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SOME PROPERTIES OF DIFFERENTIAL OPERATOR ASSOCIATED WITH GENERALIZED HYPERGEOMETRIC FUNCTIONS

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Abstract. In the present investigation, new subclasses of analytic functions in the open unit disk which are defined using generalized derivative operator are introduced. Several interesting properties of these classes are obtained.

1. Introduction

Let $\mathscr{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$. Also let $\mathscr{S}$ be the the subclass of $\mathscr{A}$ consisting of all functions which are univalent in $\mathbb{U}$. If $f(z) \in \mathscr{A}$ satisfies

$$\left|\arg\left(\frac{zf'(z)}{f(z)} - \alpha\right)\right| < \frac{\pi}{2}\beta,\ (z \in U, 0 \le \alpha < 1, 0 < \beta \le 1), \tag{1.2}$$

then $f(z)$ is said to be strongly starlike of order β and type α in $\mathbb{U}$, and denoted by $S^*(\alpha,\beta)$. If $f(z) \in \mathscr{A}$ satisfies

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right)\right| < \frac{\pi}{2}\beta,\ (z \in U, 0 \le \alpha < 1, 0 < \beta \le 1), \tag{1.3}$$

then $f(z)$ is said to be strongly convex of order β and type α in $\mathbb{U}$, and denoted by $C(\alpha,\beta)$.

It is obvious that $f(z) \in \mathscr{A}$ belongs to $C(\alpha,\beta)$ if and only if $zf'(z) \in S^*(\alpha,\beta)$. Further, we note that $S^*(\alpha,1) \equiv S^*(\alpha)$ and $C(\alpha,1) \equiv C(\alpha)$ which are, respectively, starlike and convex univalent functions of order α. Let $\mathscr{P}$ denote the class of functions of the form $p(z) = 1 + p_1(z) + \cdots$ analytic in $\mathbb{U}$ which satisfy the condition $\Re\{p(z)\} > 0$.

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For two functions given by $f(z)=\sum_{n=2}^{\infty} a_n z^n$ and $g(z)=\sum_{n=2}^{\infty} b_n z^n$ be analytic in $\mathbb{U}$. Then the Hadamard product (or convolution) $f*g$ of the two functions f, g is defined by

$$f(z)*g(z)=z+\sum_{n=2}^{\infty} a_n b_n z^n.$$

Recall that the function F is subordinate to G if there exists a function ω, analytic in $\mathbb{U}$, with $\omega(0)=0$ and $|\omega(z)|<1$ such that $F(z)=G(\omega(z)), z\in\mathbb{U}$. We denote this subordination by $F(z)\prec G(z)$. If $G(z)$ is univalent in $\mathbb{U}$, then the subordination is equivalent to $F(0)=G(0)$ and $F(\mathbb{U})\subset G(\mathbb{U})$.

For complex parameters $a_i, b_j, (i=1,\dots,r, j=1,\dots,s, b_j\in\mathbb{C}\backslash\{0,-1,-2,\dots\})$, we shall use the generalized hypergeometric function ${}_r\Phi_s(a_i,b_j;z)$

$${}_r\Phi_s(a_i,b_j;z)=\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_r)_n}{(b_1)_n\cdots(b_s)_n}\frac{z^n}{n!},$$

where $r\le s+1; r,s\in\mathbb{N}_0=\mathbb{N}\cup\{0\}; z\in\mathbb{U}, \mathbb{N}$ denotes the set of positive integers and $(x)_n$ is the Pochhammer symbol defined in terms of the Gamma function Γ, by

$$(x)_n=\frac{\Gamma(x+n)}{\Gamma(x)}=\begin{cases}1, & n=0,\\ x(x+1)\cdots(x+n-1), & n=\{1,2,3,\dots\}.\end{cases}$$

Corresponding to a function ${}_r\mathscr{G}_s(a_i,b_j;z)$ defined by

$${}_r\mathscr{G}_s(a_i,b_j;z)=z\,{}_r\Phi_s(a_i,b_j;z). \tag{1.4}$$

Dziok and Srivastava [1] introduced a convolution operator on $\mathscr{A}$ such that

$$\mathscr{H}_{r,s}(a_i,b_j):\mathscr{A}\to\mathscr{A},$$

is defined by

$$\begin{aligned}\mathscr{H}_{r,s}(a_i,b_j)f(z) &= {}_r\mathscr{G}_s(a_i,b_j;z)*f(z)\\ &= z+\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}\cdots(a_r)_{n-1}}{(b_1)_{n-1}\cdots(b_s)_{n-1}}\frac{a_n z^n}{(n-1)!}.\end{aligned}$$

We now define the following operator $\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f:\mathscr{A}\to\mathscr{A}$ by the following Hadamard product:

$$\begin{aligned}\mathscr{D}^{0,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z) &= {}_r\mathscr{G}_s(a_i,b_j;z)*f(z),\\ (1+b)\mathscr{D}^{1,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z) &= (1-(\lambda_1+\lambda_2)+b)(\varphi^b_{\lambda_2}*{}_r\mathscr{G}_s(a_i,b_j;z)*f(z))\\ &\quad+(\lambda_1+\lambda_2)z(\varphi^b_{\lambda_2}*{}_r\mathscr{G}_s(a_i,b_j;z)*f(z))',\end{aligned} \tag{1.5}$$

$$\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z) = \mathscr{D}^{b}_{\lambda_1,\lambda_2}(\mathscr{D}^{m-1,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z)), \tag{1.6}$$

where $\varphi^b_{\lambda_2} = z + \sum_{n=2}^{\infty} \frac{z^n}{1+\lambda_2(n-1)+b}$.

From (1.5) and (1.6) we may easily deduce the following linear operator:

$$\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z) = z + \sum_{n=2}^{\infty}\left[\frac{1+(\lambda_1+\lambda_2)(n-1)+b}{1+\lambda_2(n-1)+b}\right]^m \frac{(a_1)_{n-1}\cdots(a_r)_{n-1}}{(b_1)_{n-1}\cdots(b_s)_{n-1}}\frac{a_n z^n}{(n-1)!}, \tag{1.7}$$

where $\lambda_2 \geq \lambda_1 \geq 0, m, b \in \mathbb{N}_0 = \{0,1,2,\ldots\}, a_i \in \mathbb{C}, b_j \in \mathbb{C}\setminus\{0,-1,-2,\ldots\})$, $(i = 1,\ldots,r, j = 1,\ldots,s$, and $r \leq s+1; r,s \in \mathbb{N}_0$.

It should be remarked that the linear operator (1.7) is a generalization of many operators considered earlier. Let us see some of the examples:

For $m = 0$ the operator $\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f$ reduces to the well-known Dziok- Srivastava operator [1]. For $\lambda_2 = b = 0$, we get the Selvaraj derivative operator [2]. For $m = 0, r = 2$ and $s = 1$ we obtain the Hohlov derivative operator [3]. For $r = 1, s = 0, a_1 = 1, \lambda_1 = 1$ and $\lambda_2 = b = 0$, we get the Salagean derivative operator [4]. For $r = 1, s = 0, a_1 = 1$ and $\lambda_2 = b = 0$, we get the generalized Salagean derivative operator introduced by Al-Oboudi [5]. For $m = 0, r = 1, s = 0$ and $a_1 = \delta+1$ we obtain the Ruscheweyh derivative operator [6]. For $r = 1, s = 0$ and $a_1 = \delta+1$ we obtain the derivative given by El-Yagubi and Darus [7]. For $m = 0, r = 2$ and $s = 1$ and $a_2 = 1$ we obtain the Carlson and Shaffer [8]. For $r = 1, s = 0, a_1 = 1$ and $\lambda_2 = 0$ we get the Cátás derivative operator [9].

Remark 1.1. It follows from the above definition that:

$$\begin{aligned}(1+b)\mathscr{D}^{m+1,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z) &= (1-(\lambda_1+\lambda_2)+b)(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j) * \varphi^b_{\lambda_2}f(z)) \\ &\quad +(\lambda_1+\lambda_2)z(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j) * \varphi^b_{\lambda_2}f(z))',\end{aligned} \tag{1.8}$$

and

$$a_1\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_1+1,b_j)f(z) = (a_1-1)\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_1,b_j)f(z) + z(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_1,b_j)f(z))'. \tag{1.9}$$

Now, we introduce the following classes in terms of the new operator $\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)$:

Definition 1.1. For $\lambda_2 \geq \lambda_1 \geq 0, m, b \in \mathbb{N}_0 = \{0,1,2,\ldots\},\ a_i \in \mathbb{C},\ b_j \in \mathbb{C}\setminus\{0,-1,-2,\ldots\}),\ (i = 1,\ldots,r,\ j = 1,\ldots,s, r \leq s+1;\ r,s \in \mathbb{N}_0,\ 0 \leq \alpha < 1$ and $0 < \beta \leq 1$, let $\mathscr{S}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j;\alpha,\beta)$ be the class of functions $f \in \mathscr{A}$ satisfying

$$\left|\arg\left(\frac{z(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z))'}{\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z)} - \alpha\right)\right| < \frac{\pi}{2}\beta \qquad (z \in \mathbb{U}),$$

where $\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z) \in S^*(\alpha,\beta)$ and $\frac{z(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z))'}{\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z)} \neq \alpha$.

Definition 1.2. For $\lambda_2 \geq \lambda_1 \geq 0$, $m, b \in \mathbb{N}_0 = \{0,1,2,\ldots\}$, $a_i \in \mathbb{C}$, $b_j \in \mathbb{C}\setminus\{0,-1,-2,\ldots\})$, $(i = 1,\ldots,r,\ j = 1,\ldots,s, r \leq s+1;\ r,s \in \mathbb{N}_0$, $0 \leq \alpha < 1$ and $0 < \beta \leq 1$, let $\mathscr{C}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j;\alpha,\beta)$ be the class of functions $f \in \mathscr{A}$ satisfying

$$\left|\arg\left(1 + \frac{z(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z))''}{(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z))'} - \alpha\right)\right| < \frac{\pi}{2}\beta \qquad (z \in \mathbb{U}),$$

where $\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z) \in C(\alpha,\beta)$ and $1 + \frac{z(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z))''}{(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z))'} \neq \alpha$.

Clearly, $f \in \mathscr{C}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j;\alpha,\beta)$ if and only if $zf'(z) \in \mathscr{S}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j;\alpha,\beta)$.

Note that $\mathscr{S}^{0,b}_{\lambda_1,\lambda_2}(1,1;\alpha,\beta) \equiv S^*(\alpha,\beta)$, $\mathscr{S}^{0,b}_{\lambda_1,\lambda_2}(1,1;\alpha,1) \equiv S^*(\alpha)$, $\mathscr{C}^{0,b}_{\lambda_1,\lambda_2}(1,1;\alpha,\beta) \equiv C(\alpha,\beta)$ and $\mathscr{C}^{0,b}_{\lambda_1,\lambda_2}(1,1;\alpha,1) \equiv C(\alpha)$.

Definition 1.3. For $\lambda_2 \geq \lambda_1 \geq 0, m, b \in \mathbb{N}_0 = \{0,1,2,\ldots\}$, $a_i \in \mathbb{C}$, $b_j \in \mathbb{C}\setminus\{0,-1,-2,\ldots\})$, $(i = 1,\ldots,r,\ j = 1,\ldots,s, r \leq s+1;\ r,s \in \mathbb{N}_0$, $0 \leq \gamma < 1$ and $-1 \leq B < A \leq 1$, let $\mathscr{Q}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j;\gamma;A,B)$ be the class of functions $f \in \mathscr{A}$ satisfying

$$\frac{1}{1-\gamma}\left(\frac{z(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z))'}{\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z)} - \gamma\right) \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}),$$

where $\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z) \in S^*(\gamma;A,B)$ and $\frac{z(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z))'}{\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z)} \neq \gamma$.

2. Main results

To derive our theorems, the following lemma will be required.

Lemma 2.1 (see [10])**.** *Let β, ν be complex numbers. Let $\phi \in \mathscr{P}$ be convex univalent in $\mathbb{U}$ with $\phi(0) = 1$ and $\Re\{\beta\phi(z) + \nu\} > 0$, $z \in \mathbb{U}$. If $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ is analytic in $\mathbb{U}$ with $p(0) = 1$, then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \nu} \prec \phi(z) \Rightarrow p(z) \prec \phi(z), \qquad (z \in \mathbb{U}).$$

Theorem 2.2. *Let $\phi(z)$ be convex univalent in $\mathbb{U}$ with $\phi(0) = 1$ and $\Re\{\phi(z)\} \geq 0$. If $f \in \mathscr{A}$ satisfies the condition*

$$\frac{1}{1-\gamma}\left(\frac{z(\mathscr{D}^{m+1,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z))'}{\mathscr{D}^{m+1,b}_{\lambda_1,\lambda_2}(\alpha_i,\beta_j)f(z)} - \gamma\right) \prec \phi(z) \qquad (z \in \mathbb{U}),$$

then

$$\frac{1}{1-\gamma}\left(\frac{z(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(\alpha_i,\beta_j)*\varphi^b_{\lambda_2}f(z))'}{\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(\alpha_i,\beta_j)*\varphi^b_{\lambda_2}f(z)}-\gamma\right)\prec\phi(z)\qquad(z\in\mathbb{U}),$$

for $\lambda_2\geq\lambda_1\geq 0, m,b\in\mathbb{N}_0, a_i\in\mathbb{C}, b_j\in\mathbb{C}\backslash\{0,-1,-2,\ldots\}), (i=1,\ldots,r, j=1,\ldots,s, r\leq s+1; r,s\in\mathbb{N}_0$ *and* $0\leq\gamma<1$.

Proof. Let

$$p(z)=\frac{1}{1-\gamma}\left(\frac{z(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)*\varphi^b_{\lambda_2}f(z))'}{\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)*\varphi^b_{\lambda_2}f(z)}-\gamma\right),\tag{2.1}$$

where $p\in\mathscr{P}$. By using equation (1.8) in (2.1) and differentiating logarithmically, we get

$$\begin{aligned}&\frac{1}{1-\gamma}\left(\frac{z(\mathscr{D}^{m+1,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z))'}{\mathscr{D}^{m+1,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z)}-\gamma\right)\\&\quad=p(z)+\frac{zp'(z)}{(1-\gamma)p(z)+\Big((1-(\lambda_1+\lambda_2)+b)+\gamma\Big)/(\lambda_1+\lambda_2)}.\end{aligned}\tag{2.2}$$

Since $\frac{1}{1-\gamma}\left(\frac{z(\mathscr{D}^{m+1,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z))'}{\mathscr{D}^{m+1,b}_{\lambda_1,\lambda_2}(\alpha_i,\beta_j)f(z)}-\gamma\right)\prec\phi(z)$, and applying Lemma 2.1, it follows that $p\prec\phi$. Hence the required result is obtained.

Theorem 2.3. *Let* $\phi(z)$ *be convex univalent in* $\mathbb{U}$ *with* $\phi(0)=1$ *and* $\Re\{\phi(z)\}\geq 0$. *If* $f\in\mathscr{A}$ *satisfies the condition*

$$\frac{1}{1-\gamma}\left(\frac{z(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_1+1,b_j)f(z))'}{\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_1+1,b_j)f(z)}-\gamma\right)\prec\phi(z)\qquad(z\in\mathbb{U}),$$

then

$$\frac{1}{1-\gamma}\left(\frac{z(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z))'}{\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z)}-\gamma\right)\prec\phi(z)\qquad(z\in\mathbb{U}),$$

for $\lambda_2\geq\lambda_1\geq 0, m,b\in\mathbb{N}_0=\{0,1,2,\ldots\}, a_i\in\mathbb{C}, b_j\in\mathbb{C}\backslash\{0,-1,-2,\ldots\})$, $(i=1,\ldots,r,\ j=1,\ldots,s, r\leq s+1$; $r,s\in\mathbb{N}_0$ *and* $0\leq\gamma<1$.

Proof. Let

$$p(z)=\frac{1}{1-\gamma}\left(\frac{z(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z))'}{\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z)}-\gamma\right),\tag{2.3}$$

where $p\in\mathscr{P}$. By using equation (1.9) in (2.3) and differentiating logarithmically, we get

$$\frac{1}{1-\gamma}\left(\frac{z(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_1+1,b_j)f(z))'}{\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_1+1,b_j)f(z)}-\gamma\right)=p(z)+\frac{zp'(z)}{(1-\gamma)p(z)+(a_1-1)+\gamma},\tag{2.4}$$

since

$$\frac{1}{1-\gamma}\left(\frac{z(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_1+1,b_j)f(z))'}{\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_1+1,b_j)f(z)}-\gamma\right)\prec\phi(z),$$

then applying Lemma 2.1, it follows that $p\prec\phi$. Hence the required result is obtained.

Taking $\phi(z)=(1+Az)/(1+Bz)$, $(-1\le B<A\le 1)$ in Theorem 2.2 and in Theorem 2.3 we have:

Corollary 2.4. *For* $\lambda_2\ge\lambda_1\ge 0, m,b\in\mathbb{N}_0, a_i\in\mathbb{C}, b_j\in\mathbb{C}\backslash\{0,-1,-2,\ldots\}),(i=1,\ldots,r, j=1,\ldots,s, r\le s+1; r,s\in\mathbb{N}_0$ *and* $0\le\gamma<1$ *and* $\Re\{a_i\}>1-\gamma$*, we have*

$$\mathscr{Q}^{m+1,b}_{\lambda_1,\lambda_2}(a_i,b_j;\gamma;A,B)\subset\mathscr{Q}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j;\gamma;A,B),$$

and

$$\mathscr{Q}^{m,b}_{\lambda_1,\lambda_2}(a_1+1,b_j;\gamma;A,B)\subset\mathscr{Q}^{m,b}_{\lambda_1,\lambda_2}(a_1,b_j;\gamma;A,B).$$

Also, by taking $\phi(z)=((1+z)/(1-z))^{\beta}$, $(0<\beta\le 1)$ in Theorem 2.2 and in Theorem 2.3 we have:

Corollary 2.5. *For* $\lambda_2\ge\lambda_1\ge 0, m,b\in\mathbb{N}_0, a_i\in\mathbb{C}, b_j\in\mathbb{C}\backslash\{0,-1,-2,\ldots\})$, *(*$i=1,\ldots,r, j=1,\ldots,s, r\le s+1; r,s\in\mathbb{N}_0$ *and* $0\le\gamma<1$ *and* $\Re\{a_i\}>1-\beta$*, we have*

$$\mathscr{S}^{m+1,b}_{\lambda_1,\lambda_2}(a_i,b_j;\gamma,\beta)\subset\mathscr{S}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j;\gamma,\beta),$$

and

$$\mathscr{S}^{m,b}_{\lambda_1,\lambda_2}(a_1+1,b_j;\gamma;\beta)\subset\mathscr{S}^{m,b}_{\lambda_1,\lambda_2}(a_1,b_j;\gamma,\beta).$$

Corollary 2.6. *For* $\lambda_2\ge\lambda_1\ge 0, m,b\in\mathbb{N}_0, a_i\in\mathbb{C}, b_j\in\mathbb{C}\backslash\{0,-1,-2,\ldots\}), (i=1,\ldots,r, j=1,\ldots,s, r\le s+1; r,s\in\mathbb{N}_0, 0\le\gamma<1$ *and* $\Re\{a_i\}>1-\beta$*, we have*

$$\mathscr{C}^{m+1,b}_{\lambda_1,\lambda_2}(a_i,b_j;\gamma,\beta)\subset\mathscr{C}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j;\gamma,\beta),$$

and

$$\mathscr{C}^{m,b}_{\lambda_1,\lambda_2}(a_1+1,b_j;\gamma;\beta)\subset\mathscr{C}^{m,b}_{\lambda_1,\lambda_2}(a_1,b_j;\gamma,\beta).$$

Proof. We will proof the first relation and by the same method we can proof the second relation

$$\begin{aligned}
f(z)\in\mathscr{C}^{m+1,b}_{\lambda_1,\lambda_2}(a_i,b_j;\gamma,\beta) &\Leftrightarrow zf'(z)\in\mathscr{S}^{m+1,b}_{\lambda_1,\lambda_2}(a_i,b_j;\gamma,\beta)\\
&\Leftrightarrow zf'(z)\in\mathscr{S}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j;\gamma,\beta)\\
&\Leftrightarrow \mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)(zf'(z))\in S^*(\gamma,\beta)\\
&\Leftrightarrow z(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z))'\in S^*(\gamma,\beta)\\
&\Leftrightarrow \mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z)\in C(\gamma,\beta)\\
&\Leftrightarrow f(z)\in\mathscr{C}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j;\gamma,\beta).
\end{aligned}$$

Theorem 2.7. *Let $\phi(z)$ be convex univalent in $\mathbb{U}$ with $\phi(0)=1$ and $\Re\{\phi(z)\}\geq 0$. If $f\in\mathscr{A}$ satisfies the condition*

$$\frac{1}{1-\gamma}\left(\frac{z(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z))'}{\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z)}-\gamma\right)\prec\phi(z)\qquad(0\leq\gamma<1;\,z\in\mathbb{U}),$$

then

$$\frac{1}{1-\gamma}\left(\frac{z(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)F_c(z))'}{\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)F_c(z)}-\gamma\right)\prec\phi(z)\qquad(0\leq\gamma<1;\,z\in\mathbb{U}),$$

where F_c be the generalised Bernardi-Libera-Livington integral operator defined by [11]-[13].

$$F_c(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)\,dt\qquad(c>-1),$$
$$=z+\sum_{n=2}^{\infty}\frac{c+1}{n+c}a_nz^n.\tag{2.5}$$

Proof. From (2.5), we have

$$z(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)F_c(z))'=(c+1)\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z)-c\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)F_c(z).\tag{2.6}$$

Now, let

$$p(z)=\frac{1}{1-\gamma}\left(\frac{z(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)F_c(z))'}{\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)F_c(z)}-\gamma\right),\tag{2.7}$$

where $p\in\mathscr{P}$. Then by using (2.6) in (2.7), we obtain

$$(1-\gamma)p(z)+c+\gamma=\frac{(c+1)\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z)}{\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)F_c(z)}.\tag{2.8}$$

Differentiating both sides of (2.8) logarithmically, we get

$$p(z)+\frac{zp'(z)}{c+\gamma+(1-\gamma)p(z)}=\frac{1}{1-\gamma}\left(\frac{z(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z))'}{\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z)}-\gamma\right),$$

since

$$\frac{1}{1-\gamma}\left(\frac{z(\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z))'}{\mathscr{D}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j)f(z)}-\gamma\right)\prec\phi(z),$$

then applying Lemma 2.1, it follows that $p\prec\phi$. Hence the required result is obtained.

Now, by letting $\phi(z)=(1+Az)/(1+Bz)$, $(-1\leq B<A)$ in Theorem 2.4, we have

Corollary 2.8. *For $\lambda_2\geq\lambda_1\geq 0, m,b\in\mathbb{N}_0, a_i\in\mathbb{C}, b_j\in\mathbb{C}\backslash\{0,-1,-2,\ldots\}),(i=1,\ldots,r, j=1,\ldots,s, r\leq s+1; r,s\in\mathbb{N}_0, c>-\gamma$ and $0\leq\gamma<1$. If $f\in\mathscr{Q}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j;\gamma;A,B)$, then $F_c\in\mathscr{Q}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j;\gamma;A,B)$, where F_c given by (2.5).*

Also, by taking $\phi(z) = ((1+z)/(1-z))^{\beta}$, $(0<\beta\le 1)$ in Theorem 2.4, we have

Corollary 2.9. *For $\lambda_2 \ge \lambda_1 \ge 0, m, b \in \mathbb{N}_0, a_i \in \mathbb{C}, b_j \in \mathbb{C}\backslash\{0,-1,-2,\ldots\}), (i=1,\ldots,r,\ j=1,\ldots,s,$ $r \le s+1$; $r,s\in\mathbb{N}_0, c>-\beta, 0<\beta\le 1$ and $0\le\gamma<1$. If $f\in\mathscr{S}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j;\gamma,\beta)$, then $F_c\in\mathscr{S}^{m,b}_{\lambda_1,\lambda_2}(a_i,$ $b_j;\gamma,\beta)$ where F_c given by (2.5).*

Corollary 2.10. *For $\lambda_2 \ge \lambda_1 \ge 0, m, b \in \mathbb{N}_0, a_i \in \mathbb{C}, b_j \in \mathbb{C}\backslash\{0,-1,-2,\ldots\}), (i=1,\ldots,r,\ j=1,\ldots,s,$ $r \le s+1$; $r,s\in\mathbb{N}_0$, $c>-\beta, 0<\beta\le 1$ and $0\le\gamma<1$. If $f\in\mathscr{C}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j;\gamma,\beta)$, then $F_c\in\mathscr{C}^{m,b}_{\lambda_1,\lambda_2}(a_i,$ $b_j;\gamma,\beta)$, where F_c given by (2.5).*

Proof. Let

$$\begin{aligned} f(z)\in\mathscr{C}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j;\gamma,\beta) &\Leftrightarrow zf'(z)\in\mathscr{S}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j;\gamma,\beta)\\ &\Leftrightarrow F_c(zf'(z))\in\mathscr{S}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j;\gamma,\beta)\\ &\Leftrightarrow z(F_c(z))'\in\mathscr{S}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j;\gamma,\beta)\\ &\Leftrightarrow F_c(z)\in\mathscr{C}^{m,b}_{\lambda_1,\lambda_2}(a_i,b_j;\gamma,\beta). \end{aligned}$$

Note: Some other work related to differential operators and hypergeometric functions can be found in [14]-[16].

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