap S^* \neq N_G[b] \cap S^*$. This implies that S^* is a differentiating set in G. Moreover, if $x \in V(G)$, then $N_G[x] \cap S \neq S$ since S is strictly differentiating. It follows that $N_G[x] \cap S^* \neq S^*$, i.e., S^* is a strictly differentiating (dominating) set. Therefore $\gamma_{SD}(G) \leq 1 + sdn(G)$. Since $sdn(G) < \gamma_{SD}(G)$, $1 + sdn(G) \leq \gamma_{SD}(G)$. This shows that $1 + sdn(G) = \gamma_{SD}(G)$. □

Lemma 2.8. *Let G be a connected graph of order $n \geq 3$ such that $\gamma_D(G) < \gamma_{SD}(G)$. Then $1 + \gamma_D(G) = \gamma_{SD}(G)$.*

Proof. Let S be a minimum differentiating dominating set in G. Then S is not a strictly differentiating set in G. Hence, there exists a $y \in V(G)$ such that $N_G[y] \cap S = S$. Since $deg(y) \leq n-2$, there exists $z \in V(G) \setminus (S \cup \{y\})$ such that $z \notin N_G(y)$. Set $S^* = S \cup \{z\}$. Since S is a dominating set, S^* is also a dominating set. If $a, b \in V(G)$, then $N_G[a] \cap S \neq N_G[b] \cap S$ since S is a differentiating set. Thus, $N_G[a] \cap S^* \neq N_G[b] \cap S^*$, showing that S^* is a differentiating set. Now let $x \in V(G)$. If $x = y$, then $z \notin N_G[x]$. This implies that $z \notin N_G[x] \cap S^*$. Hence $N_G[x] \cap S^* \neq S^*$. If $x \neq y$, then $N_G[x] \cap S \neq S$ since S is differentiating. This implies that there exists $w \in S$ such that $w \notin N_G[x]$. It follows that $N_G[x] \cap S^* \neq S^*$. Therefore S^* is a strictly differentiating (dominating) set in G. Consequently, $\gamma_{SD}(G) \leq 1 + \gamma_D(G)$. Since $\gamma_D(G) < \gamma_{SD}(G)$, $1 + \gamma_D(G) \leq \gamma_{SD}(G)$. Accordingly, $1 + \gamma_D(G) = \gamma_{SD}(G)$. □

3. Differentiating dominating sets in the join of graphs

The *join* $G + H$ of two graphs G and H is the graph with $V(G + H) = V(G) \cup V(H)$ (disjoint union) and $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}$.

Theorem 3.9. *Let G and H be non-trivial graphs of orders $m \geq 2$ and $n \geq 2$, respectively. Then $S \subseteq V(G + H)$ is a differentiating dominating set in $G + H$ if and only if $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$ are differentiating sets in G and H, respectively, and either S_G or S_H is strictly differentiating.*

Proof. Let $S \subseteq V(G + H)$ be a differentiating-dominating set in $G + H$. Let $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$. Suppose $S_G = \emptyset$. Pick distinct vertices u and v of G. Then $N_{G+H}[u] \cap S = S = N_{G+H}[v] \cap S$, contrary to the assumption that S is a differentiating set for $G + H$. Thus, $S_G \neq \emptyset$. Similarly, $S_H \neq \emptyset$. Suppose now that one of S_G and S_H is not a differentiating set, say S_G is not a differentiating set in G. Then there exist distinct vertices $a, b \in V(G)$ such that $N_G[a] \cap S_G = N_G[b] \cap S_G$. Since $S_H \subseteq N_{G+H}[a]$ and $S_H \subseteq N_{G+H}[b]$, it follows that $N_{G+H}[a] \cap S = (N_G[a] \cap S_G) \cup S_H = N_{G+H}[b] \cap S$. This is impossible since S is a differentiating set for $G + H$. Therefore, S_G and S_H are differentiating sets in G and H, respectively.

Next, suppose that both S_G and S_H are not strictly differentiating sets in G and H, respectively. Then there exist $z \in V(G) \setminus S_G$ and $w \in V(H) \setminus S_H$ such that $N_G[z] \cap S_G = S_G$ and

$N_H[w] \cap S_H = S_H$. It follows that $N_{G+H}[z] \cap S = S = N_{G+H}[w] \cap S$, contrary to the fact that S a differentiating set in $G+H$. Thus, S_G is a strictly differentiating set in G or S_H is a strictly differentiating set in H.

For the converse, suppose $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$ are differentiating sets in G and H, respectively, and where S_G or S_H is a strictly differentiating set. Let x and y be distinct vertices in $V(G+H)$. If $x, y \in V(G)$, then $N_G[x] \cap S_G \neq N_G[y] \cap S_G$. It follows that $N_{G+H}[x] \cap S = (N_G[x] \cap S_G) \cup S_H \neq (N_G[y] \cap S_G) \cup S_H = N_{G+H}[y] \cap S$. Similarly, $N_{G+H}[x] \cap S \neq N_{G+H}[y] \cap S$ if $x, y \in V(H)$. Suppose $x \in V(G)$ and $y \in V(H)$. Suppose, without loss of generality, that S_G is a strictly differentiating set in G. Then S_G is not contained in $N_{G+H}[x]$. Since $S_G \subseteq N_{G+H}[y]$, it follows that $N_{G+H}[x] \cap S \neq N_{G+H}[y] \cap S$. Accordingly, S is a differentiating set in $G+H$. Clearly, S is a dominating set in $G+H$. □

The next results are direct consequences of Theorem 3.9 or of its proof.

Corollary 3.10. *Let G and H be connected non-trivial graphs of orders $m \geq 2$ and $n \geq 2$, respectively. Then every differentiating set in $G+H$ is dominating.*

Corollary 3.11. *Let G and H be connected non-trivial graphs of orders $m \geq 2$ and $n \geq 2$, respectively. Then $\gamma_D(G+H) = \min\{sdn(H) + dn(G), sdn(G) + dn(H)\}$.*

Theorem 3.12. *Let $G = K_1 = \langle v \rangle$ and H a non-trivial graph. Then $S \subseteq V(G+H)$ is a differentiating dominating set in $G+H$ if and only if $v \in S$ and $V(H) \cap S$ is a strictly differentiating set in H or $v \notin S$ and S is a strictly differentiating dominating set in H.*

Proof. Suppose S is a differentiating dominating set in $G+H$ and suppose $v \in S$. Since S is differentiating and $|V(H)| \geq 2$, $V(H) \cap S \neq \emptyset$. Also, since $N_{G+H}[v] \cap S = S$, $V(H) \cap S$ must be a strictly differentiating set in H. Suppose now that $v \notin S$. Then $S \subseteq V(H)$ must be a dominating set in H. Since $N_{G+H}[u] \cap S = N_H[u] \cap S$ for every $u \in V(H)$ and $N_{G+H}[v] \cap S = S$, S is a strictly differentiating set in H. Hence S is a strictly differentiating dominating set in H.

The converse is clear. □

Corollary 3.13. *Let $G = K_1 = \langle v \rangle$ and H a non-trivial graph. Then $\gamma_D(G+H) = \gamma_{SD}(H)$.*

Proof. Let S be a minimum differentiating dominating set in $G+H$. Suppose first that $v \in S$. Then $V(H) \cap S$ is a strictly differentiating set in H, by Theorem 3.12. Hence, $sdn(G) + 1 \leq |S| = \gamma_D(G+H)$. By Remark 2.4 and Lemma 2.7, $\gamma_{SD}(H) \leq \gamma_D(G+H)$. If $v \notin S$, then S is a strictly differentiating dominating set in H by Theorem 3.12. It follows that $\gamma_{SD}(H) \leq |S| = \gamma_D(G+H)$. Thus $\gamma_D(G+H) \geq \gamma_{SD}(H)$.

Now let S be a minimum strictly differentiating dominating set in H. Then S is a differentiating dominating set in $G+H$ by Theorem 3.12. Thus $\gamma_D(G+H) \leq |S| = \gamma_{SD}(H)$.

Therefore $\gamma_D(G+H) = \gamma_{SD}(H)$. □

4. Differentiating dominating sets in the corona of graphs

The *corona* $G \circ H$ of two graphs G and H is the graph obtained by taking one copy of G of order n and n copies of H, and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H. For every $v \in V(G)$, we denote by H^v the copy of H whose vertices are attached one by one to the vertex v. Subsequently, we denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle v \rangle + H^v$, where $v \in V(G)$.

Theorem 4.14. *Let G (not necessarily point distinguishing) and let H be non-trivial connected graphs. Then $C \subseteq V(G \circ H)$ is a differentiating dominating set in $G \circ H$ if and only if for every $v \in V(G)$, one of the following is true:*

(i) *$v \in C$, $N_G(v) \cap C \neq \emptyset$, and $C \cap V(H^v)$ is a differentiating set in H^v;*
(ii) *$v \in C$, $N_G(v) \cap C = \emptyset$, and $C \cap V(H^v)$ is a strictly differentiating set in H^v;*
(iii) *$v \notin C$, $N_G(v) \cap C \neq \emptyset$, and $C_1 = V(H^v) \cap C$ is a differentiating dominating set in H^v; or*
(iv) *$v \notin C$, $N_G(v) \cap C = \emptyset$ and $C_1 = V(H^v) \cap C$ is a strictly differentiating dominating set in H^v.*

Proof. Suppose C is a differentiating dominating set in $G \circ H$. Let $v \in V(G)$, $C_1 = V(H^v) \cap C$, and let $x, y \in V(H^v)$, where $x \neq y$. Then $N_{G\circ H}[x] \cap C = (N_{H^v}[x] \cap C_1) \cup (C \cap \{v\})$, $N_{G\circ H}[y] \cap C = (N_{H^v}[y] \cap C_1) \cup (C \cap \{v\})$, and $N_{G\circ H}[v] \cap C = (N_G(v) \cap C_2) \cup (N_{v+H^v}[v] \cap C_1) \cup (C \cap \{v\})$, where $C_2 = C \cap V(G)$.

Suppose first that $v \in C$. If $N_G(v) \cap C \neq \emptyset$, then, since C is differentiating,

$$(N_{H^v}[x] \cap C_1) \cup \{v\} = N_{G\circ H}[x] \cap C \neq N_{G\circ H}[y] \cap C = (N_{H^v}[y] \cap C_1) \cup \{v\}.$$

It folows that $N_{H^v}[x] \cap C_1 \neq N_{H^v}[y] \cap C_1$. Thus C_1 is differentiating in H^v, i.e., (i) holds. Suppose $N_G(v) \cap C = \emptyset$. Then, again, since C is differentiating, C_1 is differentiating in H^v. Now, since $N_{G\circ H}[v] \cap C = N_{v+H^v}[v] \cap C_1 = \{v\} \cup C_1$, C_1 must be strictly differentiating in H^v. Hence, (ii) holds.

Next, suppose that $v \notin C$. If $N_G(v) \cap C \neq \emptyset$, then, since C is differentiating,

$$N_{H^v}[x] \cap C_1 = N_{G\circ H}[x] \cap C \neq N_{G\circ H}[y] \cap C = N_{H^v}[y] \cap C_1.$$

This implies that C_1 is differentiating in H^v. Since $v \notin C$ and C is dominating, C_1 is dominating in H^v. Therefore, (iii) holds. Suppose $N_G(v) \cap C = \emptyset$. Since C is a differentiating dominating set, $v \notin C$, and $N_{G\circ H}[v] = C_1$, it follows that C_1 is a strictly differentiating dominating set in H^v, i.e., (iv) holds.

For the converse, suppose that C satisfies (i), (ii), (iii), or (iv) for every $v \in V(G)$. Let $x \in V(G \circ H) \backslash C$ and $v \in V(G)$ be such that $x \in V(v + H^v)$. If $v \in C$, then $xv \in E(G \circ H)$ $(x \neq v)$. If

$v \notin C$ (say when $x = v$), then, by (iii), or (iv), $C_1 = V(H^v) \cap C$ is a dominating set in H^v. Hence, there exists $y \in C_1$ such that $xy \in E(H^v) \subseteq E(G \circ H)$. Therefore, C is a dominating set in $G \circ H$.

Next, let $a, b \in V(G \circ H)$ with $a \neq b$. Let $u, v \in V(G)$ such that $a \in V(u + H^u)$ and $b \in V(v + H^v)$. Consider the following cases:

Case 1: Suppose that $u = v$.

If $a, b \in V(H^v)$, then $N_{H^v}[a] \cap C_1 \neq N_{H^v}[b] \cap C_1$ since C_1 is differentiating in H^v by (i), (ii), (iii), and (iv). Therefore,

$$(N_{G\circ H}[a] \cap C) \cup (\{v\} \cap C) \neq (N_{G\circ H}[b] \cap C) \cup (\{v\} \cap C), \text{ i.e.,}$$

$N_{G\circ H}[a] \cap C \neq N_{G\circ H}[b] \cap C$.

Suppose $a = v$ and $b \in V(H^v)$. If $N_G(v) \cap C \neq \emptyset$, say $z \in N_G(v) \cap C$, then $z \in [N_{G\circ H}[a] \cap C] \setminus [N_{G\circ H}[b] \cap C]$. Thus, $N_{G\circ H}[a] \cap C \neq N_{G\circ H}[b] \cap C$. If $N_G(v) \cap C = \emptyset$, then $V(H^v) \cap C$ is strictly differentiating in H^v by (ii) and (iv). Hence, there exists $w \in V(H^v) \cap C$ such that $w \notin N_{G\circ H}[b] \cap C$. Since $w \in N_{G\circ H}[a] \cap C$, it follows that $N_{G\circ H}[a] \cap C \neq N_{G\circ H}[b] \cap C$.

Case 2: Suppose that $u \neq v$.

Since $V(H^u) \cap C$ and $V(H^v) \cap C$ are non-empty disjoint sets, and $V(H^u) \cap C \cap N_{G\circ H}[a] \cap C \neq \emptyset$ and $V(H^v) \cap C \cap N_{G\circ H}[b] \cap C \neq \emptyset$, it follows that $N_{G\circ H}[a] \cap C \neq N_{G\circ H}[b] \cap C$.

Accordingly, C is a differentiating dominating set in $G \circ H$. □

Corollary 4.15. *Let G (not necessarily point distinguishing) and let H (point distinguishing) be non-trivial connected graphs. Then*

$$|V(G)|\gamma_D(H) \leq \gamma_D(G \circ H) \leq |V(G)|\gamma_{SD}(H).$$

Proof. Let C be a minimum differentiating dominating set in G. Then

$$\gamma_D(G \circ H) = |C| = \sum_{v \in V(G)\cap C} (1 + |V(H^v) \cap C|) + \sum_{v \in V(G)\setminus C} |V(H^v) \cap C|.$$

From Theorem 4.14(i) and (ii), Remark 2.4, and Lemma 2.5, $1 + |V(H^v) \cap C| \geq 1 + dn(H) \geq \gamma_D(H)$ for every $v \in V(G) \cap C$. Now, if $v \in V(G) \setminus C$ and $N_G(v) \cap C \neq \varnothing$, then $|V(H^v) \cap C| \geq \gamma_D(H)$ by Theorem 4.14(iii). If $N_G(v) \cap C = \varnothing$, then $|V(H^v) \cap C| \geq \gamma_{SD}(H) \geq \gamma_D(H)$ by Theorem 4.14(iv) and Remark 2.4. Thus, $|V(H^v) \cap C| \geq \gamma_D(H)$ for every $v \in V(G) \setminus C$. Therefore $\gamma_D(G \circ H) = |C| \geq |V(G)|\gamma_D(H)$.

Next, let S be a minimum strictly differentiating-dominating set in H. For each $v \in V(G)$, pick $S_v \subseteq V(H^v)$, where $\langle S_v \rangle \cong \langle S \rangle$. Then $C = \cup_{v \in V(G)} S_v$ is a differentiating dominating set in $G \circ H$ by Theorem 4.14. Hence, $\gamma_D(G \circ H) \leq |C| = |V(G)|\gamma_{SD}(H)$.

Therefore, $|V(G)|\gamma_D(H) \leq \gamma_D(G \circ H) \leq |V(G)|\gamma_{SD}(H)$. □

5. Differentiating dominating sets in the lexicographic product of graphs

The *lexicographic product* $G[H]$ of two graphs G and H is the graph with $V(G[H]) = V(G)\times V(H)$ and $(u,u')(v,v') \in E(G[H])$ if and only if either $uv \in E(G)$ or $u = v$ and $u'v' \in E(H)$.

Observe that any subset C of $V(G)\times V(H)$ (in fact, any set of ordered-pairs) can be written as $C = \cup_{x\in S}(\{x\}\times T_x)$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$. Henceforth, we shall use this form to denote any subset C of $V(G)\times V(H)$.

Theorem 5.16. *Let G (not necessarily point distinguishing) and H be non-trivial connected graphs. Then $C = \cup_{x\in S}(\{x\}\times T_x)$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, is a differentiating-dominating set in $G[H]$ if and only if*

(i) *$S = V(G)$;*
(ii) *T_x is a differentiating set in H for every $x \in V(G)$;*
(iii) *T_x or T_y is strictly differentiating in H whenever x and y are adjacent vertices of G with $N_G[x] = N_G[y]$; and*
(iv) *T_x or T_y is (differentiating) dominating in H whenever x and y are distinct non-adjacent vertices of G with $N_G(x) = N_G(y)$.*

Proof. Suppose C is a differentiating dominating set in $G[H]$. Suppose there exists $x \in V(G)\setminus S$. Pick $a,b \in V(H)$, where $a \neq b$. Then $(x,a),(x,b) \notin C$ and $(x,a) \neq (x,b)$. Since $\{(x,c) : c \in V(H)\}\cap C = \emptyset$, it follows that $N_{G[H]}[(x,a)]\cap C = N_{G[H]}[(x,b)]\cap C$. This implies that C is not a differentiating set in $G[H]$, contrary to our assumption. Therefore, $S = V(G)$.

Now let $x \in V(G)$ and suppose that T_x is not differentiating in H. Then there exists distinct vertices p and q in $V(H)$ such that $N_H[p]\cap T_x = N_H[q]\cap T_x$. Let $D_x = N_H[p]\cap T_x$. Since $N_{G[H]}[(x,p)]\cap C = \cup\{\{y\}\times T_y : y \in N_G(x)\}\cup(\{x\}\times D_x) = N_{G[H]}[(x,q)]\cap C$, it follows that C is a not a differentiating set in $G[H]$. Again, this gives a contradiction. Therefore, T_x is a differentiating set in H.

To prove (iii), let x and y be adjacent vertices of G with $N_G[x] = N_G[y]$. Suppose that T_x and T_y are not strictly differentiating in H. Then there exist $c,d \in V(H)$ such that $N_H[c]\cap T_x = T_x$ and $N_H[d]\cap T_y = T_y$. It follows that $(\{x\}\times T_x)\cup(\{y\}\times T_y) \subseteq N_{G[H]}[(x,c)]\cap N_{G[H]}[(y,d)]$. Since $N_G[x] = N_G[y]$, it follows that $N_{G[H]}[(x,c)]\cap C = N_{G[H]}[(y,d)]\cap C$, i.e., C is not a differentiating set in $G[H]$. This contradicts our assumption. Therefore, T_x or T_y is strictly differentiating in H.

To prove (iv), let x and y be distinct non-adjacent vertices of G with $N_G(x) = N_G(y)$. Suppose that T_x is not a dominating set in H. Then there exists $a \in V(H)\setminus T_x$ such that $ab \notin E(H)$ for all $b \in T_x$. It follows that $(x,a) \notin C$ and $N_{G[H]}[(x,a)]\cap C = N_{G[H]}((x,a))\cap C = \cup\{\{z\}\times T_z : z \in N_G(x)\}$. Let $c \in V(H)\setminus T_y$. Then $(y,c) \notin C$. Since $N_G(x) = N_G(y)$, it follows that

$\cup\{\{z\}\times T_z : z\in N_G(x)\}\subseteq N_{G[H]}((y,c))\cap C = N_{G[H]}[(y,c)]\cap C$. Since C is a differentiating set in $G[H]$, there exists $(y,d)\in\{y\}\times T_y$ such that $(y,d)(y,c)\in E(G[H])$. This implies that $d\in T_y$ and $cd\in E(H)$. Therefore, T_y is a dominating set in H. This shows that (iv) holds.

For the converse, suppose that conditions (i), (ii), (iii), and (iv) hold. By (i) and the fact that G is connected, it follows that C is a dominating set in $G[H]$. Now let $(x,a),(y,b)\in V(G[H])$ with $(x,a)\neq(y,b)$. Consider the following cases:

Case 1. Suppose $x=y$.

Then $a\neq b$. Since T_x is a differentiating set in H, $N_H[a]\cap T_x = A\neq B = N_H[b]\cap T_y$. Now, since $(\{x\}\times T_x)\cap(N_{G[H]}[(x,a)]\cap C) = \{x\}\times A$ and $(\{y\}\times T_y)\cap(N_{G[H]}[(y,b)]\cap C) = \{y\}\times B$, it follows that $N_{G[H]}[(x,a)]\cap C\neq N_{G[H]}[(y,b)]\cap C$.

Case 2. Suppose $x\neq y$.

Consider the following sub-cases:

Sub-case 1. Suppose $xy\notin E(G)$.

Suppose first that $N_G(x)\neq N_G(y)$, say $z\in N_G(x)\backslash N_G(y)$. Pick $d\in T_z$. Then $(z,d)\in C$ and $(z,d)\in(N_{G[H]}[(x,a)]\cap C)\backslash(N_{G[H]}[(y,b)]\cap C)$. Next, suppose that $N_G(x)=N_G(y)$. By (iv), we may assume that T_x is a dominating set in H. If $(x,a)\in C$, then $(x,a)\in(N_{G[H]}[(x,a)]\cap C)\backslash(N_{G[H]}[(y,b)]\cap C)$. If $(x,a)\notin C$, then $a\notin T_x$. Hence, there exists $c\in T_x$ such that $ac\in E(H)$. This implies that $(x,c)\in C$ and $(x,c)\in(N_{G[H]}[(x,a)]\cap C)\backslash(N_{G[H]}[(y,b)]\cap C)$.

Sub-case 2. Suppose $xy\in E(G)$.

If $N_G[x]\neq N_G[y]$, then $N_G(x)\neq N_G(y)$; hence, as in a previous case, $N_{G[H]}[(x,a)]\cap C\neq N_{G[H]}[(y,b)]\cap C$. If $N_G[x]=N_G[y]$, then, by (iii), it can be assumed that T_x is strictly differentiating in H. Hence $N_H[a]\cap T_x\neq T_x$. This implies that there exists $q\in T_x$ such that $q\notin N_H[a]$. It follows that $(x,q)\in C$ and $(x,q)\in(N_{G[H]}[(y,b)]\cap C)\backslash(N_{G[H]}[(x,a)]\cap C)$.

Accordingly, C is a differentiating-dominating set in $G[H]$. □

The following is a direct consequence of Theorem 5.16.

Corollary 5.17. *Let G be a non-trivial connected totally point determining graph and H a non-trivial connected point distinguishing graph with $\Delta(H)\leq|V(H)|-2$. Then $C=\cup_{x\in S}(\{x\}\times T_x)$, where $S\subseteq V(G)$ and $T_x\subseteq V(H)$ for each $x\in S$, is a minimum differentiating-dominating set in $G[H]$ if and only if $S=V(G)$ and each T_x is a minimum differentiating set in H.*

The next result is immediate from Corollary 5.17.

Corollary 5.18. *Let G be a non-trivial connected totally point determining graph and H a non-trivial connected point distinguishing graph with $\Delta(H)\leq|V(H)|-2$. Then $\gamma_D(G[H]) = |V(G)|\gamma_D(H)$.*

Proof. Let $C = \cup_{x\in S}(\{x\} \times T_x)$ be a minimum differentiating dominating set in $G[H]$. Then $S = V(G)$ and T_x is a minimum differentiating set in H for every $x \in V(G)$, by Corollary 5.17. It follows that $\gamma_D(G[H]) = |C| = |V(G)|dn(H)$. □

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