



ON CERTAIN NEW CAUCHY-TYPE FRACTIONAL INTEGRAL INEQUALITIES AND OPIAL-TYPE FRACTIONAL DERIVATIVE INEQUALITIES

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Abstract. The aim of this paper is to establish several new fractional integral and derivative inequalities for non-negative and integrable functions. These inequalities related to the extension of general Cauchy type inequalities and involving Saigo, Riemann-Louville type fractional integral operators together with multiple Erdelyi-Kober operator. Furthermore the Opial-type fractional derivative inequality involving H-function is also established. The generosity of H-function could leads to several new inequalities that are of great interest of future research.

1. Introduction

In last few years the fractional integral and derivative inequalities and their applications have been addressed extensively by several authors like Anastassiou [1, 2], Beesack [3], Handley et al. [8], Opial [10] etc., by using the Riemann Liouville fractional integrals and Opial type fractional Derivatives. Researchers have great interest in this field due to vast applications of these inequalities in fractional differential equation in establishing the uniqueness of the solution of initial value problems, giving upper bounds to their solutions.

In the presented paper authors established certain theorems based on general Cauchy type inequality, involving Riemann-Liouville fractional integral operators, Saigo operator and multiple Erdely-Kober operator, then finally in last section Opial type fractional derivative inequality involving H-function is established, which is capable of yielding various results in the theory of Opial type integral inequalities.

2. Preliminaries and definitions

In this section, we will present some definitions that will be used in the proof of our main results.

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Definition 2.1. If a and b are positive real numbers, satisfying $a + b = 1$ and f and g are monotonic functions defined on $(0, \infty)$, then Cauchy's general inequality [9] is defined as

$$af(x) + bg(y) = [f(x)]^a [g(y)]^b \quad (2.1)$$

Definition 2.2. The standard Riemann-Liouville fractional integral [12] of order $\alpha \in \mathbb{C}$ for function $f : (0, \infty) \rightarrow R$ is given as

$$I^\alpha f(t) = I_{0+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x) dx, \operatorname{Re}(\alpha) > 0 \quad (2.2)$$

provided that integral on the right-hand side converges.

Definition 2.3. Saigo [11] defined the fractional integration operator as follows:

$$I_{0+}^{\alpha, \beta, \eta} f(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2F_1(\alpha + \beta, -\eta; \alpha; 1 - \frac{t}{x}) f(t) dt \quad (2.3)$$

where $\alpha \in \mathbb{C}, \operatorname{Re}(\alpha) > 0, \beta$ and η are real numbers and $f(x)$ is a real valued and continuous function defined on the interval $(0, \infty)$.

For $\beta = -\alpha$, in (2.3), we get Riemann - Liouville operator (2.2).

Definition 2.4. Erdelyi [6] defined the space of functions C_{α^*} for arbitrary real number α^* with set of real valued function $C'[0, \infty)$, as follows:

$$C_{\alpha^*} = \left\{ f(x) = x^q g(x); q < \alpha^*, g(x) \in C'[0, \infty) \right\} \quad (2.4)$$

where $\alpha^* = \min_{1 \leq k \leq m} (\beta \tau_k)$ with $m \in \mathbb{Z}^+, \beta > 0, k = 1, \dots, m; \tau_1, \tau_2, \tau_3, \dots, \tau_m$ be arbitrary real numbers.

Definition 2.5. Erdelyi [6] also defined the linear space of function C_α for arbitrary real numbers α , with set of real valued functions $C'[0, \infty)$, as follows:

$$C_\alpha = \left\{ f(x) = x^p \tilde{f}(x) : p > \alpha, \tilde{f}(x) \in C'[0, \infty) \right\} \quad (2.5)$$

where $\alpha = \max_{1 \leq k \leq m} [-\beta(\gamma_k + 1)]$, $m \in \mathbb{Z}^+, \beta > 0, k = 1, \dots, m; \gamma_1, \gamma_2, \gamma_3, \dots, \gamma_m$ be arbitrary real numbers.

Definition 2.6. A multiple Erdelyi-Kober operator of Riemann - Liouville type is defined in the form, [6]

$$I_{(\beta_k), (\lambda_k), m}^{(\gamma_k), (\delta_k)} f(x) = \frac{1}{x} \int_0^x H_{m, m}^{m, 0} \left(\frac{t}{x} \left| \begin{matrix} \gamma_k + \delta_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k} \\ \gamma_k + 1 - \frac{1}{\lambda_k}, \frac{1}{\lambda_k} \end{matrix} \right. \right)_1^m f(t) dt \quad (2.6)$$

where $m \in z^+; \beta_k, \lambda_k, \delta_k, \gamma_k > 0, k = 1, \dots, m; \gamma_1, \gamma_2, \gamma_3, \dots, \gamma_m$ be arbitrary real numbers. Furthermore,

$$\sum_{k=1}^m \frac{1}{\lambda_k} = \sum_{k=1}^m \frac{1}{\beta_k}$$

and $f(x) \in C_\alpha, \alpha = \max_{1 \leq k \leq m} \{-\lambda_k(\gamma_k + 1)\}$. Here $H_{p,q}^{m,n}[x]$ is H-function due to Fox [7], introduced and defined via a Mellin-Barnes type integral for integers m, n, p, q such that $0 \leq m \leq q, 0 \leq n \leq p$, for $a_i, b_j \in \mathbb{C}$ with \mathbb{C} , the set of complex numbers and for $\alpha_i, \beta_j \in R_+ = (0, \infty), (i = 1, 2, \dots, p; j = 1, 2, \dots, q)$, as

$$H_{p,q}^{m,n}(z) \equiv H_{p,q}^{m,n} \left[z \middle| \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right] = \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{H}_{p,q}^{m,n}(s) z^{-s} ds \tag{2.7}$$

with

$$\mathcal{H}_{p,q}^{m,n}(s) \equiv \mathcal{H}_{p,q}^{m,n} \left[\begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \middle| s \right] = \frac{\prod_{j=1}^m \Gamma(b_j + \beta_j s) \prod_{i=1}^n \Gamma(1 - a_i - \alpha_i s)}{\prod_{i=n+1}^p \Gamma(a_i + \alpha_i s) \prod_{j=m+1}^q \Gamma(1 - b_j - \beta_j s)} \tag{2.8}$$

Asymptotic expansions and analytic continuations of the H-function have been discussed by Braaksma [4].

Definition 2.7. Canavati [5] defined the generalized ν -fractional Riemann-Liouville integral for $x, x_0 \in [a, b]$ such that $x = x_0, x_0$ is fixed, $\nu = 1$, for the function $f \in C([a, b])$, as follows

$$(I_\nu^{x_0} f)(x) = \frac{1}{\Gamma(\nu)} \int_{x_0}^x (x-t)^{\nu-1} f(t) dt, x_0 = x = b. \tag{2.9}$$

Further the generalized ν -fractional derivative [5] of f over $[x_0, b]$ is given as

$$D_{x_0}^\nu f := DI_{1-\alpha}^{x_0} f^{(n)} (f^{(n)} := D^n f). \tag{2.10}$$

where $n = [\nu]$ the integral part, $\nu > 0$ and $\alpha = \nu - n \quad (0 < \alpha < 1)$.

Furthermore, Anastassiou [1] defined subspace $C_{x_0}^\nu([a, b])$ of $C^n([a, b])$ for $f \in C_{x_0}^\nu([a, b])$; as follows:

$$C_{x_0}^\nu([a, b]) = \{ f \in C^n([a, b]) : I_{1-\alpha}^{x_0} D^n f \in C^1([x_0, b]) \}. \tag{2.11}$$

3. Cauchy type fractional integral inequalities

Theorem 1. If a and b are positive real numbers satisfying $a + b = 1$ and f and g are monotonic functions defined on $[0, \infty)$, then for all $\alpha, \beta \in \mathbb{C}, \text{Re}(\alpha) > 0, \text{Re}(\beta) > 0, t > 0$, the following inequality holds

$$\frac{at^\beta}{\Gamma(\beta+1)} I^\alpha f(t) + \frac{bt^\alpha}{\Gamma(\alpha+1)} I^\beta g(t) \geq I^\beta g^b(t) I^\alpha f^a(t) \tag{3.1}$$

Proof. Multiplying both side of Cauchy general inequality (2.1) by $\frac{(t-x)^{\alpha-1}}{\Gamma(\alpha)}$ and integrating with respect to x , between the limits 0 to t , we get

$$\begin{aligned} & \frac{a}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x) dx + \frac{b}{\Gamma(\alpha)} g(y) \int_0^t (t-x)^{\alpha-1} dx \\ & \geq [g(y)]^b \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} [f(x)]^a dx \end{aligned}$$

By using (2.2), we obtained

$$aI^\alpha f(t) + bg(y)I^\alpha(1) \geq [g(y)]^b I^\alpha [f(t)]^a \quad (3.2)$$

multiplying both sides of equation (3.2) by $\frac{(t-y)^{\beta-1}}{\Gamma(\beta)}$ and integrating w.r.t. to y , between the limits 0 to t and finally by virtue of (2.2), we obtained inequality (3.1).

Remark 1. For $\alpha = \beta$ in equation (3.1), we obtained

$$aI^\alpha f(t) + bI^\alpha g(t) = \Gamma(1+\alpha) t^{-\alpha} I^\alpha f^a(t) I^\alpha g^b(t). \quad (3.3)$$

Theorem 2. If a and b are positive real numbers satisfying $a + b = 1$ and f and g are monotonic functions defined over $[0, \infty)$, then for all $\alpha, \beta, \eta, \gamma \in C, \operatorname{Re}(\alpha) > 0, \operatorname{Re}(\gamma) > 0, t > 0$ the following inequality holds

$$\begin{aligned} & a \frac{\Gamma(1-\delta+\sigma)}{\Gamma(1-\delta)\Gamma(1+\gamma+\sigma)} t^{-\delta} I_{0+}^{\alpha,\beta,\eta} f(t) + b \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} t^{-\beta} I_{0+}^{\gamma,\delta,\sigma} g(t) \\ & \geq I_{0+}^{\alpha,\beta,\eta} f^a(t) I_{0+}^{\gamma,\delta,\sigma} g^b(t) \end{aligned} \quad (3.4)$$

Proof. Multiplying both sides of equation (2.1) by $\frac{t^{-\alpha-\beta}}{\Gamma(\alpha)} (t-x)^{\alpha-1} {}_2F_1(\alpha+\beta, -\eta; \alpha; 1-\frac{x}{t})$ then integrating w.r.t. x from 0 to t and by virtue of (2.3), we have

$$aI_{0+}^{\alpha,\beta,\eta} f(t) + bg(y)I_{0+}^{\alpha,\beta,\eta}(1) \geq g^b(y)I_{0+}^{\alpha,\beta,\eta} f^a(t)$$

now multiplying both sides of above equation by $\frac{t^{-\gamma-\delta}(t-y)^{\gamma-1}}{\Gamma(\gamma)} {}_2F_1(\gamma+\delta, -\sigma, \gamma; 1-\frac{y}{t})$ then integrating w.r.t. y from 0 to t and by virtue of (2.3), we obtained inequality (3.4).

Remark 2. If $\alpha = \gamma, \beta = \delta$ and $\eta = \sigma$, inequality (3.4) becomes

$$aI_{0+}^{\alpha,\beta,\eta} f(t) + bI_{0+}^{\alpha,\beta,\eta} g(t) \geq \frac{\Gamma(1-\beta)\Gamma(1+\alpha+\eta) t^{+\beta}}{\Gamma(1-\beta+\eta)} I_{0+}^{\alpha,\beta,\eta} f^a(t) I_{0+}^{\alpha,\beta,\eta} g^b(t). \quad (3.5)$$

Remark 3. Putting $\beta = -\alpha$, in inequality (3.4) we obtained inequality (3.3).

Theorem 3. *If a and b are positive real numbers, satisfying $a + b = 1$ and $f(x), g(x) \in C_\alpha, m, n \in \mathbb{Z}^+; \lambda_k, \beta_k, \delta_k, \gamma_k > 0$ where, $\beta_k, \lambda_k, \delta_k, \gamma_k$ be arbitrary real numbers with*

$$\sum_{k=1}^m \frac{1}{\lambda_k} = \sum_{k=1}^m \frac{1}{\beta_k}, \alpha = \max_{1 \leq k \leq m} [-\lambda_k (\gamma_k + 1)]$$

then the following inequality holds

$$\begin{aligned} & a I_{(\beta_k)(\lambda_k), m}^{(\gamma_k), (\delta_k)} [f(t)] I_{(\beta_k)(\lambda_k), n}^{(\gamma_k), (\delta_k)} (1) + b I_{(\beta_k)(\lambda_k), m}^{(\gamma_k), (\delta_k)} (1) I_{(\beta_k)(\lambda_k), n}^{(\gamma_k), (\delta_k)} [g(t)] \\ & \geq I_{(\beta_k)(\lambda_k), m}^{(\gamma_k), (\delta_k)} [f(t)]^a I_{(\beta_k)(\lambda_k), n}^{(\gamma_k), (\delta_k)} [g(t)]^b \end{aligned} \tag{3.6}$$

where $k = 1, \dots, m$ for operator $I_{(\beta_k)(\lambda_k), m}^{(\gamma_k), (\delta_k)} [\cdot]$ and $k = 1, \dots, n$ for operator $I_{(\beta_k)(\lambda_k), n}^{(\gamma_k), (\delta_k)} [\cdot]$.

Proof. Multiply both side of general Cauchy inequality (2.1) by

$\frac{1}{t} H_{m, m}^{m, 0} \left[\frac{x}{t} \left| \begin{matrix} \left(\gamma_k + \delta_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k} \right)_1^m \\ \left(\gamma_k + 1 - \frac{1}{\lambda_k}, \frac{1}{\lambda_k} \right)_1 \end{matrix} \right. \right]$ then integrating with the limits 0 to t and using equation (2.6), we get

$$a I_{(\beta_k)(\lambda_k), m}^{(\gamma_k), (\delta_k)} f(t) + b g(y) I_{(\beta_k)(\lambda_k), m}^{(\gamma_k), (\delta_k)} (1) \geq [g(y)]^b I_{(\beta_k)(\lambda_k), m}^{(\gamma_k), (\delta_k)} [f(t)]^a$$

now multiply both side by $\frac{1}{t} H_{n, n}^{n, 0} \left[\frac{y}{t} \left| \begin{matrix} \left(\gamma_k + \delta_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k} \right)_1^n \\ \left(\gamma_k + 1 - \frac{1}{\lambda_k}, \frac{1}{\lambda_k} \right)_1 \end{matrix} \right. \right]$ and integrating with the limits 0 to t , we obtained (3.6).

4. Opial type fractional derivative inequalities

Theorem 4. *Let $f \in C_{x_0}^v ([a, b])$, $v = 1$ and $f^{(i)}(x_0) = 0, i = 0, 1, \dots, n - 1, n = [v]; x, x_0 \in [a, b]: x = x_0$. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds*

$$\begin{aligned} & \int_{x_0}^x |f(w)| |(D_0^v f)(w)| dw \leq \frac{2^{-1/q} (x - x_0)^{(pv - p + 2)/p}}{\Gamma(v) ((pv - p + 1)(pv - p + 2))^{1/p}} \\ & \times \left[\int_{x_0}^x \left(t^{w + \alpha} H_{P+1, Q+1}^{M, N+1} \left(t^\sigma \left| \begin{matrix} (-\omega, \sigma), (a_j, \alpha_j)_{1, P} \\ (b_j, \beta_j)_{1, Q}, (-\omega - \alpha, \sigma) \end{matrix} \right. \right) \right)^q dt \right]^{\frac{2}{q}} \end{aligned} \tag{4.1}$$

where, $f(x) = x^w \left| H_{P, Q}^{M, N} \left(x^\sigma \left| \begin{matrix} (a_j, \alpha_j)_{1, P} \\ (b_j, \beta_j)_{1, Q} \end{matrix} \right. \right) \right|$.

Proof. Let

$$|f(x)| = \left| x^w H_{P, Q}^{M, N} \left(x^\sigma \left| \begin{matrix} (a_j, \alpha_j)_{1, P} \\ (b_j, \beta_j)_{1, Q} \end{matrix} \right. \right) \right| \leq \frac{1}{\Gamma(v)} \int_{x_0}^x (x - t)^{v-1} |D_{x_0}^v f(t)| dt$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(v)} \left(\int_{x_0}^x [(x-t)^{v-1}]^p dt \right)^{1/p} \left(\int_{x_0}^x [|D_{x_0}^v f|(t)]^q dt \right)^{1/q} \\
&= \frac{1}{\Gamma(v)} \frac{(x-x_0)^{(pv-p+1)/p}}{(pv-p+1)^{1/p}} \left(\int_{x_0}^x [|D_{x_0}^v f|(t)]^q dt \right)^{1/q}
\end{aligned} \tag{4.2}$$

on setting

$$z(x) = \left(\int_{x_0}^x (|D_{x_0}^v f|(t)]^q dt \right), \quad (z(x_0) = 0) \tag{4.3}$$

then $\frac{d}{dx} z(x) = z'(x) = (|D_{x_0}^v f|(x)]^q$, i.e.,

$$|D_{x_0}^v f|(x) = (z'(x))^{1/q} = x^{\omega+\alpha} H_{P+1, Q+1}^{M, N+1} \left(x^\sigma \left| \begin{array}{c} (-\omega, \sigma), (a_j, \alpha_j)_{1, p} \\ (b_j, \beta_j)_{1, q}, (-\omega-\alpha, \sigma) \end{array} \right. \right) \tag{4.4}$$

now by virtue of (4.2) and (4.4), we have

$$|f(w)| |D_{x_0}^v f|(w) = \frac{1}{\Gamma(v)} \frac{(\omega-x_0)^{(pv-p+1)/p}}{(pv-p+1)^{1/p}} \left[\left\{ \int_{x_0}^\omega (|D_{x_0}^v f|(t)]^q dt \right\} \times z'(w) \right]^{1/q},$$

$x_0 = \omega = x$, further integrating over $[x_0, x]$, we get

$$\begin{aligned}
&\int_{x_0}^x |f(w)| |D_{x_0}^v f|(w) dw \\
&= \frac{1}{\Gamma(v) (pv-p+1)^{1/p}} \int_{x_0}^x (w-x_0)^{(pv-p+1)/p} (z(w) \cdot z'(w))^{1/q} dw \\
&\leq \frac{1}{\Gamma(v) (pv-p+1)^{1/p}} \left(\int_{x_0}^x (w-x_0)^{(pv-p+1)} dw \right)^{1/p} \left(\int_{x_0}^x z(w) z'(w) dw \right)^{1/q} \\
&= \frac{(x-x_0)^{(pv-p+2)/p}}{\Gamma(v) [(pv-p+1)(pv-p+2)]^{1/p}} \cdot \frac{[z(x)]^{2/q}}{2^{1/q}}
\end{aligned}$$

now using equation (4.3), we obtained

$$|f(w)| |D_{x_0}^v f|(w) = \frac{2^{-1/q} (x-x_0)^{(pv-p+2)/p}}{\Gamma(v) [(pv-p+1)(pv-p+2)]^{1/p}} \left[\int_{x_0}^x (|D_{x_0}^v f|(t)]^q dt \right]^{2/q}$$

finally using equation (4.4), we arrived at (4.1).

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