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A STRENGTHENED SCHWARZ-PICK INEQUALITY FOR DERIVATIVES OF THE HYPERBOLIC METRIC*

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Abstract. This paper is to investigate the Schwarz-Pick inequality for the hyperbolic derivative. Our result is not only a contraction but also a contraction minus a positive constant and this improves Beardon's theorem greatly.

1. Introduction

In the open unit disk $D \subset C$ (where C is the complex plane), the hyperbolic metric ρ is defined by

$$\rho(z,w) = \frac{1}{2}\log\frac{1+|\varphi_w(z)|}{1-|\varphi_w(z)|},$$

where $\tau \in D$, $\varphi_\tau \in Aut(D)$, $Aut(D)$ denotes the automorphism on D given by

$$\varphi_\tau(\lambda) = \frac{\tau-\lambda}{1-\overline{\tau}\lambda}.$$

The *Schwarz-Pick Lemma* says that any analytic function $f : D \to D$ is nonincreasing under ρ, equivalently

$$\rho(f(z), f(w)) \leq \rho(z,w), \quad \forall\ z, w \in D.$$

Mercer[4],[5] proved *a strengthened Schwarz-Pick inequality*:

Lemma 1. *Let $f : D \to D$ be analytic and $\tau \in D$. Then*

$$\rho(f(z), f(w)) \leq \rho(z,w) - B, \qquad (B \geq 0) \tag{1}$$

i.e.

$$\rho(f(z), f(w)) \leq \rho(z,w) + \frac{1}{2}\log\Big[1-(1-A)\frac{2|\varphi_z(w)|}{(1+|\varphi_z(w)|)^2}\Big], \quad \forall\ z, w \in D$$

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where

$$B = -\frac{1}{2}\log\Big[1-(1-A)\frac{2|\varphi_z(w)|}{(1+|\varphi_z(w)|)^2}\Big] \tag{2}$$

and

$$A = \begin{cases} \dfrac{\alpha+|\varphi_\tau(w)|}{1+\alpha|\varphi_\tau(w)|} & \text{if } \rho(z,\tau)\le\rho(z,w), \\[2ex] \dfrac{\alpha+|\varphi_\tau(z)|}{1-\alpha|\varphi_\tau(z)|} & \text{if } \rho(\tau,w)\le\rho(z,w), \\[2ex] \dfrac{(|\varphi_w(z)|-\alpha)(u^2+1)+2u(\alpha|\varphi_w(z)|-1)}{2u(\varphi_w(z)-\alpha)+(\alpha\varphi_w(z)-1)(u^2+1)} & \text{otherwise } (u=\max\{|\varphi_\tau(z)|,|\varphi_\tau(w)|\}). \end{cases} \tag{3}$$

Note that $A=\varphi_\alpha(-2u/(u^2+1))$ in the last line and hence $0\le A\le 1$ in all cases, where $a=\varphi_\alpha(|\varphi_w(z)|)$.

Dieudone[3] proved the following *result*:

Lemma 2. *If $f: D\to D$ is analytic and $f(0)=0$, then*

$$|f'(z)|\le \begin{cases} 1 & \text{if } |z|\le\sqrt{2}-1, \\[1ex] \dfrac{(1+|z|^2)^2}{4|z|(1-|z|^2)} & \text{if } |z|>\sqrt{2}-1; \end{cases}$$

Later he pointd out that the above inequality, i.e., the so-called Schwarz Lemma for the derivative of f is best possible for each value of $z\in D$.

The dick D is endowed with the hyperbolic metric $ds^* = 2dz/(1-|z|^2)$, and the hyperbolic derivative $f^*(z)$ of f at z is given by

$$f^*(z) = \Big(\frac{1-|z|^2}{1-|f(z)|^2}\Big)f'(z).$$

Beardon[1],[2] obtained a Schwarz-Pick Lemma for derivatives, i.e.

Lemma 3. *If $f: D\to D$ is analytic but not a conformal automorphism of D with $f(0)=0$, then*

$$\rho(f^*(0), f^*(z)) \le 2\rho(0,z). \tag{4}$$

Furthermore, "=" holds for each z if $f(z)=z^2$.

Below we use *Mercer*'s result to improve *Beardon*'s.

2. Main Result and Its Proof

Let $w = 0$, then from (2) and (3), we get

$$B_0 = -\frac{1}{2}\log\Big[1-(1-A_0)\frac{2|z|}{(1+|z|)^2}\Big], \tag{5}$$

where

$$A_0 = \begin{cases} \dfrac{\alpha+|\tau|}{1+\alpha|\tau|} & \text{if } \rho(z,\tau)\le\rho(z,0), \\ \dfrac{\alpha+|\varphi_\tau(z)|}{1+\alpha|\varphi_\tau(z)|} & \text{if } \rho(\tau,0)\le\rho(z,0), \\ \dfrac{(|z|-\alpha)(u^2+1)+2u(\alpha|z|-1)}{2u(|z|-\alpha)+(\alpha|z|-1)(u^2+1)} & \text{otherwise } \ u=\max\{|\varphi_\tau(z)|,|\tau|\}. \end{cases} \tag{6}$$

We have as a consequence:

Theorem. *If $f : D \to D$ is analytic but not a conformal automorphism of D with $f(0) = 0$, $z \in D$, then*

$$\rho(f^*(0), f^*(z)) \le 2\rho(0,z) - 2B_0, \tag{7}$$

where B_0 and A_0 are defined as in (5) *and* (6), *and $f^*(z) = \dfrac{1-|z|^2}{1-|f(z)|^2}\cdot f'(z)$ is the hyperbolic derivative. Furthermore, "=" holds for each z if $f(z) = z^2$.*

We begin with a preliminary *Lemma*:

Lemma 4. *Let $z_0, w_0 \in D$ and $|w_0| < |z_0|$. If $f : D \to D$ is analytic with $f(0)=0$, $f(z_0) = w_0$, then both $f^*(0)$ and $f^*(z_0)$ lie in the closed hypertolic disc $D=\{z|\rho(z, w_0/z_0) \le \rho(0, z_0) - B_0\}$.*

Proof. As in [1], we are given z_0 and w_0 in D, so we define maps $h : D \to D$ and $g : D \to D$ by

$$h = \frac{f(z)}{z}, \quad \frac{f(z)-f(z_0)}{1-f(z)\cdot\overline{f(\overline{z}_0)}} = g(z)\Big(\frac{z-z_0}{1-\overline{z}_0 z}\Big),$$

Then

$$h(0) = f'(0) = f^*(0), \quad h(z_0) = \frac{w_0}{z_0}, \quad g(0) = \frac{w_0}{z_0}, \quad g(z_0) = f^*(z_0).$$

Using (1) and (5), then we get

$$\begin{aligned} \rho(f^*(0), w_0/z_0) &= \rho(h(0), h(z_0)) \le \rho(0,z_0) - B_0 \\ \rho(f^*(z_0), w_0/z_0) &= \rho(g(0), g(z_0)) \le \rho(0,z_0) - B_0 \end{aligned} \qquad (B_0 \ge 0) \tag{8}$$

This completes the proof of Lemma 4.

Proof of Theorem.

From the Lemma, we have

$$\begin{aligned}\rho(f^*(0), f^*(z_0)) &\le \rho(f^*(0), \frac{w_0}{z_0}) + \rho(f^*(z_0), \frac{w_0}{z_0}) \le \rho(0, z_0) - B_0 + \rho(0, z_0) - B_0 \\ &= 2\rho(0, z_0) - 2B_0,\end{aligned}$$

where B_0 and A_0 are given by (5) and (6) respectively. If $f(z) = z^2$, then $f^*(0) = 0$, $f^*(z) = \dfrac{2z}{1+|z|^2}$, and $B_0 = 0$. Therefore $\rho(0, z) = \log \dfrac{1+|z|}{1-|z|}$ and this completes the proof.

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