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MULTIPLE SOLUTIONS OF BOUNDARY VALUE PROBLEM

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Abstract. By means of variational structure and Z_2 group index theory, we obtain multiple solutions of boundary value problems for second-order ordinary differential equations

$$\begin{cases} -(ru')' + qu = \lambda f(t,u), & 0 < t < 1 \\ u'(0) = 0 = \gamma u(1) + u'(1), & \text{where } \gamma \geq 0. \end{cases}$$

1. Introduction

Lynn H. Erbe and Ronald M. Mathsen [6] study the following boundary value problem: $-(ru')' + qu = \lambda f(t,u), 0 < t < 1, \alpha u(0) - \beta u'(0) = 0 = \gamma u(1) + \delta u'(1)$, where $\lambda > 0$ is a parameter, $\alpha, \beta, \gamma, \delta \geq 0$ and $\alpha\delta + \alpha\gamma + \beta\gamma > 0, f \in C((0,1) \times R, R), r \in C([0,1], (0,\infty))$ and $q \in C([0,1], [0,\infty))$.

In this paper we are interested in the study of boundary value problems

$$\begin{cases} -(ru')' + qu = \lambda f(t,u), & 0 < t < 1, \\ u'(0) = 0 = \gamma u(1) + u'(1), & \text{where } \gamma \geq 0. \end{cases} \tag{1.1}$$

By means of variational structure and Z_2 group index theory, we obtain multiple solutions of boundary value problems for (1.1).

Let E be a real Banach space, $S_\rho = \{x \in E : \|x\| = \rho\}$ be the unit sphere of E. A mapping I from E to R will be called a functional. We all know that a critical point of I is a point where $I'(x_0) = 0$ and a critical value of I is a number c such that $I(x_0) = c$ for some critical point x_0. Next, we recall the definition of the Palais-Smale condition.

Definition 1.1. Let $I \in C^1(E,R)$, we say that f satisfies the Palais-Smale condition if every sequence $\{x_n\} \subset E$ such that $\{I(x_n)\}$ is bounded and $I'(x_n) \to 0 (n \to \infty)$ has a converging subsequence.

We let $K = \{x \in E : I'(x) = 0\}$, $K_c = \{x \in E : I'(x) = 0,\ I(x) = c\}$ and $I_c = \{x \in E : I(x) \leq c\}$. $\sum$ denote the set { A: A is a symmetric closed subset of E}, where symmetry means that $x \in A$ implies $-x \in A$. The Z_2 index is defined as following.

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Definition 1.2.([5]) A function $i : \sum \to Z_+ \bigcup \{+\infty\}$ is called Z_2-index, if for $A \in \sum$, $i(A)$ is defined by

(1) If $A = \emptyset, i(A) = 0$.
(2) If $A \neq \emptyset$, there exists a positive number m and a continuous odd map $\varphi : A \to R^m \setminus \{0\}$, then define $i(A)$ to be the minimum of this kind of m. i.e

$$i(A) = \min\{m \in Z_+ : \text{ there is a continuous odd map } \varphi : A \to R^m \backslash \{0\}\}.$$

(3) If $A \neq \emptyset$, and there is none positive integer satisfies (2), define $i(A) = +\infty$.

Denote $i_1(I) = \lim_{c \to -0} i(I_c)$ and $i_2(I) = \lim_{c \to -\infty} i(I_c)$.

We know that if $A \in \sum$ and if there exists an odd homeomorphism of n-sphere onto A then $i(A) = n + 1$; If X is a Hilbert space, and E is an $n-$dimensional subspace of X, and $A \in \sum$ is such that $A \cap E^\perp = \emptyset$ then $i(A) \leq n$. The following Lemma plays an important role in proving our main results.

Lemma 1.3.([5]) *Let $I \in C^1(X, R^1)$ be an even functional which satisfies the Palais-Smale condition and $I(0) = 0$. Then*

(1) *If there exists an m dimensional subspace E of X and $\rho > 0$ with*

$$\sup_{x \in E \bigcap S_\rho} I(x) < 0, \text{ we have } i_1(I) \geq m;$$

(2) *If there exists a j dimensional subspace $\widetilde{E}$ of X with*

$$\inf_{x \in \widetilde{E}^\perp} I(x) > -\infty, \text{ we have } \ i_2(I) \leq j;$$

(3) *If $m \geq j$, (1) and (2) hold, then I at least has $2(m - j)$ distinct critical points.*

2. Main Results

Theorem 2.1. *Let f, $r(t)$ and $q(t)$ be the function satisfying the following conditions:*

(1) $f \in C([0, 1] \times R^1, R^1)$;
(2) $0 < m \leq q(t) \leq M$ *for all* $t \in [0, 1]$;
(3) *There exists* $\alpha > 0$, *such that* $f(t, \alpha) = 0$ *and* $f(t, u) > 0$, $\forall u \in (0, \alpha)$;
(4) $f(t, u)$ *is odd in* u;
(5) $r \in C^1[0, 1]$ *and* $0 \leq r(t) - q(t) \leq N$.

Then for any integer n, there exists λ_n, such that (1.1) has at least $2n$ nontrivial solutions in $C^2[0, 1]$ whenever $\lambda \geq \lambda_n$.

Proof. Set $h : [0,1] \times R^1 \to R^1$

$$h(t,u) = \begin{cases} f(t,\alpha), & u > \alpha, \\ f(t,u), & |u| \le \alpha, \\ f(t,-\alpha), & u < -\alpha \end{cases}$$

Let us consider the functional defined on $H_0^1(0,1)$

$$I(u) = \int_0^1 [\frac{1}{2}r(t)|u'(t)|^2 + \frac{1}{2}q(t)|u(t)|^2 - \lambda G(t,u)]dt + \frac{r(1)}{2}\gamma u^2(1),\ u \in H_0^1(0,1), \quad (2.1)$$

where $G(t,u) = \int_0^u h(t,v)dv$.

The norm $\| \cdot \|$ and inner product $(\ , \ )$ can be defined respectively by

$$\|u\| = \Big(\int_0^1 (|u'(t)|^2 + |u(t)|^2)dt \Big)^{\frac{1}{2}}; \quad (u,v) = \int_0^1 (u'(t)v'(t) + u(t)v(t))dt.$$

Thus $H_0^1(0,1) = W_0^{1,2}(0,1)$ will be a Hilbert space.

Let $E = H_0^1(0,1)$, since $h(t,u)$ is an odd continuous map in u, we know that $I \in C^1(E,R)$ is even in u and $I(0) = 0$.

First, we will show that the critical points of the $I(u)$ are the solutions of (1.1) in $C^2[0,1]$.

By

$$\begin{aligned} &I(u+sv) \\ &= I(u)+s\Big\{ \int_0^1 [r(t)u'(t)v'(t)+q(t)u(t)v(t)-\lambda h(t,u+\theta(t)sv)v(t)]dt + r(1)\gamma u(1)v(1)\Big\} \\ &\quad +\frac{s^2}{2}\Big\{ \int_0^1 (r(t)|v'(t)|^2 + q(t)|v(t)|^2)dt + r(1)\gamma v^2(1)\Big\} \quad \forall u,v \in E, 0 < \theta < 1. \end{aligned} \quad (2.2)$$

We have

$$(I'(u),v) = \int_0^1 [r(t)u'(t)v'(t) + q(t)u(t)v(t) - \lambda h(t,u(t))v]dt + r(1)\gamma u(1)v(1),\ \forall u,v \in E. \quad (2.3)$$

By $I'(u) = 0$, one gets

$$\int_0^1 [r(t)u'(t)v'(t) + q(t)u(t)v(t) - \lambda h(t,u(t))v]dt + r(1)\gamma u(1)v(1) = 0 \quad (2.4)$$

for all $v \in E$.

On the other hand

$$\begin{aligned} &\int_0^1 r(t)u'(t)v'(t)dt + \int_0^1 \frac{d}{dt}(r(t)\frac{du}{dt})vdt \\ &= \int_0^1 r(t)u'(t)v'(t)dt + u'(t)v(t)r(t)|_0^1 - \int_0^1 r(t)u'(t)dv(t) \\ &= r(1)v(1)u'(1) - r(0)u'(0)v(0). \end{aligned} \quad (2.5)$$

So, it is easy to see that

$$\int_0^1 v[\frac{d}{dt}(r(t)\frac{du}{dt}) - q(t)u(t) + \lambda h(t,u(t))]dt$$
$$= r(1)v(1)(u'(1) + \gamma u(1)) - r(0)u'(0)v(0) = 0.$$

Hence we obtain

$$-(ru')' + qu = \lambda h(t,u)$$

Thus the critical points of the $I(u)$ are the solutions of (1.1) in $C^2[0,1]$.

Next, we show that $I(u)$ is bounded from below.
Since $h(t,u(t)) = 0$ whenever $|u(t)| \geq \alpha$, we have

$$\int_0^1 G(t,u(t)) = \int_0^1 \int_0^{u(t)} h(t,v)dvdt \leq \int_0^1 \int_{-\alpha}^{\alpha} |h(t,v)|dvdt$$

Let c $= \int_{-\alpha}^{\alpha} |h(t,v)|dvdt$, then

$$I(u) = \int_0^1 [\frac{1}{2}q(t)(|u'(t)|^2 + |u(t)|^2) + \frac{1}{2}(r(t) - q(t))|u'(t)|^2 - \lambda G(t,u)]dt + \frac{r(1)}{2}\gamma u^2(1).$$

Since $0 < m \leq q(t) \leq M$ and $0 \leq r(t) - q(t) \leq N$ and $\gamma \geq 0$, we have

$$I(u) \geq \frac{m}{2}[\int_0^1 (|u'(t)|^2 + |u(t)|^2)dt] - \int_0^1 \lambda G(t,u(t))dt$$

Thus

$$I(u) \geq \frac{m}{2}\|u\|^2 - \lambda c, \quad \forall u \in E. \tag{2.6}$$

Hence $I(u)$ is bounded from below. Thus $i_2(I) = 0$.

Third, we will verify that $I(u)$ satisfies the Palais-Smale condition.
Suppose that $u_n \subset E$ with $c_1 \leq I(u_n) \leq c_2$ and $I'(u_n) \to 0$ as $n \to \infty$. Then we have

$$\sup\Big\{\int_0^1 [r(t)u_n'(t)v'(t) + q(t)u_n(t)v(t) - \lambda h(t,u_n)v(t) + \gamma r(1)u_n(1)v(1)]dt\Big\} \to 0, \tag{2.7}$$

as $n \to \infty$, for all $u, v \in E$, $\|v\| = 1$.
By

$$I(u) \geq \frac{m}{2}\|u\|^2 - \lambda c$$

we have

$$\|u_n\| \leq c_3 \text{ for some } c_3.$$

Thus $\|u_n\|$ is bounded in $H_0^1(0,1)$. Since $H_0^1(0,1)$ is reflexive, $\{u_n\}$ has a weak converging subsequence $\{u_{n_k}\}$. By [8], we know the convergence is uniform in $C([0,1],R)$, By (2.7) and standard arguments, we have $\{u_{n_k}\}$ is a converging sequence in $H_0^1(0,1)$, hence $I(u)$ satisfies the Palais-Smale condition.

Fourth, we show that Theorem 2.1 holds by Lemma 1.3.
Denote $\beta_k(t) = \frac{\sqrt{2}}{k\pi}\cos k\pi t$, $k = 1, 2, 3, \ldots, n, \ldots$. Then

$$\int_0^1 |\beta_k(t)|^2 dt = \frac{1}{k^2\pi^2},$$
$$\int_0^1 |\beta_k'(t)|^2 dt = 1$$

Consider the n-dimensional subspace

$$E_n = span\{\beta_1(t), \beta_2(t), \ldots, \beta_n(t)\}$$

It is easy to see that E_n is the subset of X symmetric with respect to the origin. For $\rho > 0$, we have

$$E_n \bigcap S_\rho = \Big\{\sum_{k=0}^{n} b_k\beta_k : \sum_{k=0}^{n} b_k^2(1 + \frac{1}{k^2\pi^2}) = \rho^2\Big\}$$

Let ρ with $0 < \rho < \alpha$, for any $u \in E_n \bigcap S_\rho$, we have

$$\max_{0\le t\le 1} u(t) \le \sum_{k=0}^{n} \frac{\sqrt{2}}{k\pi}|b_k| \le \Big(\sum_{k=0}^{n} b_k^2(1 + \frac{1}{k^2\pi^2})\Big)^{\frac{1}{2}} = \|u\| = \rho < \alpha$$

and

$$\int_0^1 (r(t) - q(t))|u(t)|^2 dt \le N\int_0^1 |u(t)|^2 dt \le N\|u\|^2 < N\rho^2$$

By

$$\int_0^1 G(t,u)dt > 0, \quad \forall u \in E_n \bigcap S_\rho$$

and S_ρ is a compact subset in E_n. Let $Q_n = \inf\limits_{u\in E_n \bigcap S_\rho} \int_0^1 G(t,u)dt$, then $Q_n > 0$. Choose $\lambda_n = \frac{1}{2}(3M + N + r(1)\gamma)Q_n^{-1}\rho^2$, it is easy to see that

$$\begin{aligned}
I(u) &= \int_0^1 [\frac{1}{2}q(t)(|u'(t)|^2 + |u(t)|^2) + \frac{1}{2}(r(t) - q(t))|u'(t)|^2 - \lambda G(t,u)]dt + \frac{r(1)}{2}\gamma u^2(1) \\
&< (\frac{M}{2} + \frac{N}{2} + \frac{r(1)}{2}\gamma)\rho^2 - \lambda Q_n \\
&= \frac{1}{2}(M + N + r(1)\gamma)\rho^2 - \frac{1}{2}(3M + N + r(1)\gamma)Q_n^{-1}\rho^2 Q_n \\
&= -M\rho^2 < 0.
\end{aligned}$$

Whenever $\lambda \geq \lambda_n$ and $u \in E_n \bigcap S_\rho$.

Thus $i_1(I) \geq n$ and $i_2(I) = 0$, by Lemma 1.3, I at least has $2(n-0)$ distinct critical points. Hence (1.1) has at least $2n$ nontrivial solutions in $C^2[0,1]$ Whenever $\lambda \geq \lambda_n$.

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