TAMKANG JOURNAL OF MATHEMATICS
Volume 37, Number 2, 123-130, Summer 2006

ON STRONG APPROXIMATION BY MODIFIED MEYER-KÖNIG AND ZELLER OPERATORS

L. REMPULSKA AND M. SKORUPKA

Abstract. We introduce certain modified Meyer-König and Zeller operators $M_{n;r}$ in the space of r-th times differentiable functions f and we study strong differences $H^q_{n;r}(f)$ for them.

This note is motivated by results on strong approximation connected with Fourier series ([7]).

1. Introduction

1.1. The Meyer-König and Zeller operators M_n, $n \in N = \{1, 2, \dots, \}$,

$$M_n(f;x) := \begin{cases} \sum_{k=0}^{\infty} p_{nk}(x) f(\frac{k}{n+k}) & \text{if } 0 \le x < 1, \\ f(1) & \text{if } x = 1, \end{cases} \tag{1}$$

$$p_{nk}(x) := \binom{n+k}{k} x^k (1-x)^{n+1}, \quad k \in N_0 = N \cup \{0\}, \tag{2}$$

associated with bounded functions $f : I = [0, 1] \to R$, were introduced in [9].

Approximation properties of M_n were given in many papers (e.g. [1, 3, 5, 8, 9].

In many papers (e.g. [2, 4, 10]) were introduced certain modifications of operators M_n and were studied their properties in various functions spaces.

Let C_I be the space of all functions $f : I \to R$, continuous on I with the norm

$$\|f\| = \sup\{|f(x)| : x \in I\}. \tag{3}$$

It is known ([1, 3, 8]) that M_n, $n \in N$, is a positive linear operator from the space C_I into C_I and for every $f \in C_I$ we have

$$\|M_n(f)\| \le \|f\|, \quad n \in N, \tag{4}$$

and

$$\|M_n(f) - f\| \le \frac{31}{27}\, \omega\left(f; \frac{1}{\sqrt{n}}\right), \quad n \in N, \tag{5}$$

Received September 20, 2004; revised December 07, 2004.
2000 *Mathematics Subject Classification*. 41A36.
Key words and phrases. Meyer-König and Zeller operator, linear operator, strong approximation.

where $\omega(\cdot,\cdot)$ is the modulus of continuity of f ([11]), i.e.

$$\omega(f;t) = \sup\{|f(x)-f(y)| : x,y\in I,\ |x-y|\le t\},\quad t\in I. \tag{6}$$

From (5) immediatelly follows

$$\lim_{n\to\infty} \|M_n(f)-f\| = 0,$$

for every $f\in C_I$.

Let $r\in N_0$ be a fixed number and let $C_I^r := \{f\in C_I : f^{(r)}\in C_I\}$ and the norm in C_I^r is defined by (3) ($C_I^0\equiv C_I$).

It is known ([1]-[3]) that if $f\in C_I^r$, $r\ge 2$, then

$$\|M_n(f)-f\| = O\Big(\frac{1}{n}\Big),\quad n\in N, \tag{7}$$

and this result cannot be improved.

1.2. In this note we shall show that certain modification of the formula (1) improves the approximation order (7) for functions $f\in C_I^r$, $r\ge 2$.

We introduce the following.

Definition. Let $r\in N_0$ be a fixed number. For $f\in C_I^r$ and $n\in N$ we define the following modified Meyer-König and Zeller operators:

$$M_n(f;x) := \begin{cases} \sum\limits_{k=0}^{\infty} p_{nk}(x) \sum\limits_{j=0}^{r} \frac{f^{(j)}(\xi_{nk})}{j!}(x-\xi_{n,k})^j & \text{if } 0\le x<1,\\ f(1) & \text{if } x=1,\end{cases} \tag{8}$$

where

$$\xi_{nk} := \frac{k}{n+k},\quad n\in N,\quad k\in N_0, \tag{9}$$

and $p_{nk}(x)$ is defined by (2).

Obviously $M_{n;0}(f;x)\equiv M_n(f;x)$ for $f\in C_I^0$, $x\in I$ and $n\in N$.

From (1), (2), (8) and (9) it follows that

$$M_{n;r}(1;x) = \sum_{k=0}^{\infty} p_{nk}(x) = 1 \quad\text{for}\quad x\in I,\quad n\in N,\quad r\in N_0. \tag{10}$$

In Section 2 we shall prove that $M_{n;r}$ is a linear operator from the space C_I^r into C_I. The main approximation theorem will be given also in Section 2.

In this paper we shall denote by $K_i(a,b)$, $i\in N$, suitable positive constants depending only on indicated paramenters a,b.

1.3. Let $r \in N_0$ and $q > 0$ be fixed numbers. For $f \in C_I^r$ and $M_{n;r}(f)$ we introduce strong differences with the power q as follows:

$$H^q_{n;r}(f;x) := \begin{cases} \Big(\sum\limits_{k=0}^{\infty} p_{nk}(x) \Big| \sum\limits_{j=0}^{r} \frac{f^{(j)}(\xi_{nk})}{j!}(x-\xi_{n,k})^j - f(x)\Big|^q\Big)^{\frac{1}{q}} & \text{if } x \in [0,1), \\ 0 & \text{if } x = 1, \end{cases} \tag{11}$$

In particular for $f \in C_I$, $n \in N$ and $q > 0$ we have

$$H^q_{n;0}(f;x) := \begin{cases} \Big(\sum\limits_{k=0}^{\infty} p_{nk}(x) |f(\xi_{nk}) - f(x)|^q\Big)^{\frac{1}{q}} & \text{if } 0 \le x < 1, \\ 0 & \text{if } x = 1, \end{cases} \tag{12}$$

The properties of $H^q_{n;r}(f)$ will be given in Section 2.

2. Lemmas and Theorem

2.1. First we shall give auxiliary results.

Lemma 1. *For every $s \in N$ there exists $K_1(s)$ =const.> 0 such that*

$$M_{n;0}(|t-x|^s;x) \equiv M_n(|t-x|^s;x) \le K_1(s)n^{-\frac{s}{2}},$$

for all $x \in I$ and $n \in N$.

Proof. In [4] was given the following inequality

$$M_n((t-x)^{2s};x) \le K_2(s)n^{-s} \quad \text{for} \quad x \in I \quad \text{and} \quad n,s \in N,$$

where $K_2(s)$ is suitable positive constant dependent only on s.

Using the Hölder inequality to $M_n(|t-x|^s;x)$ and by (10) and the above result, we immediately obtain the desired inequality.

Now we shall prove analogue of the inequality (4).

Lemma 2. *Let $n, r \in N$ be fixed numbers. Then $M_{n;r}(f)$ is a linear operator from the space C_I^r into C_I and*

$$\|M_{n;r}(f)\| \le \sum_{j=0}^{r} \|f^{(j)}\|, \tag{13}$$

for every $f \in C_I^r$.

Proof. Let $f \in C_I^r$ with $r \in N$. By (2), (3) and (9) we have

$$\Big|\sum_{j=0}^{r} \frac{f^{(j)}(\xi_{nk})}{j!}(x-\xi_{nk})^j\Big| \le \sum_{j=0}^{r} \frac{\|f^{(j)}\|}{j!}|x-\xi_{nk}|^j \le \sum_{j=0}^{r} \frac{\|f^{(j)}\|}{j!},$$

for all $x \in I$, $k \in N_0$ and $n \in N$. From this and by (8) - (10) we deduce the continuity of $M_{n;r}(f)$ on interval [0,1) and

$$|M_{n;r}(f;x)| \leq \sum_{j=0}^{r} \|f^{(j)}\| \quad \text{for} \quad x \in [0,1), \quad n \in N. \tag{14}$$

Now we shall prove the continuity of $M_{n;r}(f)$ at $x = 1$.

If $f \in C_I^r$, $r \in N$, then the functions $h_{j,s}(x) = x^s f^{(j)}(x)$, $x \in I$, $j, s = 0, 1, \ldots, r$, belong to C_I and by properties of operators M_n given in Section 1 we have also $M_n(h_{j,s}) \in C_I$ and $\lim_{x \to 1-} M_n(h_{j,s}(t);x) = h_{j,s}(1)$, i.e.

$$\lim_{x \to 1-} M_n(t^s f^{(j)}(t);x) = f^{(j)}(1), \quad 0 \leq j, \quad s \leq r. \tag{15}$$

From the above and by (8) and (1) we get

$$M_{n;r}(f;x) = \sum_{j=0}^{r} \frac{1}{j!} \sum_{s=0}^{j} \binom{j}{s} (-1)^s x^{j-s} M_n(t^s f^{(j)}(t);x) \tag{16}$$

for $x \in [0,1)$, which by (15) and the equality

$$\sum_{s=0}^{j} \binom{j}{s} (-1)^s = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{if } j \in N, \end{cases}$$

implies that

$$\lim_{x \to 1-} M_n(f;x) = \sum_{j=0}^{r} \frac{f^{(j)}(1)}{j!} \sum_{s=0}^{j} \binom{j}{s} (-1)^s = f(1). \tag{17}$$

The formulas (8), (16) and (17) show that $M_{n;r}$ with $n, r \in N$ is a linear operator from the space C_I into C_I. Moreover from (8) and (14) immedialely follow (13).

Applying Lemma 1 and Lemma 2 we can prove the following.

Lemma 3. *Let $r \in N_0$ be a fixed number. Then $H_{n;r}^q(f;x)$ is well-defined for every $f \in C_I^r$, $x \in I$, $n \in N$ and $q > 0$. Moreover the formula* (11) *can be written in the form:*

$$H_{n;r}^q(f;x) := \begin{cases} \left(M_n\left(\Big| \sum_{j=0}^{r} \frac{f^{(j)}(t)}{j!}(x-t)^j - f(x) \Big|^q ; x \right) \right)^{\frac{1}{q}} & \text{if } x \in [0,1), \\ 0 & \text{if } x = 1, \end{cases} \tag{18}$$

By elementary calculations we obtain.

Lemma 4. *Suppose that $f \in C_I^r$ with a fixed $r \in N_0$. Then for $x \in I$ and $n \in N$ we have*

$$|M_{n;r}(f;x) - f(x)| \leq H_{n;r}^1(f;x) \tag{19}$$

and

$$H^p_{n;r}(f;x) \le H^q_{n;r}(f;x) \quad \textit{if} \;\; 0 < p < q < \infty. \tag{20}$$

Proof. The formulas (8)-(10) and (1) imply that

$$\begin{aligned}|M_{n;r}(f;x) - f(x)| &= \Big| M_n\Big(\sum_{j=0}^{r} \frac{f^{(j)}(t)}{j!}(x-t)^j - f(x);x\Big)\Big| \\ &\le M_n\Big(\Big|\sum_{j=0}^{r} \frac{f^{(j)}(t)}{j!}(x-t)^j - f(x)\Big|;x\Big)\end{aligned}$$

for $0 \le x < 1$ and $M_{n;r}(f;1) - f(1) = 0$, which by (18) yield (19).

Applying the Hölder inequality and (10), we get

$$(M_n(|g(t)|^p;x))^{\frac{1}{p}} \le (M_n(|g(t)|^q;x))^{\frac{1}{q}}, \quad x \in I, \quad n \in N, \tag{21}$$

for every $g \in C_I$ and $0 < p < q < \infty$. From (18) and (21) immediately follows (20).

2.2. Applying the above lemmas we shall prove the main theorem.

Theorem. *Let $r \in N_0$ and $q > 0$ be fixed numbers. Then there exists $K_5(q,r) =$ const.> 0 such that for every $f \in C^r_I$ and $n \in N$ we have*

$$\|H^q_{n;r}(f;\cdot)\| \le K_5(q,r) n^{-\frac{r}{2}} \; \omega \, \Big(f^{(r)}; \frac{1}{\sqrt{n}}\Big). \tag{22}$$

Proof. First let $r \in N$ and $q \in N$. Analogously to [6] we apply the following modified Taylor formula of $f \in C^r_I$ at a fixed point $x_0 \in I$:

$$\begin{aligned}f(x) = &\sum_{j=0}^{r} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j \\ &+ \frac{(x-x_0)^r}{(r-1)!}\int_0^1 (1-t)^{r-1}(f^{(r)}(x_0 + t(x-x_0)) - f^{(r)}(x_0))dt, \quad x \in I.\end{aligned} \tag{23}$$

Setting $x_0 = \frac{k}{n+k} \equiv \xi_{nk}$ and using (23) to (11), we can write

$$H^q_{n;r}(f;x) = \Big(\sum_{k=0}^{\infty} p_{nk}(x) \Big| \frac{(x-\xi_{nk})^r}{(r-1)!} I_{n,k,r}\Big|^q\Big)^{\frac{1}{q}} \quad \text{for} \;\; x \in [0,1),$$

where

$$I_{n,k,r} := \int_0^1 (1-t)^{r-1}(f^{(r)}(\xi_{nk} + t(x-\xi_{nk})) - f^{(r)}(\xi_{nk}))dt.$$

By (6) and properties of modulus of continuity ([11]) we have

$$\begin{aligned}&|f^{(r)}(\xi_{nk} + t(x-\xi_{nk})) - f^{(r)}(\xi_{nk})| \le \; \omega \; (f^{(r)}; t|x-\xi_{nk}|) \\ &\le \; \omega \; (f^{(r)}; |x-\xi_{nk}|) \le \; \omega \; \Big(f^{(r)}; \frac{1}{\sqrt{n}}\Big)(\sqrt{n}\; |x-\xi_{nk}| + 1),\end{aligned}$$

for $0 \le t \le 1$ and further

$$|I_{n,k,r}| \le \frac{1}{r}\ \omega\ \Big(f^{(r)}; \frac{1}{\sqrt{n}}\Big)(\sqrt{n}\ |x-\xi_{nk}|+1).$$

From the above and by the Minkowski inequality and (1) we get

$$\begin{aligned} H^q_{n;r}(f;x) &\le \frac{1}{r!}\ \omega\ \Big(f^{(r)}; \frac{1}{\sqrt{n}}\Big)\Big(\sum_{k=0}^{\infty} p_{nk}(x)(\sqrt{n}\ |x-\xi_{nk}|^{r+1}+|x-\xi_{nk}|^r)^q\Big)^{\frac{1}{q}} \\ &\le \frac{1}{r!}\ \omega\ \Big(f^{(r)}; \frac{1}{\sqrt{n}}\Big)\Big(\sqrt{n}\ (M_n(|t-x|^{(r+1)q};x))^{\frac{1}{q}}+(M_n(|t-x|^{qr};x))^{\frac{1}{q}}\Big), \end{aligned}$$

for $x \in [0,1)$, $n \in N$. Now applying Lemma 1, we easily obtain

$$H^q_{n;r}(f;x) \le K_6(q,r)n^{-\frac{r}{2}}\ \omega\ \Big(f^{(r)}; \frac{1}{\sqrt{n}}\Big),$$

for $x \in [0,1)$ and $\in N$. From this and (11) and (3) follows (22) for $q \in N$ and $r \in N$.

If $r \in N$ and $0 < q \notin N$, then $[q]+1$ belongs to N and $q < [q]+1$ ($[q]$ is the integral part of q). Applying (20) and (3), we get

$$\|H^q_{n;r}(f;\cdot)\| \le \|H^{[q]+1}_{n;r}(f;\cdot)\|, \quad n \in N,$$

which by (22) for $\|H^{[q]+1}_{n;r}(f;\cdot)\|$ implies (22) for $r \in N$ and $0 < q \notin N$.

b) If $r = 0$ and $f \in C_I$, then by (12) and (6) we have

$$H^q_{n;0}(f;x) := \begin{cases} (M_n(|f(t)-f(x)|^q;x))^{\frac{1}{q}} & \text{if } 0 \le x < 1, \\ 0 & \text{if } x = 1, \end{cases} \tag{24}$$

and

$$|f(t)-f(x)| \le \omega(f;|t-x|) \le \omega\Big(f;\frac{1}{\sqrt{n}}\Big)(\sqrt{n}\ |t-x|+1) \tag{25}$$

for $t, x \in I$. Arguing as in the case $r \in N$ and using (25) to (24), we obtain (22) for $r = 0$.

Thus the proof is completed.

2.3. Finaly we shall give some corollaries and remarks.

From imequalities (19) and (20) and by (3) we deduce that

$$\|M_{n;r}(f)-f\| \le \|H^1_{n;r}(f)\| \le \|H^q_{n;r}(f)\|, \quad n \in N, \tag{26}$$

for every $f \in C_I$, $r \in N_0$, and $q > 1$.

The inequality (26) shows that our theorem on the strong approximation (with the power $q \ge 1$) of $f \in C^r_I$ by $M_{n;r}(f)$ implies the classical approximation theorem for them.

From Theorem and (26) we derive the following two corollaries.

Corollary 1. *For every* $f \in C_I^r$, $r \in N_0$, *and* $q > 0$ *we have*

$$\lim_{n\to\infty} n^{\frac{r}{2}} \|H_{n;r}^q(f)\| = 0$$

and

$$\lim_{n\to\infty} n^{\frac{r}{2}} \|M_{n;r}(f) - f\| = 0. \tag{27}$$

Corollary 2. *Suppose that* $f \in C_I^r$, $r \in N_0$, *and* $f^{(r)} \in \mathrm{Lip}\ \alpha$ *with* $0 < \alpha \le 1$, *i.e.* $\omega(f^{(r)}; t) = O(t^\alpha)$, $t \in (0, 1]$. *Then*

$$\|H_{n;r}^q(f)\| = O\Big(n^{-\frac{r+\alpha}{2}}\Big), \quad n \in N,$$

for every fixed $q > 0$. *Consequently we have*

$$\|M_{n;r}(f) - f\| = O\Big(n^{-\frac{r+\alpha}{2}}\Big), \quad n \in N. \tag{28}$$

Remark. The given theorem (also the above corollaries) shows that the order of strong approximation of $f \in C_I^r$, $r \in N$, by $M_{n;r}(f)$ is better than for classical Meyer-König and Zeller operators $M_n(f)$. Moreover, Theorem shows that the order of strong approximation of $f \in C_I^r$ by $M_{n;r}(f)$ improves if r increases.

The identical properties we deduce from (27), (28) and (5) for ordinary approximation of $f \in C_I^r$ by $M_{n;r}$, $r \in N$.

References

[1] M. Becker and R. J. Nessel, *A global approximation theorem for Meyer-König and Zeller operators*, Math. Z. **160**(1978), 195-206.

[2] W. Chen, *On the integral type Meyer-König and Zeller operators*, Approx. Theory and its Applic. **2**(1986), 7-18.

[3] R. A. De Vore, The Approximation of Continuous Functions by Positive Linear Operators, Springer, new York, 1972.

[4] S. Guo, *On the rate of convergence of integrated Meyer-König and Zeller operators for functions of bounded variation*, J. Approx. Theory **56**(1989), 245-255.

[5] V. Gupta, *A note on Meyer-König and Zeller operators for functions of bounded variation*, Approx. Theory and its Applic. **18**(2002), 99-102.

[6] G. H. Kirov, *A generalization of the Bernstein polynomials*, Math. Balkanica **6**(1992), 147-153.

[7] L. Leindler, Strong Approximation by Fourier Series, Akad. Kiado, Budapest, 1985.

[8] A. Lupas and M. W. Müller, *Approximation properties of the M_n-operators*, Aequationes Math. **5**(1970), 19-37.

[9] W. Meyer-König and K. Zeller, *Bernsteiniche Potenzreihen*, Studia Math. **19**(1960), 89-94.

[10] M. W. Müller, *L_p-approximation by the method of integral Meyer-König and Zeller operators*, Studia Math. **63**(1978), 81-88.

[11] A. F. Timan, Theory of Approximation of Functions of Real Variable, New York, 1963.

Institute of Mathematics, Poznań University of Technology, ul. Piotrowo 3A, 60-965 Poznań, Poland.

E-mail: lrempuls@math.put.poznan.pl

Institute of Mathematics, Poznań University of Technology, ul. Piotrowo 3A, 60-965 Poznań, Poland.

E-mail: mariolas@math.put.poznan.pl