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ON CORRECTED BULLEN-SIMPSON'S 3/8 INEQUALITY

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Abstract. The aim of this paper is to derive corrected Bullen-Simpson's 3/8 inequality, starting from corrected Simpson's 3/8 and corrected Maclaurin's formula. By corrected we mean formulae that approximate the integral not only with the values of the function in certain points but also with the value of the first derivative in end points of the interval. These formulae will have a higher degree of exactness than formulae derived in [3].

1. Introduction

For any convex function $f : [0,1] \to \mathbf{R}$, the following pair of inequalities, usually referred in literature as Hadamard's inequalities, holds:

$$f\Big(\frac{1}{2}\Big) \le \int_0^1 f(t)dt \le \frac{f(0)+f(1)}{2}. \tag{1.1}$$

If f is concave, inequalities are reversed.

In [1], it was shown by a simple geometric argument that for a convex function f the following inequality is valid:

$$0 \le \int_0^1 f(t)dt - f\Big(\frac{1}{2}\Big) \le \frac{f(0)+f(1)}{2} - \int_0^1 f(t)dt. \tag{1.2}$$

An elementary analytic proof of (1.1) and (1.2), but stated on the interval $[-1,1]$, was given in [2]. Another interesting result of a similar type was given in that same paper. Namely, provided f is 4-convex, we have:

$$\begin{aligned} 0 &\le \int_0^1 f(t)dt - \frac{1}{8}\Big[3f\Big(\frac{1}{6}\Big) + 2f\Big(\frac{1}{2}\Big) + 3f\Big(\frac{5}{6}\Big)\Big] \\ &\le \frac{1}{8}\Big[f(0) + 3f\Big(\frac{1}{3}\Big) + 3f\Big(\frac{2}{3}\Big) + f(1)\Big] - \int_0^1 f(t)dt. \end{aligned} \tag{1.3}$$

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This implies that Maclaurin's quadrature rule is more accurate than Simpson's 3/8 quadrature rule. This inequality is sometimes called Bullen-Simpson's 3/8 inequality and was generalized for a class of $(2k)$-convex functions in [3].

The aim of this paper is to derive similar type inequalities, only this time starting from corrected Simpson's 3/8 and corrected Maclaurin's formula. By corrected we mean formulae that approximate the integral not only with the values of the function in certain points but also with the value of the first derivative in end points of the interval. These formulae will have a higher degree of exactness than formulae derived in [3].

Using identities named the extended Euler formulae (see [4]), corrected Euler- Simpson's 3/8 formulae were derived in [5].

Theorem 1. *Let* $f : [0,1] \to \mathbf{R}$ *be such that* $f^{(n-1)}$ *is a continuous function of bounded variation on* $[0,1]$*, for some* $n \geq 1$*. Then*

$$\int_0^1 f(t)dt = \frac{1}{80}\Big[13f(0) + 27f\Big(\frac{1}{3}\Big) + 27f\Big(\frac{2}{3}\Big) + 13f(1)\Big] - T_n^S(f) + \sigma_n^1(f), \tag{1.4}$$

and

$$\int_0^1 f(t)dt = \frac{1}{80}\Big[13f(0) + 27f\Big(\frac{1}{3}\Big) + 27f\Big(\frac{2}{3}\Big) + 13f(1)\Big] - T_{n-1}^S(f) + \sigma_n^2(f), \tag{1.5}$$

where $T_0^S(f) = T_1^S(f) = 0$*,*

$$T_2^S(f) = T_3^S(f) = T_4^S(f) = T_5^S(f) = \frac{1}{120}[f'(1) - f'(0)]$$

and, for $m \geq 6$

$$T_m^S(f) = \frac{1}{120}[f'(1) - f'(0)] + \frac{1}{80}\sum_{k=3}^{[m/2]} \frac{B_{2k}}{(2k)!}(3^{4-2k} - 1)\Big[f^{(2k-1)}(1) - f^{(2k-1)}(0)\Big], \tag{1.6}$$

where $[m/2]$ *is the greatest integer less than or equal to* $m/2$*. Further*

$$\sigma_n^1(f) = \frac{1}{80n!}\int_0^1 G_n^S(t)df^{(n-1)}(t),$$

and

$$\sigma_n^2(f) = \frac{1}{80n!}\int_0^1 F_n^S(t)df^{(n-1)}(t),$$

where, for $t \in \mathbf{R}$*,*

$$G_n^S(t) = 27B_n^*\Big(\frac{1}{3} - t\Big) + 27B_n^*\Big(\frac{2}{3} - t\Big) + 26B_n^*(1-t), \quad n \geq 1 \tag{1.7}$$

$$F_1^S(t) = G_1^S(t), \quad F_n^S(t) = G_n^S(t) - G_n^S(0), \quad n \geq 2.$$

Applying the same idea, corrected Euler-Maclaurin's formulae were derived. This was done in [6].

Theorem 2. *Let $f : [0,1] \to \mathbf{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$, for some $n \geq 1$. Then*

$$\int_0^1 f(t)dt = \frac{1}{80}\Big[27f\Big(\frac{1}{6}\Big) + 26f\Big(\frac{1}{2}\Big) + 27f\Big(\frac{5}{6}\Big)\Big] - T_n^D(f) + \tau_n^1(f), \tag{1.8}$$

and

$$\int_0^1 f(t)dt = \frac{1}{80}\Big[27f\Big(\frac{1}{6}\Big) + 26f\Big(\frac{1}{2}\Big) + 27f\Big(\frac{5}{6}\Big)\Big] - T_{n-1}^D(f) + \tau_n^2(f), \tag{1.9}$$

where $T_0^D(f) = T_1^D(f) = 0$,

$$T_2^D(f) = T_3^D(f) = T_4^D(f) = T_5^D(f) = -\frac{1}{240}[f'(1) - f'(0)]$$

and for $m \geq 6$

$$\begin{aligned} T_m^D(f) = &-\frac{1}{240}[f'(1) - f'(0)] \\ &+\frac{1}{80}\sum_{k=3}^{[m/2]} \frac{B_{2k}}{(2k)!}(1-2^{1-2k})(1-3^{4-2k})\Big[f^{(2k-1)}(1) - f^{(2k-1)}(0)\Big]. \end{aligned} \tag{1.10}$$

Further,

$$\tau_n^1(f) = \frac{1}{80n!}\int_0^1 G_n^D(t)df^{(n-1)}(t),$$

and

$$\tau_n^2(f) = \frac{1}{80n!}\int_0^1 F_n^D(t)df^{(n-1)}(t)$$

where for $t \in \mathbf{R}$ *and* $n \geq 1$,

$$\begin{aligned} G_n^D(t) &= 27B_n^*\Big(\frac{1}{6} - t\Big) + 26B_n^*\Big(\frac{1}{2} - t\Big) + 27B_n^*\Big(\frac{5}{6} - t\Big), \\ F_n^D(t) &= G_n^D(t) - G_n^D(0). \end{aligned} \tag{1.11}$$

Here, as in the rest of the paper, we write $\int_0^1 g(t)d\varphi(t)$ to denote the Riemann-Stieltjes integral with respect to a function $\varphi : [0,1] \to \mathbf{R}$ of bounded variation, and $\int_0^1 g(t)dt$ for the Riemann integral.

Finally, we'll say a few words about Bernoulli polynomials since they play an improtant role in this paper. Bernoulli polynomials $B_k(t)$, $k \geq 0$, are uniquely determined by the following identities

$$B_k'(t) = kB_{k-1}(t), \quad k \geq 1; \quad B_0(t) = 1 \tag{1.12}$$

and

$$B_k(t+1) - B_k(t) = kt^{k-1}, \quad k \geq 0. \tag{1.13}$$

We have

$$\begin{aligned} &B_0(t) = 1, \quad B_1(t) = t - \frac{1}{2}, \quad B_2(t) = t^2 - t + \frac{1}{6}, \quad B_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t \\ &B_4(t) = t^4 - 2t^3 + t^2 - \frac{1}{30}, \quad B_5(t) = t^5 - \frac{5}{2}t^4 + \frac{5}{3}t^3 - \frac{1}{6}t. \end{aligned} \tag{1.14}$$

$B_k^*(t)$, $k \geq 0$, are periodic functions of period 1 such that

$$B_k^*(t) = B_k(t), \quad 0 \leq t < 1.$$

$B_k = B_k(0)$ are Bernoulli numbers. From (1.13) it follows that

$$B_k(1) = B_k(0) = B_k, \quad k \geq 2.$$

For further details on Bernoulli polynomials and Bernoulli numbers see [7] or [8].

2. Corrected Bullen-Simpson's 3/8 Formulae of Euler Type

For $k \geq 1$ and $t \in \mathbf{R}$, we define functions

$$G_k(t) = G_k^S(t) + G_k^D(t), \quad F_k(t) = F_k^S(t) + F_k^D(t),$$

where $G_k^S(t)$, $G_k^D(t)$, $F_k^S(t)$ and $F_k^D(t)$ are defined as in Introduction. So,

$$\begin{aligned} G_k(t) &= 27B_k^*\Big(\frac{1}{6} - t\Big) + 27B_k^*\Big(\frac{1}{3} - t\Big) + 26B_k^*\Big(\frac{1}{2} - t\Big) \\ &\quad + 27B_k^*\Big(\frac{2}{3} - t\Big) + 27B_k^*\Big(\frac{5}{6} - t\Big) + 26B_k^*(1 - t), \quad k \geq 1, \\ F_1(t) &= G_1(t), \quad F_k(t) = G_k(t) - G_k(0), \quad k \geq 2. \end{aligned}$$

Introduce notation $\tilde{B}_k = G_k(0)$. By direct calculation we get

$$\tilde{B}_2 = 2/3 \quad \text{and} \quad \tilde{B}_3 = \tilde{B}_4 = \tilde{B}_5 = 0.$$

Using the properties of Bernoulli polynomials, it is easy to check that $\tilde{B}_{2k-1} = 0$, $k \geq 2$, no matter which symmetrical linear combination they were obtained by. The reason these specific coefficients were chosen is because they give $\tilde{B}_4 = 0$ and that is an interesting case to study.

Now, let $f : [0, 1] \to \mathbf{R}$ be such that $f^{(n-1)}$ exists on [0, 1] for some $n \geq 1$. Introduce the following notation

$$D(0,1) = \frac{1}{160}\Big[13f(0) + 27f\Big(\frac{1}{6}\Big) + 27f\Big(\frac{1}{3}\Big) + 26f\Big(\frac{1}{2}\Big) + 27f\Big(\frac{2}{3}\Big) + 27f\Big(\frac{5}{6}\Big) + 13f(1)\Big].$$

Define $T_0(f) = 0$ and for $1 \le m \le n$

$$T_m(f) = \frac{1}{2}[T_m^S(f) + T_m^D(f)],$$

where $T_m^S(f)$ and $T_m^D(f)$ are given by (1.6) and (1.10), respectively. So, we have $T_1(f) = 0$,

$$T_2(f) = T_3(f) = T_4(f) = T_5(f) = \frac{1}{480}[f'(1) - f'(0)]$$

and, for $m \ge 6$,

$$T_m(f) = \frac{1}{480}[f'(1) - f'(0)] + \frac{1}{80}\sum_{k=3}^{[m/2]} \frac{B_{2k}}{(2k)!} \cdot 2^{-2k}(3^{4-2k} - 1)\Big[f^{(2k-1)}(1) - f^{(2k-1)}(0)\Big].$$

In the next theorem we establish two formulae which play the key role in this paper. We call them corrected Bullen-Simpson's 3/8 formulae of Euler type.

Theorem 3. *Let $f : [0,1] \to \mathbf{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$, for some $n \ge 1$. Then*

$$\int_0^1 f(t)dt = D(0,1) - T_n(f) + \tilde{R}_n^1(f), \tag{2.1}$$

and

$$\int_0^1 f(t)dt = D(0,1) - T_{n-1}(f) + \tilde{R}_n^2(f), \tag{2.2}$$

where

$$\tilde{R}_n^1(f) = \frac{1}{160n!}\int_0^1 G_n(t)df^{(n-1)}(t),$$

and

$$\tilde{R}_n^2(f) = \frac{1}{160n!}\int_0^1 F_n(t)df^{(n-1)}(t).$$

Proof. Indentity (2.1) is produced after adding formulae (1.4) and (1.8), and dividing them by 2. Identity (2.2) is obtained from (1.5) and (1.9) analogously.

Remark 1. Interval [0, 1] is used for simplicity and involves no loss in generality. In what follows, Theorem 3 and others will be applied, without comment, to any interval that is convenient.

It is easy to see that if $f : [a,b] \to \mathbf{R}$ is such that $f^{(n-1)}$ is continuous of bounded variation on $[a,b]$, for some $n \ge 1$, then

$$\int_a^b f(t)dt = D(a,b) - \tilde{T}_n(f) + \frac{(b-a)^n}{160n!}\int_a^b G_n\Big(\frac{t-a}{b-a}\Big)df^{(n-1)}(t)$$

and

$$\int_a^b f(t)dt = D(a,b) - \tilde{T}_{n-1}(f) + \frac{(b-a)^n}{160n!}\int_a^b F_n\Big(\frac{t-a}{b-a}\Big)df^{(n-1)}(t),$$

where

$$D(a,b) = \frac{b-a}{160}\Big[13f(a) + 27f\Big(\frac{5a+b}{6}\Big) + 27f\Big(\frac{2a+b}{3}\Big) + 26f\Big(\frac{a+b}{2}\Big) \\ +27f\Big(\frac{a+2b}{3}\Big) + 27f\Big(\frac{a+5b}{6}\Big) + 13f(b)\Big],$$

$$\tilde{T}_0(f) = \tilde{T}_1(f) = 0,$$

$$\tilde{T}_2(f) = \tilde{T}_3(f) = \tilde{T}_4(f) = \tilde{T}_5(f) = \frac{(b-a)^2}{480}[f'(b) - f'(a)]$$

and for $m \geq 6$

$$\tilde{T}_m(f) = \frac{(b-a)^2}{480}[f'(b) - f'(a)] \\ + \frac{1}{80}\sum_{k=3}^{[m/2]}\frac{(b-a)^{2k}}{(2k)!}\cdot 2^{-2k}(3^{4-2k}-1)B_{2k}[f^{(2k-1)}(b) - f^{(2k-1)}(a)].$$

Remark 2. Suppose that $f : [0,1] \to \mathbf{R}$ is such that $f^{(n)}$ exists and is integrable on [0, 1], for some $n \geq 1$. In this case (2.1) holds with

$$\tilde{R}_n^1(f) = \frac{1}{160n!}\int_0^1 G_n(t)f^{(n)}(t)dt,$$

while (2.2) holds with

$$\tilde{R}_n^2(f) = \frac{1}{160n!}\int_0^1 F_n(t)f^{(n)}(t)dt.$$

Remark 3. For $n = 6$, (2.2) yields

$$\int_0^1 f(t)dt - D(0,1) + \frac{1}{480}[f'(1) - f'(0)] = \frac{1}{115200}\int_0^1 F_6(t)df^{(5)}(t).$$

From this identity it is clear that corrected Bullen-Simpson's 3/8 formula of Euler type is exact for all polynomials of order ≤ 5.

3. Main Result

Before we state our main result, we will need to prove some properties of functions G_k and F_k. Notice that it is enough to know the values of those functions on the interval $[0,\ \frac{1}{2}]$, since $G_k(t + \frac{1}{2}) = G_k(t)$.

direct calculations show that

$$G_1(t) = F_1(t) = \begin{cases} -160t + 13, & 0 < t \le 1/6 \\ -160t + 40, & 1/6 < t \le 1/3 \\ -160t + 67, & 1/3 < t \le 1/2 \end{cases} \tag{3.1}$$

$$G_2(t) = \begin{cases} 160t^2 - 26t + 2/3, & 0 \le t \le 1/6 \\ 160t^2 - 80t + 29/3, & 1/6 \le t \le 1/3 \\ 160t^2 - 134t + 83/3, & 1/3 \le t \le 1/2 \end{cases} \tag{3.2}$$

$$F_2(t) = G_2(t) - 2/3 = \begin{cases} 160t^2 - 26t, & 0 \le t \le 1/6 \\ 160t^2 - 80t + 9, & 1/6 \le t \le 1/3 \\ 160t^2 - 134t + 27, & 1/3 \le t \le 1/2 \end{cases} \tag{3.3}$$

$$G_3(t) = F_3(t) = \begin{cases} -160t^3 + 39t^2 - 2t, & 0 \le t \le 1/6 \\ -160t^3 + 120t^2 - 29t + 9/4, & 1/6 \le t \le 1/3 \\ -160t^3 + 201t^2 - 83t + 45/4, & 1/3 \le t \le 1/2 \end{cases} \tag{3.4}$$

$$G_4(t) = F_4(t) = \begin{cases} 160t^4 - 52t^3 + 4t^2, & 0 \le t \le 1/6 \\ 160t^4 - 160t^3 + 58t^2 - 9t + 1/2, & 1/6 \le t \le 1/3 \\ 160t^4 - 268t^3 + 166t^2 - 45t + 9/2, & 1/3 \le t \le 1/2 \end{cases} \tag{3.5}$$

$$\begin{aligned} &G_5(t) = F_5(t) \\ &= \begin{cases} -160t^5 + 65t^4 - 20/3 \cdot t^3, & 0 \le t \le 1/6 \\ -160t^5 + 200t^4 - 290/3 \cdot t^3 + 45/2 \cdot t^2 - 5/2 \cdot t + 5/48, & 1/6 \le t \le 1/3 \\ -160t^5 + 335t^4 - 830/3 \cdot t^3 + 225/2 \cdot t^2 - 45/2 \cdot t + 85/48, & 1/3 \le t \le 1/2 \end{cases} \end{aligned} \tag{3.6}$$

Lemma 1. *For $k \ge 3$, function $G_{2k-1}(t)$ has no zeros in the interval $(0, \frac{1}{4})$. The sign of this function is determined by*

$$(-1)^k G_{2k-1}(t) > 0, \quad 0 < t < \frac{1}{4}. \tag{3.7}$$

Proof. For $k = 3$, $G_5(t)$ is given by (3.6) and it is elementary to see that

$$G_5(t) < 0, \quad 0 < t < 1/4, \tag{3.8}$$

so our first assertion is true for $k = 3$. Assuming the opposite, by induction, it follows easily that the assertion is true for all $k \ge 4$.

Further, if $G_{2k-3}(t) > 0$, $0 < t < 1/4$, then since

$$G''_{2k-1}(t) = (2k-1)(2k-2)G_{2k-3}(t)$$

it follows that G_{2k-1} is convex and hence $G_{2k-1}(t) < 0$ on $(0,\, 1/4)$. Similarly, we conclude that if $G_{2k-3}(t) < 0$, then $G_{2k-1}(t) > 0$ on $(0,\, 1/4)$. (3.7) now follows from (3.8).

Corollary 1. *For* $k \geq 3$*, functions* $(-1)^{k-1}F_{2k}(t)$ *and* $(-1)^{k-1}G_{2k}(t)$ *are strictly increasing on the interval* $(0,1/4)$ *and strictly decreasing on the interval* $(1/4,1/2)$*. Consequently,* 0 *and* $1/2$ *are the only zeros of* $F_{2k}(t)$ *on* $[0,1/2]$ *and*

$$\max_{t\in[0,1]} |F_{2k}(t)| = 2^{2-2k}(1-2^{-2k})(1-3^{4-2k})|B_{2k}|,$$
$$\max_{t\in[0,1]} |G_{2k}(t)| = 2^{1-2k}(1-3^{4-2k})|B_{2k}|.$$

Proof. Since

$$[(-1)^{k-1}F_{2k}(t)]' = [(-1)^{k-1}G_{2k}(t)]' = (-1)^k \cdot 2k \cdot G_{2k-1}(t),$$

from Lemma 1 we conclude that $(-1)^{k-1}F_{2k}(t)$ and $(-1)^{k-1}G_{2k}(t)$ are strictly increasing on $(0, 1/4)$. It is easy to check that for $k \geq 2$ and $0 \leq t \leq 1/2$,

$$G_k(1/2-t) = (-1)^k G_k(t) \quad \text{and} \quad F_k(1/2-t) = (-1)^k F_k(t).$$

From there we conclude that $(-1)^{k-1}F_{2k}(t)$ and $(-1)^{k-1}G_{2k}(t)$ are strictly decreasing on $(1/4, 1/2)$. Further, $F_{2k}(0) = F_{2k}(1/2) = 0$, which implies $|F_{2k}(t)|$ achieves maximum at $t = 1/4$ and thus, the first assertion is proved.

On the other hand,

$$\max_{t\in[0,1]} |G_{2k}(t)| = \max\left\{|G_{2k}(0)|, \left|G_{2k}\left(\frac{1}{4}\right)\right|\right\} = |G_{2k}(0)|.$$

The proof is now complete.

Corollary 2. *For* $k \geq 3$*, we have*

$$\int_0^1 |F_{2k-1}(t)|dt = \int_0^1 |G_{2k-1}(t)|dt = \frac{2^{3-2k}}{k}(1-2^{-2k})(1-3^{4-2k})|B_{2k}|,$$
$$\int_0^1 |F_{2k}(t)|dt = |\tilde{B}_{2k}| = 2^{1-2k}(1-3^{4-2k})|B_{2k}|$$
$$\int_0^1 |G_{2k}(t)|dt \leq 2|\tilde{B}_{2k}| = 2^{2-2k}(1-3^{4-2k})|B_{2k}|.$$

Proof. Using the properties of functions G_k, i.e. properties of Bernoulli polynomials, we get

$$\int_0^1 |G_{2k-1}(t)|dt = 4\left|\int_0^{1/4} G_{2k-1}(t)dt\right| = \frac{2}{k}\left|F_{2k}\left(\frac{1}{4}\right)\right|,$$

which proves the first assertion. Since $F_{2k}(0) = F_{2k}(1/2) = 0$, from Corollary 1 we conclude that $F_{2k}(t)$ does not change sign on $(0, 1/2)$. Therefore,

$$\int_0^1 |F_{2k}(t)|dt = 2\left|\int_0^{1/2} G_{2k}(t)dt - \frac{1}{2}\tilde{B}_{2k}\right| = |\tilde{B}_{2k}|,$$

which proves the second assertion. Finally, we use the triangle inequality to obtain

$$\int_0^1 |G_{2k}(t)|dt \le \int_0^1 |F_{2k}(t)|dt + |\tilde{B}_{2k}| = 2|\tilde{B}_{2k}|,$$

which proves the third assertion.

Theorem 4. *If $f : [0,1] \to \mathbf{R}$ is such that $f^{(2k)}$ is a continuous function on $[0,1]$, for some $k \ge 3$, then there exists a point $\eta \in [0,1]$ such that*

$$\tilde{R}^2_{2k}(f) = \frac{2^{-2k}}{80(2k)!}(1 - 3^{4-2k})B_{2k} \cdot f^{(2k)}(\eta). \tag{3.9}$$

Proof. We can rewrite $\tilde{R}^2_{2k}(f)$ as

$$\tilde{R}^2_{2k}(f) = \frac{(-1)^{k-1}}{160(2k)!}J_k, \tag{3.10}$$

where

$$J_k = \int_0^1 (-1)^{k-1}F_{2k}(t)f^{(2k)}(t)dt. \tag{3.11}$$

From Corollary 1 we know that $(-1)^{k-1}F_{2k}(t) \ge 0$, $0 \le t \le 1$, so the claim follows from the mean value theorem for integrals and Corollary 2.

Remark 4. For $k = 3$ formula (3.9) reduces to

$$\tilde{R}^2_6(f) = \frac{1}{174182400} \cdot f^{(6)}(\eta).$$

Now, we prove our main result:

Theorem 5. *Let $f : [0,1] \to \mathbf{R}$ be such that $f^{(2k)}$ is a continuous function on $[0,1]$ for some $k \ge 3$. If f is a $(2k)$-convex function, then for even k we have*

$$\begin{aligned} 0 &\le \int_0^1 f(t)dt - \frac{1}{80}\Big[27f\Big(\frac{1}{6}\Big) + 26f\Big(\frac{1}{2}\Big) + 27f\Big(\frac{5}{6}\Big)\Big] + T^D_{2k-1}(f) \\ &\le \frac{1}{80}\Big[13f(0) + 27f\Big(\frac{1}{3}\Big) + 27f\Big(\frac{2}{3}\Big) + 13f(1)\Big] - T^S_{2k-1}(f) - \int_0^1 f(t)dt \end{aligned} \tag{3.12}$$

while for odd k inequalities are reversed.

Proof. Denote the middle part and the right-hand side of (3.14) by LHS and DHS, respectively. Then we have

$$LHS = \tau^2_{2k}(f)$$

and

$$RHS - LHS = -2\tilde{R}^2_{2k}(f)$$

where $\tau_{2k}^2(f)$ and $\tilde{R}_{2k}^2(f)$ are defined as in Theorems 2 and 3. In [6], we proved that under given assumptions on f, there exists a point $\xi \in [0,1]$ such that

$$\tau_{2k}^2(f) = -\frac{1}{80(2k)!}(1-2^{1-2k})(1-3^{4-2k})B_{2k} \cdot f^{(2k)}(\xi). \tag{3.13}$$

Recall that if f is $(2k)$-convex on [0, 1], then $f^{(2k)}(x) \geq 0$, $x \in [0,1]$. Now, having in mind that $(-1)^{k-1}B_{2k} > 0$ $(k \in \mathbf{N})$, from (3.13) and (3.9) we get

$$\begin{array}{lll} LHS \geq 0, & RHS - LHS \geq 0, & \text{for even } k \\ LHS \leq 0, & RHS - LHS \leq 0, & \text{for odd } k \end{array}$$

and thus the proof is complete.

Remark 5. From (3.14) for $k = 3$ we get

$$\begin{aligned} 0 &\leq \int_0^1 f(t)dt - \frac{1}{80}\Big[27f\Big(\frac{1}{6}\Big) + 26f\Big(\frac{1}{2}\Big) + 27f\Big(\frac{5}{6}\Big)\Big] - \frac{1}{240}[f'(1) - f'(0)] \\ &\leq \frac{1}{80}\Big[13f(0) + 27f\Big(\frac{1}{3}\Big) + 27f\Big(\frac{2}{3}\Big) + 13f(1)\Big] - \frac{1}{120}[f'(1) - f'(0)] - \int_0^1 f(t)dt \end{aligned}$$

Theorem 6. *If $f : [0,1] \to \mathbf{R}$ is such that $f^{(2k)}$ is a continuous function on $[0,1]$ and f is either $(2k)$-convex or $(2k)$-concave, for some $k \geq 3$, then there exists a point $\theta \in [0,1]$ such that*

$$\tilde{R}_{2k}^2(f) = \theta \cdot \frac{2^{-2k}}{40(2k)!}(1-2^{-2k})(1-3^{4-2k})B_{2k}[f^{(2k-1)}(1) - f^{(2k-1)}(0)]. \tag{3.14}$$

Proof. Suppose f is $(2k)$-convex, so $f^{(2k)}(t) \geq 0$, $0 \leq t \leq 1$. If J_k is given by (3.11), using Corollary 1, we obtain

$$0 \leq J_k \leq (-1)^{k-1} F_{2k}\Big(\frac{1}{4}\Big) \cdot \int_0^1 f^{(2k)}(t)dt.$$

which means that there must exist a point $\theta \in [0,1]$ such that

$$J_k = \theta \cdot (-1)^{k-1} \cdot 2^{2-2k}(1-2^{-2k})(1-3^{4-2k})B_{2k}[f^{(2k-1)}(1) - f^{(2k-1)}(0)].$$

When f is $(2k)$-concave, the statement follows similarly.

Now define

$$\Delta_{2k}(f) = \frac{2^{-2k}}{80(2k)!} \cdot (1-3^{4-2k})B_{2k}[f^{(2k-1)}(1) - f^{(2k-1)}(0)].$$

Clearly,

$$\tilde{R}_{2k}^2(f) = \theta \cdot (2-2^{1-2k}) \cdot \Delta_{2k}(f).$$

Theorem 7. *Suppose that $f : [0,1] \to \mathbf{R}$ is such that $f^{(2k+2)}$ is a continuous function on $[0,1]$ for some $k \geq 3$. If f is either $(2k)$-convex and $(2k+2)$-convex or $(2k)$-concave and $(2k+2)$-concave, then the remainder $\tilde{R}^2_{2k}(f)$ has the same sign as the first meglected term $\Delta_{2k}(f)$ and*

$$|\tilde{R}^2_{2k}(f)| \leq |\Delta_{2k}(f)|.$$

Proof. We have

$$\Delta_{2k}(f) = \tilde{R}^2_{2k}(f) - \tilde{R}^2_{2k+2}(f).$$

From Corollary 1 it follows that for all $t \in [0,1]$

$$(-1)^{k-1} F_{2k}(t) \geq 0 \quad \text{and} \quad (-1)^{k-1}[-F_{2k+2}(t)] \geq 0,$$

so we conclude that $\tilde{R}^2_{2k}(f)$ has the same sign as $-\tilde{R}^2_{2k+2}(f)$. Therefore, $\Delta_{2k}(f)$ must have the same sign as $\tilde{R}^2_{2k}(f)$ and $-\tilde{R}^2_{2k+2}(f)$. Moreover, it follows that

$$|\tilde{R}^2_{2k}(f)| \leq |\Delta_{2k}(f)| \quad \text{and} \quad |\tilde{R}^2_{2k+2}(f)| \leq |\Delta_{2k}(f)|.$$

4. Some Inequalities Related to Corrected Bullen-Simpson's 3/8 Formulae of Euler Type

In this section, using formulae derived in Theorem 3, we shall prove a number of inequalities for various classes of functions.

Theorem 8. *Assume (p,q) is a pair of conjugate exponents, that is $1 \leq p,\ q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Let $|f^{(n)}|^p : [0,1] \to \mathbf{R}$ be R-integrable function for some $n \geq 1$. Then we have*

$$\Big| \int_0^1 f(t)dt - D(0,1) + T_{n-1}(f) \Big| \leq K(n,p) \cdot \|f^{(n)}\|_p, \tag{4.1}$$

and

$$\Big| \int_0^1 f(t)dt - D(0,1) + T_n(f) \Big| \leq K^*(n,p) \cdot \|f^{(n)}\|_p, \tag{4.2}$$

where

$$K(n,p) = \frac{1}{160n!} \Big[\int_0^1 |F_n(t)|^q dt \Big]^{\frac{1}{q}} \quad \text{and} \quad K^*(n,p) = \frac{1}{160n!} \Big[\int_0^1 |G_n(t)|^q dt \Big]^{\frac{1}{q}}.$$

Proof. Applying the Hölder inequality we get

$$\Big| \frac{1}{160n!} \int_0^1 F_n(t) f^{(n)}(t) dt \Big| \leq \frac{1}{160n!} \Big[\int_0^1 |F_n(t)|^q dt \Big]^{\frac{1}{q}} \cdot \|f^{(n)}\|_p$$

Having in mind Remark 2, from (2.2) and the above inequality, we obtain (4.1). Similarly, from (2.1) we obtain (4.2).

Remark 6. Taking $p = \infty$ and $n = 1, 2$ in Theorem 8, i.e. (4.1) we get

$$\Big| \int_0^1 f(t)dt - D(0,1) \Big| \le K(n,\infty) \cdot \|f^{(n)}\|_\infty,$$

where

$$K(1,\infty) = \frac{2401}{57600}, \quad K(2,\infty) = \frac{597 + 320\sqrt{10}}{768000}.$$

Taking $p = 1$ and $n = 1, 2$, we get

$$\Big| \int_0^1 f(t)dt - D(0,1) \Big| \le K(n,1) \cdot \|f^{(n)}\|_1,$$

where

$$K(1,1) = \frac{41}{480}, \quad K(2,1) = \frac{169}{51200}.$$

Comparison of these estimates and estimates obtained in [3] in Remarks 9 and 10 shows that ours are better in all cases except for $n = 2$ and $p = \infty$.

Moreover, for $p = \infty$ and $n = 3, 4, 5$ we obtain

$$\Big| \int_0^1 f(t)dt - D(0,1) + \frac{1}{480}[f'(1) - f'(0)] \Big| \le K(n,\infty) \cdot \|f^{(n)}\|_\infty,$$

where

$$K(3,\infty) = \frac{48693 + 3133\sqrt{241}}{3932160000}, \quad K(4,\infty) = \frac{1}{1179648}, \quad K(5,\infty) = \frac{1}{22118400},$$

and for $p = 1$ and $n = 3, 4, 5$ we get

$$\Big| \int_0^1 f(t)dt - D(0,1) + \frac{1}{480}[f'(1) - f'(0)] \Big| \le K(n,1) \cdot \|f^{(n)}\|_1,$$

where

$$K(3,1) = \frac{1053 + 187\sqrt{561}}{110592000}, \quad K(4,1) = \frac{1}{614400}, \quad K(5,1) = \frac{1}{9437184}.$$

Finally, for $p = 2$ we get

$$\Big| \int_0^1 f(t)dt - D(0,1) \Big| \le K(n,2) \cdot \|f^{(n)}\|_2,$$

where

$$K(1,2) = \frac{\sqrt{534}}{480}, \quad K(2,2) = \frac{\sqrt{5}}{960},$$

and

$$\Big| \int_0^1 f(t)dt - D(0,1) + \frac{1}{480}[f'(1) - f'(0)] \Big| \le K(n,2) \cdot \|f^{(n)}\|_2,$$

where

$$K(3,2) = \frac{\sqrt{1155}}{1209600}, \quad K(4,2) = \frac{\sqrt{210}}{14515200}, \quad K(5,2) = \frac{\sqrt{116655}}{5748019200}.$$

Remark 7. Note that $K^*(1,p) = K(1,p)$, for $1 < p \leq \infty$, since $G_1(t) = F_1(t)$. Also, for $1 < p \leq \infty$, we can easily calculate $K(1,p)$. Namely,

$$K(1,p) = \frac{1}{480}\Big[\frac{39^{q+1} + 40^{q+1} + 41^{q+1}}{120(q+1)}\Big]^{\frac{1}{q}}.$$

In the limit case when $p \to 1$, that is when $q \to \infty$, we have

$$\lim_{p\to 1} K(1,p) = \frac{41}{480} = K(1,1).$$

Now we use formula (2.1) and a Grüss type inequality to obtain estimations of corrected Bullen-Simpson's 3/8 formulae in terms of oscillation of derivatives of a function. To do this, we need the following two technical lemmas. The first one was proved in [9] and the second one is the key result from [10].

Lemma 2. *Let $k \geq 1$ and $\gamma \in \mathbf{R}$. Then*

$$\int_0^1 B_k^*(\gamma - t)dt = 0.$$

Lemma 3. *Let $F, G : [0,1] \to \mathbf{R}$ be two integrable functions. If*

$$m \leq F(t) \leq M, \quad 0 \leq t \leq 1$$

and

$$\int_0^1 G(t)dt = 0,$$

then

$$\Big|\int_0^1 F(t)G(t)dt\Big| \leq \frac{M-m}{2} \cdot \int_0^1 |G(t)|dt. \tag{4.3}$$

Theorem 9. *Let $f : [0,1] \to \mathbf{R}$ be such that $f^{(n)}$ exists and is integrable on $[0,1]$, for some $n \geq 1$. Suppose*

$$m_n \leq f^{(n)}(t) \leq M_n, \quad 0 \leq t \leq 1,$$

for some constants m_n and M_n. Then

$$\Big|\int_0^1 f(t)dt - D(0,1) + T_n(0,1)\Big| \leq C_n(M_n - m_n) \tag{4.4}$$

where

$$C_1 = \frac{2401}{115200}, \quad C_2 = \frac{320\sqrt{30} + 187\sqrt{561}}{27648000},$$
$$C_3 = \frac{48693 + 3133\sqrt{241}}{7864320000}, \quad C_4 = \frac{1}{2359296},$$
$$C_{2k-1} = \frac{2^{-2k}}{20(2k)!}(1 - 2^{-2k})(1 - 3^{4-2k})|B_{2k}|, \quad k \geq 3,$$
$$C_{2k} = \frac{2^{-2k}}{80(2k)!}(1 - 3^{4-2k})|B_{2k}|, \quad k \geq 3.$$

Proof. Lemma 2 ensures that the second condition of Lemma 3 is satisfied. Having in mind Remark 2, apply inequality (4.3) to obtain the estimate for $|\tilde{R}_n^1(f)|$. Now our statement follows easily from Corollary 2 for $n \geq 5$ and direct calculation for $n = 1, 2, 3, 4$.

References

[1] P. C. Hammer, *The mid-point method of numerical integration*, Math. Mag. **31** (1957-58), 193-195.

[2] P. S. Bullen, *Error estimates for some elementary quadrature rules*, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz. **602-633** (1978), 97-103.

[3] M. Matić, J. Pečarić and A. Vukelić, *Generalization of Bullen-Simpson's* 3/8 *inequality*, Mathematical and Computer Modelling **41** (2005), 463-483.

[4] Lj. Dedić, M. Matić and J. Pečarić, *On generalizations of Ostrowski inequality via some Euler-type identities*, Math. Inequal. Appl. **3** (2000), 337-353.

[5] I. Franjić and J. Pečarić, *On corrected Euler-Simpson's* 3/8 *formulae* (to appear in Nonlinear Studies).

[6] I. Franjić and J. Pečarić, *Corrected Euler-Maclaurin's formulae*, Rend. Circ. Mat. Palermo (2), **54**(2005), 259-272.

[7] V. I. Krylov, Approximate calculation of integrals, Macmillan, New york-London, 1962.

[8] M. Abramowitz and I. A. Stegun (Eds), Handbook of Mathematical Functions with Formulae, Graphs and Mathematical Tables, National Bureau of Standards, Applied Math. Series 55, 4th printing, Washington, 1965.

[9] Lj. Dedić, M. Matić and J. Pečarić, *Some inequalities of Euler-Grüss type*, Computers and Mathematics with Applications **41** (2001), 843-856.

[10] M. Matić, *Improvement of some inequalities of Euler-Grüs type*, Computers and Mathematics with Applications **46** (2003), 1325-1336.

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