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COMPOSITE FUNCTIONS WITH ALLEN DETERMINANTS AND THEIR APPLICATIONS TO PRODUCTION MODELS IN ECONOMICS

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Abstract. In this paper, we derive an explicit formula for the Allen determinants of composite functions of the form:

$$f(\mathbf{x}) = F(h_1(x_1) \times \cdots \times h_n(x_n)).$$

We completely classify the composite functions by using their Allen determinants. Some applications of Allen determinants to production models are also given.

1. Introduction

In economics, a *production function* is a mathematical expression which denotes the physical relations between the output generated of a firm, an industry or an economy and inputs that have been used. Explicitly, a production function is a map which has non-vanishing first derivatives defined by

$$f : \mathbb{R}_+^n \longrightarrow \mathbb{R}_+, \ \ f = f(x_1, \ldots, x_n),$$

where f is the quantity of output, n are the number of inputs and $x_1, \ldots, x_n$ are the inputs. The production functions satisfy the following conditions:

1. f is equivalently zero in absence of an input.
2. $\frac{\partial f}{\partial x_i} > 0$, for $i = 1, \ldots, n$, which means that the production function is strictly increasing with respect to any factor of production.
3. $\frac{\partial^2 f}{\partial x_i^2} < 0$, for $i = 1, \ldots, n$, i.e., the production has decrasing efficiency with respect to any factor of production.
4. $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$, $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}_+^n$, which implies that the production has non-decreasing global efficiency. For more detailed properties of production functions, see [2, 13, 15, 17, 19, 21, 22].

A production function $f(x_1, \ldots, x_n)$ is said to be *homogeneous of degree p or p−homogenous* if

$$f(tx_1, \ldots, tx_n) = t^p f(x_1, \ldots, x_n) \tag{1.1}$$

holds for each $t \in \mathbb{R}_+$ for which (1.1) is defined. A homogeneous function of degree one is called *linearly homogeneous*. Many important properties of homogeneous production functions in economics were interpreted in terms of the geometry of their graphs by [4, 5, 11, 17, 18].

A. D. Vilcu and G. E. Vilcu [23] gave an exact classification for homogeneous production functions with proportional marginal rate of substitution and with constant elasticity of labor and capital.

The most common quantitative indices of production factor substitutability are forms of the elasticity of substitution. R.G.D. Allen and J.R. Hicks [1] suggested two generalizations of Hicks' original two variable elasticity concept.

The first concept, called Hicks elasticity of substitution, is defined as follows.

Let $f(x_1, \ldots, x_n)$ be a production function. Then *Hicks elasticity of substitution* of the i−th production variable with respect to the j−th production variable is given by

$$H_{ij}(\mathbf{x}) = -\frac{\dfrac{1}{x_i f_i} + \dfrac{1}{x_j f_j}}{\dfrac{f_{ii}}{(f_i)^2} - \dfrac{2 f_{ij}}{f_i f_j} + \dfrac{f_{jj}}{(f_j)^2}} \quad \left(\mathbf{x} \in \mathbb{R}_+^n, \ i, j = 1, \ldots, n, \ i \neq j\right),$$

where $f_i = \partial f / \partial x_i$, $f_{ij} = \partial^2 f / \partial x_i \partial x_j$.

L. Losonczi [16] classified homogeneous production functions of 2 variables, having constant Hicks elasticiy of substitution. Then, the classification of L. Losonczi was extended to n variables by B-Y. Chen [6].

The second concept, investigated by R.G.D. Allen and H. Uzawa [20], is the following:

Let f be a production function. Then *Allen elasticity of substitution* of the i−th production variable with respect to the j−th production variable is defined by

$$A_{ij}(\mathbf{x}) = -\frac{x_1 f_1 + x_2 f_2 + \cdots + x_n f_n}{x_i x_j} \frac{D_{ij}}{D} \quad \left(\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}_+^n, \ i, j = 1, \ldots, n, \ i \neq j\right),$$

where D is the determinant of the matrix

$$M(f) = \begin{pmatrix} 0 & f_1 & \cdots & f_n \\ f_1 & f_{11} & \cdots & f_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ f_n & f_{n1} & \cdots & f_{nn} \end{pmatrix} \tag{1.2}$$

and D_{ij} is the co-factor of the element f_{ij} in the determinant D ($D \neq 0$ is assumed). $M(f)$ is called the *Allen's matrix* and we call $\det(M(f))$ the *Allen determinant.*

It is a simple calculation to show that in case of two variables Hicks elasticity of substitution coincides with Allen elasticity of substitution.

In this paper, by analogy to the Hessian Determinant Formula given by B.-Y. Chen in [10] for composite functions of the form $f(\mathbf{x}) = F(h_1(x_1) + \cdots + h_n(x_n))$, we give an explicit formula for Allen determinant of the composite functions of the form

$$f(\mathbf{x}) = F(h_1(x_1) \times \cdots \times h_n(x_n)). \tag{1.3}$$

We classify the composite functions given by (1.3) using their Allen determinants. Some geometric applications of Allen determinants are also given.

2. Allen determinant formula and classification of composite functions

Throughout this article, we assume that $h_1, \ldots, h_n : \mathbb{R} \longrightarrow \mathbb{R}$ are thrice differentiable functions and $F : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is a twice differentiable function with $F'(u) \neq 0$ such that $I \subset \mathbb{R}$ is an interval of positive length.

The following provides an explicit formula for the Allen determinant of a composite function given by (1.3).

Allen Determinant Formula. The determinant of the Allen matrix $M(f)$ of the composite function $f = F(h_1(x_1) \times \cdots \times h_n(x_n))$ is given by

$$\det(M(f)) = -u^{n+1}(F')^{n+1} \sum_{j=1}^{n} \left(\frac{h'_1}{h_1}\right)' \cdots \left(\frac{h'_{j-1}}{h_{j-1}}\right)' \left(\frac{h'_j}{h_j}\right)^2 \left(\frac{h'_{j+1}}{h_{j+1}}\right)' \cdots \left(\frac{h'_n}{h_n}\right)', \tag{2.1}$$

where $h'_j = \frac{dh_j}{dx_j}$ and $F' = F'(u)$ for $u = h_1(x_1) \times \cdots \times h_n(x_n)$.

Proof. Let f be a twice differentiable composite function given by

$$f(\mathbf{x}) = F(h_1(x_1) \times \cdots \times h_n(x_n)), \tag{2.2}$$

where $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. It follows from (2.2) that

$$f_i = \frac{\partial f}{\partial x_i} = \frac{h'_i}{h_i} u F', \ f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{h'_i h'_j}{h_i h_j} u \left[F' + u F''\right], \ 1 \leq i \neq j \leq n, \tag{2.3}$$

and

$$f_{ii} = \frac{\partial^2 f}{\partial x_i^2} = \frac{h''_i}{h_i} u F' + \left(\frac{h'_i}{h_i}\right)^2 u^2 F''. \tag{2.4}$$

By using (2.3) and (2.4), the determinant of Allen matrix $M(f)$ of the composite function given by (2.2) is

$$\det\left(M\left(f\right)\right) = \begin{vmatrix} 0 & \frac{h_1'}{h_1}uF' & \frac{h_2'}{h_2}uF' & \ldots & \frac{h_n'}{h_n}uF' \\ \frac{h_1'}{h_1}uF' & \frac{h_1''}{h_1}uF'+\left(\frac{h_1'}{h_1}\right)^2u^2F'' & \frac{h_1'h_2'}{h_1h_2}u\left[F'+uF''\right] & \ldots & \frac{h_1'h_n'}{h_1h_n}u\left[F'+uF''\right] \\ \frac{h_2'}{h_2}uF' & \frac{h_1'h_2'}{h_1h_2}u\left[F'+uF''\right] & \frac{h_2''}{h_2}uF'+\left(\frac{h_2'}{h_2}\right)^2u^2F'' & \ldots & \frac{h_2'h_n'}{h_2h_n}u\left[F'+uF''\right] \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ \frac{h_n'}{h_n}uF' & \frac{h_1'h_n'}{h_1h_n}u\left[F'+uF''\right] & \frac{h_2'h_n'}{h_2h_n}u\left[F'+uF''\right] & \ldots & \frac{h_n''}{h_n}uF'+\left(\frac{h_n'}{h_n}\right)^2u^2F'' \end{vmatrix} \quad (2.5)$$

Now we apply Gauss elimination method for the determinant from the formula (2.5). We replace the second column by second column minus $\frac{h_1'}{h_1}\left(1+\frac{uF''}{F'}\right)$ times the first column; then we derive

$$\det\left(M\left(f\right)\right) = \begin{vmatrix} 0 & \frac{h_1'}{h_1}uF' & \frac{h_2'}{h_2}uF' & \ldots & \frac{h_n'}{h_n}uF' \\ \frac{h_1'}{h_1}uF' & \left(\frac{h_1'}{h_1}\right)'uF' & \frac{h_1'h_2'}{h_1h_2}u\left[F'+uF''\right] & \ldots & \frac{h_1'h_n'}{h_1h_n}u\left[F'+uF''\right] \\ \frac{h_2'}{h_2}uF' & 0 & \frac{h_2''}{h_2}uF'+\left(\frac{h_2'}{h_2}\right)^2u^2F'' & \ldots & \frac{h_2'h_n'}{h_2h_n}u\left[F'+uF''\right] \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ \frac{h_n'}{h_n}uF' & 0 & \frac{h_2'h_n'}{h_2h_n}u\left[F'+uF''\right] & \ldots & \frac{h_n''}{h_n}uF'+\left(\frac{h_n'}{h_n}\right)^2u^2F'' \end{vmatrix}.$$

By similar elementary transformations, we get

$$\det\left(M\left(f\right)\right) = \begin{vmatrix} 0 & \frac{h_1'}{h_1}uF' & \frac{h_2'}{h_2}uF' & \ldots & \frac{h_n'}{h_n}uF' \\ \frac{h_1'}{h_1}uF' & \left(\frac{h_1'}{h_1}\right)'uF' & 0 & \ldots & 0 \\ \frac{h_2'}{h_2}uF' & 0 & \left(\frac{h_2'}{h_2}\right)'uF' & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ \frac{h_n'}{h_n}uF' & 0 & 0 & \ldots & \left(\frac{h_n'}{h_n}\right)'uF' \end{vmatrix} \quad (2.6)$$

After calculating the determinant from the formula (2.6), we finally obtain

$$\det\left(M\left(f\right)\right) = -u^{n+1}\left(F'\right)^{n+1}\sum_{j=1}^{n}\left(\frac{h_1'}{h_1}\right)'\cdots\left(\frac{h_{j-1}'}{h_{j-1}}\right)'\left(\frac{h_j'}{h_j}\right)^2\left(\frac{h_{j+1}'}{h_{j+1}}\right)'\cdots\left(\frac{h_n'}{h_n}\right)',$$

which completes the proof.   □

Since it is assumed that $F' \neq 0$, the Allen Determinant Formula follows immediately

Corollary 2.1. *Let $f = F(h_1(x_1) \times \cdots \times h_n(x_n))$ be a twice differentiable composite function. Then the singularity of Allen's matrix of f only depends on the functions $h_1, \ldots, h_n$.*

Next result completely classifies the composite functions in the form

$$f = F(h_1(x_1) \times \cdots \times h_n(x_n))$$

whose Allen's matrices are singular.

Theorem 2.2. *Let $F(u)$ be a twice differentiable function with $F'(u) \neq 0$ and let f be a composite function given by*

$$f = F(h_1(x_1) \times \cdots \times h_n(x_n)),$$

where $h_1, \ldots, h_n$ are thrice differentiable and nonzero functions. Then the Allen matrix $M(f)$ of f is singular if and only if f is one of the following:

(i) $f = F\left(\gamma e^{\alpha_1 x_1 + \alpha_2 x_2} \times h_3(x_3) \times \cdots \times h_n(x_n)\right)$, *where $\gamma, \alpha_1, \alpha_2$ are nonzero constants;*

(ii) $f = F\left(\gamma \left(x_1 + \beta_1\right)^{\alpha_1} \times \cdots \times \left(x_n + \beta_n\right)^{\alpha_n}\right)$, *where γ, α_i are nonzero constants satisfying $\alpha_1 + \cdots + \alpha_n = 0$ and β_i some constants.*

Proof. Let $f = F(h_1(x_1) \times \cdots \times h_n(x_n))$ be a twice differentiable composite function. Let us assume that the Allen matrix $M(f)$ of f is singular. Then, by Allen Determinant formula, we get

$$0 = \det\left(M(f)\right) = -u^{n+1}\left(F'\right)^{n+1} \sum_{j=1}^{n} \left(\frac{h_1'}{h_1}\right)' \cdots \left(\frac{h_{j-1}'}{h_{j-1}}\right)' \left(\frac{h_j'}{h_j}\right)^2 \left(\frac{h_{j+1}'}{h_{j+1}}\right)' \cdots \left(\frac{h_n'}{h_n}\right)', \qquad (2.7)$$

where $u = h_1(x_1) \times \cdots \times h_n(x_n)$. Because of $F'(u) \neq 0$, the equation (2.7) takes the form:

$$\sum_{j=1}^{n} \left(\frac{h_1'}{h_1}\right)' \cdots \left(\frac{h_{j-1}'}{h_{j-1}}\right)' \left(\frac{h_j'}{h_j}\right)^2 \left(\frac{h_{j+1}'}{h_{j+1}}\right)' \cdots \left(\frac{h_n'}{h_n}\right)' = 0. \qquad (2.8)$$

Case (i): At least one of $\left(\frac{h_1'}{h_1}\right)', \ldots, \left(\frac{h_n'}{h_n}\right)'$ vanishes. Without loss of generality, we may assume that $\left(\frac{h_1'}{h_1}\right)' = 0$. Hence from (2.8) we get

$$\left(\frac{h_1'}{h_1}\right)^2 \left(\frac{h_2'}{h_2}\right)' \left(\frac{h_3'}{h_3}\right)' \cdots \left(\frac{h_n'}{h_n}\right)' = 0. \qquad (2.9)$$

Without loss of generality, we may assume from (2.9) that $\left(\frac{h_2'}{h_2}\right)' = 0$. Thus we have

$$h_1(x_1) = \gamma_1 e^{\alpha_1 x_1}, \quad h_2(x_2) = \gamma_2 e^{\alpha_2 x_2},$$

for some nonzero constants $\alpha_1, \alpha_2, \gamma_1, \gamma_2$. This gives case (i) of the theorem.

Case (ii): $\left(\frac{h_1'}{h_1}\right)', \ldots, \left(\frac{h_n'}{h_n}\right)'$ are nonzero. Then from (2.8), by dividing with the product $\left(\frac{h_1'}{h_1}\right)' \cdots \left(\frac{h_n'}{h_n}\right)'$, we write

$$\frac{\left(\frac{h_1'}{h_1}\right)^2}{\left(\frac{h_1'}{h_1}\right)'} + \cdots + \frac{\left(\frac{h_n'}{h_n}\right)^2}{\left(\frac{h_n'}{h_n}\right)'} = 0. \qquad (2.10)$$

Taking the partial derivative of (2.10) with respect to x_i, we obtain

$$2 - \left(\frac{h_i'}{h_i}\right) \frac{\left(\frac{h_i'}{h_i}\right)''}{\left[\left(\frac{h_i'}{h_i}\right)'\right]^2} = 0. \tag{2.11}$$

By solving (2.11), we find

$$h_i = \gamma_i \left(x_i + \beta_i\right)^{\alpha_i},$$

for some nonzero constants γ_i, α_i satisfying $\alpha_1 + \cdots + \alpha_n = 0$ and some constants β_i. Therefore, we complete first part of the proof.

The converse can be verified directly.

3. Some applications of Allen determinants to production models

In economics, goods that are completely substitutable with each other are called perfect substitutes. Mathematically, a production function is a perfect substitute if it is of the form:

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i x_i,$$

where $a_1, \ldots, a_n$ are nonzero constants [12].

In 1928, C. W. Cobb and P. H. Douglas introduces in [13] a famous two-factor production function

$$Y = bL^k C^{1-k},$$

where b presents the total factor productivity, Y the total production, L the labor input and C the capital input. This function is nowadays called *Cobb-Douglas production function*. In its generalized form the Cobb-Douglas production function may be expressed as

$$f(x_1, \ldots, x_n) = \gamma x_1^{\alpha_1} \ldots x_n^{\alpha_n},$$

where $\gamma, \alpha_1, \ldots, \alpha_n > 0$.

Corollary 3.1. *Let $h_1(x_1), \ldots, h_n(x_n)$ be thrice differentiable functions with $h_i(x_i) \neq \gamma_i e^{\alpha_i x_i}$ for $i = 1, \ldots, n$ and $F(u) = u$. Then the Allen matrix $M(f)$ of composite function $f(\mathbf{x}) = F(h_1(x_1) \times \cdots \times h_n(x_n))$ is singular if and only if, up to suitable translations of $x_1, \ldots, x_n$,*

$$f(\mathbf{x}) = \gamma x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

where γ_i, α_i are some nonzero constants satisfying $\alpha_1 + \cdots + \alpha_n = 0$.

Proof. Let $f = F(h_1(x_1) \times \cdots \times h_n(x_n))$ be a twice differentiable composite function. First, let us assume that the Allen matrix $M(f)$ is singular. Under hypothesis of the theorem, it follows from Theorem 2.2 that

$$h_i = \gamma_i x_i^{\alpha_i}, \; i = 1, \ldots, n$$

for some nonzero constants γ_i, α_i satisfying $\alpha_1 + \cdots + \alpha_n = 0$. Thus, in our case, it means that

$$f(\mathbf{x}) = \gamma x_1^{\alpha_1} \times \cdots \times x_n^{\alpha_n}.$$

Converse is easy to verify.

Remark 3.2. We have that for a generalized Cobb-douglas production function the values of $\alpha_1, \ldots, \alpha_n$ are positive constants. Hence, while Corollary 3.1 is correct in mathematical perspective, in reality such a generealized Cobb-Douglas production function does not exist.

Corollary 3.3. *Let $h_1(x_1), \ldots, h_n(x_n)$ be thrice differentiable functions and let f be a composite function given by*

$$f(\mathbf{x}) = \ln(h_1(x_1) \times \cdots \times h_n(x_n)). \tag{3.1}$$

Then the Allen matrix $M(f)$ is singular if f is a perfect substitute.

Proof. Let $f(\mathbf{x}) = \ln(h_1(x_1) \times \cdots \times h_n(x_n))$ be a twice differentiable composite function. If f is a perfect substitute, then we write

$$f(\mathbf{x}) = \ln(h_1(x_1) \times \cdots \times h_n(x_n)) = \sum_{i=1}^{n} a_i x_i, \tag{3.2}$$

and it follows from (3.2) that

$$h_i(x_i) = e^{a_i x_i}, \; i = 1, \ldots, n.$$

It means from statement (i) of Theorem 2.2 that the Allen matrix $M(f)$ is singular.

Remark 3.4. The converse of Corollary 3.3 is not always true.

4. Some geometric interpretations of Allen determinants

Let (N, g) be a Riemannian manifold. For more detailed properties of geometric structures on Riemannian manifolds, see [3, 14]. A *Riemannian connection*, also called Levi-Civita connection, on the Riemannian manifold (N, g) is an affine connection which is compatible with metric, i.e, $\nabla g = 0$ and symmetric, i.e, $\nabla_X Y - \nabla_Y X = [X, Y]$ for any vector fields X and Y on N, where $[,]$ is the Lie bracket.

The *Riemannian curvature tensor R* is given in terms of ∇ by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

A Riemannian manifold is called a *flat space* if its Riemann curvature tensor vanishes identically.

Let σ be a two dimensional subspace of the tangent space T_pN and let $u, v \in \sigma$ be two linearly independent vectors such that $\sigma = Sp(u, v)$. Then the *sectional curvature* of σ at the point $p \in N$ is a real number defined by

$$K(u, v) = K(\sigma) = \frac{g(R(u, v)v, u)}{g(u, u)g(v, v) - g(u, v)^2}.$$

The *Ricci tensor* of a Riemannian manifold N at a point $p \in N$ is defined to be the trace of the linear map $T_pN \longrightarrow T_pN$ given by

$$w \longmapsto R(w, u)v.$$

A Riemannian manifold is called *Ricci-flat* if its Ricci tensor vanishes identically.

One of the B.-Y. Chen's geometric interpretations regarding $h-$homogeneous production functions from [4, 5] is as follows

Theorem 4.1. *An $h-$homogeneous production function f has flat production hypersurface if and only if either f is a perfect substitute or it is of the form: $f = (c_1 x_1 + \cdots + c_n x_n)^h$, for some constants $c_1, \ldots, c_n$.*

Corollary 4.2. *Let $h_1(x_1), \ldots, h_n(x_n)$, $n \geq 3$, be thrice differentiable functions and let f be an $h-$homogeneous composite function given by*

$$f(\mathbf{x}) = \ln(h_1(x_1) \times \cdots \times h_n(x_n)).$$

If the graph of f is a flat space, then the Allen matrix $M(f)$ is singular.

Proof. Let $f(\mathbf{x}) = \ln(h_1(x_1) \times \cdots \times h_n(x_n))$ be an twice differentiable $h-$homogeneous composite function. Let us assume that the graph of f is a flat space. Then, from Theorem 4.1, we have that f is a perfect substitute. This means from Corollary 3.3 that the Allen matrix $M(f)$ is singular.

Remark 4.3. Corollary 4.2 holds also in case the graph of f has vanishing sectional curvature function.

Since Ricci-flat 3-manifolds are always flat, Corollary 4.2 follows immediately

Corollary 4.4. *Let $h_1(x_1), h_2(x_2), h_3(x_3)$ be thrice differentiable functions and let f be an $h-$homogeneous composite function given by*

$$f(\mathbf{x}) = \ln(h_1(x_1) \times h_2(x_2) \times h_3(x_3)).$$

If the graph of f is Ricci-flat, then the Allen matrix $M(f)$ is singular.

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