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ON THE GENERALIZED FUGLEDE-PUTNAM THEOREM

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Abstract. In this paper, we prove the following assertions:

(1) If the pair of operators (A, B^*) satisfies the Fuglede-Putnam Property and $S \in \ker(\delta_{A,B})$, where $S \in \mathbf{B}(\mathscr{H})$, then we have
$$\|\delta_{A,B}X + S\| \geq \|S\|.$$

(2) Suppose the pair of operators (A, B^*) satisfies the Fuglede-Putnam Property. If $A^2X = XB^2$ and $A^3X = XB^3$, then $AX = XB$.

(3) Let $A, B \in \mathbf{B}(\mathscr{H})$ be such that A, B^* are p-hyponormal. Then for any $X \in \mathscr{C}_2$, $AX - XB \in \mathscr{C}_2$ implies $A^*X - XB^* \in \mathscr{C}_2$.

(4) Let $T, S \in \mathbf{B}(\mathscr{H})$ be such that T and S^* are quasihyponormal operators. If $X \in \mathbf{B}(\mathscr{H})$ and $TX = XS$, then $T^*X = XS^*$.

1. Introduction

Let $\mathscr{H}$ denote a separable, infinite dimensional Hilbert space. Let $\mathbf{B}(\mathscr{H})$, $\mathbf{C}_2$ and $\mathbf{C}_1$ denote the algebra of all bounded operators acting on $\mathscr{H}$, the Hilbert-Schmidt class and the trace class in $\mathbf{B}(\mathscr{H})$ respectively. It is known that $\mathbf{C}_2$ and $\mathbf{C}_1$ each form a two-sided $^*-$ ideal in $\mathbf{B}(\mathscr{H})$ and $\mathbf{C}_2$ is itself a Hilbert space with inner product
$$\langle X, Y\rangle = \sum \langle Xe_i, Ye_i\rangle = tr(XY^*),$$
where $\{e_i\}$ is any orthonormal basis of $\mathscr{H}$ and $tr(.)$ is the natural trace on $\mathbf{C}_1$. The Hilbert-Schmidt norm of $X \in \mathscr{C}_2$ is given by $\|X\|_2$.

For operators $A, B \in \mathbf{B}(\mathscr{H})$, the generalized derivation $\delta_{A,B}(X)$ as an operator on $\mathbf{B}(\mathscr{H})$ is defined as follows:
$$\delta_{A,B}(X) = AX - XB$$
for all $X \in \mathbf{B}(\mathscr{H})$. When $A = B$, we simply write δ_A for $\delta_{A,A}$.

In [3], Anderson proved that if $N \in \mathbf{B}(\mathscr{H})$ is normal, S is an operator such that $NS = SN$, then for all $X \in \mathbf{B}(\mathscr{H})$
$$\|\delta_N X + S\| \geq \|S\|,$$

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where $\|.\|$ is the usual operator norm. Thus in the sense of [3, Definition 1.2], Anderson's result says that the range of δ_N is orthogonal to the kernel of δ_N, which is just the commutant $\{N\}'$ of N.

Kittaneh [10] extended this result to an arbitrary unitarily invariant norm (for more information about this norm the reader should refer to [10]), and he proved the following theorem.

Theorem 1.1. *If the operator $A, B \in \mathbf{B}(\mathscr{H})$ are normal, then for all $X, S \in \mathbf{B}(\mathscr{H})$ such that $AS = SB$ we have*

$$|||AX - XB + S||| \geq |||S|||.$$

Recall that an operator T in $\mathbf{B}(\mathscr{H})$ is called normal if $T^*T = TT^*$ and p-hyponormal if $(T^*T)^p \geq (TT^*)^p$, where $0 < p \leq 1$. In particular 1-hyponormal is called hyponormal and $\frac{1}{2}$-hyponormal is called semi-hyponormal. The Löwner-Heinz inequality implies that if T is $p-hyponormal$, then it is $q-hyponormal$ for any $0 < q \leq p$. Let $T = U|T|$ be the polar decomposition of T, where U is partial isometry. Then $|T|$ is a positive square root of T^*T and $\ker T = \ker|T| = \ker U$, where $\ker(T)$ denotes the kernel of T. Aluthge [2] introduced the operator $\widetilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ which is called the Aluthge transform, and also shown to satisfy the following result.

Theorem 1.2. *Let $A = U|A|$ be the polar decomposition of a p-hyponormal for $0 < p < 1$ and U is unitary. Then the following assertion holds:*

(1) $\widetilde{A} = |A|^{\frac{1}{2}} U |A|^{\frac{1}{2}}$ is $p + \frac{1}{2}$-hyponormal if $0 < p < \frac{1}{2}$.

(2) $\widetilde{A} = |A|^{\frac{1}{2}} U |A|^{\frac{1}{2}}$ is hyponormal if $\frac{1}{2} \leq p < 1$.

According to [13], $T \in \mathbf{B}(\mathscr{H})$ is called dominant if $ran(T - zI) \subseteq ran(T - zI)^*$, for all $z \in \sigma(T)$, where $ran(T)$ and $\sigma(T)$ denotes the range and the spectrum of T.

It has been shown in [4, Theorem 3.4] that if A, B^* and S are operators in $\mathbf{B}(\mathscr{H})$ such that B^* is p-hyponormal or log-hyponormal, A is dominant and $AS = SB$, then for all $X \in \mathbf{B}(\mathscr{H})$ we have

$$\|AX - XB + S\| \geq \|S\|.$$

2. Main Results

We begin by the following definition of the orthogonality in the sense of Anderson [3, Definition 1.2] which generalize the idea of orthogonality in Hilbert space.

Definition 2.1. Let $\mathbb{C}$ be the field of complex numbers and let X be a normed linear space. If $\|x - \lambda y\| \geq \|\lambda y\|$ for all $\lambda \in \mathbb{C}$ whenever $x, y \in X$, then x is said to be orthogonal to y. Let U and W be two subspace in X. If $\|x + y\| \geq \|y\|$, for all $x \in U$ and for all $y \in W$, then U is said to be orthogonal to W.

Definition 2.2. Let $A, B \in \mathbf{B}(\mathscr{H})$. We say that the pair (A, B^*) satisfies the Fuglede-Putnam Property, if whenever $S \in \ker(\delta_{A,B})$, where $S \in \mathbf{B}(\mathscr{H})$ implies that $S \in \ker(\delta_{A^*,B^*})$.

Theorem 2.3. *Let $A, B, X \in \mathbf{B}(\mathscr{H})$. If the pair of operators (A, B^*) satisfies the Fuglede-Putnam Property and $S \in \ker(\delta_{A,B})$, where $S \in \mathbf{B}(\mathscr{H})$ then we have*

$$\|\delta_{A,B}X + S\| \geq \|S\|.$$

Proof. Since the pair (A, B^*) satisfies the Fuglede-Putnam Property, it follows that $\overline{ranS}$ reduces A, $\ker^{\perp} S$ reduces B. Let $A_1 = A|_{\overline{ranS}}$, $B_1 = B|_{\ker^{\perp} S}$ and let $S_1 : \ker^{\perp} S \longrightarrow \overline{ranS}$ be the quasi-affinity defined by setting $S_1 x = Sx$ for each $x \in \ker^{\perp} S$. Then $\delta_{A_1,B_1}(S_1) = 0 = \delta_{A_1^*,B_1^*}(S_1)$, and it follows that $0 \notin \sigma(A_1)$ and $0 \notin \sigma(B_1)$.
Since the pair (A, B^*) satisfies the Fuglede-Putnam Property, then $\delta_{A,B}(AS) = 0$ implies $A_1^* B_1^* S_1^* = A_1 S_1 B_1^{*-1} = A_1 A_1^* S_1$, then A_1 is normal, similarly B_1 is normal.
Then, with respect to the orthogonal decompositions $\mathscr{H} = \overline{ranS} \oplus (\overline{ranS})$ and $\mathscr{H} = \ker^{\perp} S \oplus \ker S$, A and B can be respectively represented as $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$, $B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$.
Now assume that the operators $S, X : \ker^{\perp} S \oplus \ker S \longrightarrow \overline{ranS} \oplus (\overline{ranS})^{\perp}$ have the matrix representations $\begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$.
Then

$$\|\delta_{A,B}(X) + S\| = \left\| \begin{pmatrix} \delta_{A_1,B_1}(X_1) + S_1 & * \\ * & * \end{pmatrix} \right\| \geq \|\delta_{A_1,B_1}(X_1) + S_1\| = \|S\|,$$

which completes the proof of the theorem.

Next, we prove some commutativity results. Al-Moadjil [1] proved that if N is a normal operator such that $N^2 X = XN^2$ and $N^3 X = XN^3$ for some $X \in \mathbf{B}(\mathscr{H})$, then $NX = XN$. Kittaneh [7] generalize this results for subnormal operators by taking A and B^* subnormal operators. This result was also generalized by Bachir [4] by taking A is a dominant operator and B^* is P-hyponormal operators. In this note, we generalize this result for any pair (A, B^*) satisfying the Fuglede-Putnam Property.

Theorem 2.4. *Let $A, B, X \in \mathbf{B}(\mathscr{H})$ and let the pair of operators (A, B^*) satisfies the Fuglede-Putnam Property. If $A^2 X = XB^2$ and $A^3 X = XB^3$, then $AX = XB$.*

Proof. Let $Y = AX - XB$, then

$$A^2 Y = A^3 X - A^2 XB = XB^3 - XB^3 = 0,$$

$$YB^2 = AXB^2 - XB^3 = A^3 X - A^3 X = 0,$$

and

$$AYB = A^2 XB - AXB^2 = XB^3 = A^3 X = 0.$$

Hence $A(AY - YB) = A^2 Y - AYB = 0$ and $(AY - YB)B = AYB - YB^2 = 0$.
This yields that $AY - YB \in \ker(\delta_{A,B}) \cap ran(\delta_{A,B}) = \{0\}$, therefore $AY - YB = 0$. Hence $Y \in$

$\ker(\delta_{A,B}) \cap ran(\delta_{A,B}) = \{0\}$ is obtained by Theorem 2.3, this implies that $Y = 0$. That is, $AX = XB$.

Kittaneh [7] proved that if $A, B \in \mathbf{B}(\mathscr{H})$ such that $A^2 = B^2$, $A^3 = B^3$, $\ker A \subset \ker A^*$ and $\ker B \subset \ker B^*$, then $A = B$. This results can be generalized to any pair (A, B^*) that satisfies the Fuglede-Putnam Property as follows.

Corollary 2.5. *Let $A, B \in \mathbf{B}(\mathscr{H})$. If the pair of operators (A, B^*) satisfies the Fuglede-Putnam Property and $A^2 = B^2$ and $A^3 = B^3$, then $A = B$.*

Proof. This is an immediate consequence of Theorem 2.3 and Theorem 2.4.

Remark 2.6. Algebraic manipulations and induction show that the powers 2 and 3 in Theorem 2.4 and Corollary 2.5 can be replaced by any two relatively prime powers n and m.

In order to obtain our next result, we require some lemmas.

Lemma 2.7. *Let $T = \begin{pmatrix} T_1 & S \\ 0 & T_2 \end{pmatrix}$ be a quasihyponormal operator on $\mathscr{H} = M \oplus M^\perp$, where M is a T-invariant such that the restriction $T_1 = T|_M$ is normal. Then the range of S is included in* $\ker T_1$. *In particular, if T is injective, every normal part of T reduces T.*

Proof. Let P be the orthogonal projection onto M. Then we have

$$\begin{aligned} PT^*TP = \begin{pmatrix} T_1^*T_1 & 0 \\ 0 & 0 \end{pmatrix} &\le P|T^2|P && \text{(since } T \text{ is quasihyponormal)} \\ &\le \begin{pmatrix} (T_1^{*2}T_1^2)^{\frac{1}{2}} & 0 \\ 0 & 0 \end{pmatrix} && \text{(by Hansen's inequality)} \\ &= \begin{pmatrix} T_1^*T_1 & 0 \\ 0 & 0 \end{pmatrix} && \text{(since } T_1 \text{ is normal).} \end{aligned}$$

Let $|T^2| = \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix}$ be the matrix representation of $|T^2|$ on $H = M \oplus M^\perp$. Then we have $X = T_1^*T_1$.
Since

$$|T^2|^2 = T^{2*}T^2 = \begin{pmatrix} X^2 + YY^* & XY + YZ \\ ZY^* + Y^*X & Y^*Y + Z^2 \end{pmatrix} = \begin{pmatrix} T_1^{*2}T_1^2 & T_1^{*2}T_1S \\ S^*T_1^*T_1^2 & S^*S + T_2^{*2}T_2^2 \end{pmatrix}.$$

Hence

$$X^2 + YY^* = T_1^{*2}T_1^2 = (T_1^*T_1)^2 = X^2.$$

This implies that $Y = 0$. Thus we have

$$|T^2| = \begin{pmatrix} (T_1^*T_1) & 0 \\ 0 & Z^2 \end{pmatrix} \qquad \ge T^*T = \begin{pmatrix} (T_1^*T_1) & T_1^*S \\ S^*T_1 & S^*S + T_2^*T_2) \end{pmatrix},$$

and hence $T_1^* S = 0$. Thus the range of S is included in $\ker T_1^* = \ker T_1$. If T is injective, then so is T_1. Thus the second statement of lemma follows from [6, Lemma 10].

Lemma 2.8. *If T is a quasihyponormal operator, then every normal part of T reduces T.*

Proof. If T is invertible, then T is hyponormal. Hence the assertion holds by Stampfli result [12]. Now, we assume that T is not invertible. Let M be a normal part of T. By Lemma 2.7 T is of the form $\begin{pmatrix} N & S \\ 0 & 0 \end{pmatrix}$ on $M \oplus M^{\perp}$, where N is normal and $ranS \subset \ker N$. It is easy to see that

$$T^{2*}T^2 = \begin{pmatrix} N^{2*}N^2 & N^{2*}NS \\ S^*N^*N^2 & S^*N^*NS \end{pmatrix}$$

and

$$(T^*T)^2 = \begin{pmatrix} (N^*N)^2 + N^*SS^*N & N^*NN^*S + N^*SS^*S \\ S^*NN^*N + S^*SS^*N & S^*NN^*S + S^*SS^*S \end{pmatrix}.$$

Then

$$0 \le T^{2*}T^2 - (T^*T)^2 = \begin{pmatrix} -N^*SS^*N & -N^*SS^*S \\ -S^*SS^*N & -S^*SS^*S \end{pmatrix}.$$

This implies that $S = 0$.

Theorem 2.9. *Let $T \in \mathbf{B}(\mathcal{H})$ be a quasihyponormal operator. Let $L \in \mathbf{B}(\mathcal{H})$ be self-adjoint which satisfies $TL = LT^*$. Then $T^*L = LT$.*

Proof. First, we will show that If $TL = LT^* = 0$, then $T^*L = LT = 0$. Since $\ker T$ reduces T, $TL = 0$ implies that $ranL \subset \ker T \subset \ker T^*$. Hence $\overline{ranT} \subset \ker T$. Therefore we have $T^*L = LT = 0$

Next, we prove the case $TL \neq 0$. Assume that T is quasihyponormal. Using the decomposition $\mathcal{H} = \overline{ranL} \oplus \ker L$, the operators L and T can be represented as follows.

$$L = \begin{pmatrix} L_1 & 0 \\ 0 & 0 \end{pmatrix}, T = \begin{pmatrix} T_1 & S \\ 0 & T_2 \end{pmatrix},$$

where L_1 is self-adjoint with $\ker L_1 = \{0\}$ and T_1 is also quasihyponormal. The assumption $TL = LT^*$ implies that $T_1L_1 = L_1T_1^*$. Since $\ker T_1$ reduces T_1 and L_1, they are of the form $T_1 = T_{11} \oplus 0$ and $L_1 = L_{11} \oplus L_{22}$ on $\overline{ranL} = \overline{\ker|T|} \oplus \ker T_1$. Hence T_{11} is an injective quasihyponormal operator and L_{11} is self-adjoint operator which satisfies $T_{11}L_{11} = L_{11}T_{11}^*$. But this implies that T_{11} is normal. Hence $T_1 = T_{11} \oplus 0$ is also normal. By Fuglede-Putnam Theorem, we see that $T_1^*L_1 = L_1T_1$. since T_1 is normal, $S = 0$, so we have $T^*L = LT$.

Corollary 2.10. *Let $T \in \mathbf{B}(\mathcal{H})$ be a quasihyponormal operator. If $X \in \mathbf{B}(\mathcal{H})$ and $TX = XT^*$, then $T^*X = XT$.*

Proof. Let $X = Y + iZ$ be the cartesian decomposition of X. Then $TX = XT^*$ implies that $TY = YT^*$ and $TZ = ZT^*$. By Theorem 2.9, we have $T^*Y = YT$ and $T^*Z = ZT$. Hence $TX = XT^*$.

Corollary 2.11. *Let $T, S \in \mathbf{B}(\mathscr{H})$ be such that T and S^* are quasihyponormal operators. If $X \in \mathbf{B}(\mathscr{H})$ and $TX = XS$, then $T^*X = XS^*$.*

Proof. Let $A = \begin{pmatrix} T & 0 \\ 0 & S^* \end{pmatrix}$ and $B = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$ on $\mathscr{H} \oplus \mathscr{H}$. Then A is quasihyponormal operator on $\mathscr{H} \oplus \mathscr{H}$ that satisfies

$$AB = \begin{pmatrix} 0 & TX \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & XS \\ 0 & 0 \end{pmatrix} = BA^*,$$

by Corollary 2.10, we have $A^*B = BA$ and therefore $T^*X = XS^*$.

The most recent generalization of the Fuglede-Putnam theorem was obtained by Uchiyama and Tanahashi [15] and can be stated as follows.

Theorem 2.12. *Let $T, S \in \mathbf{B}(\mathscr{H})$ be such that T and S^* are p-hyponormal operators. If $X \in \mathbf{B}(\mathscr{H})$ and $TX = XS$, then $T^*X = XS^*$.*

As an application of the above results we have

Lemma 2.13. *Let V, A and X be operators in $\mathbf{B}(\mathscr{H})$. If V is an isometry, A^* is p-hyponormal, and X is one-one, then $VX = XA$ implies A is unitary.*

Proof. By Corollary 11, $VX = XA$ implies that $V^*X = XA^*$. Multiply the first equation on the left by V^* to get $X = V^*A^*A$. Therefore, $X = XA^*A$. Let $X = UP$ be the polar decomposition of X, then U is unitary and P is one-one. But this implies that $1 = A^*A$. Since A^* is p-hyponormal and $A^*A = 1$, it follows that A is normal and hence unitary.

An attempt to generalize Theorem [7, Theorem 1] to the hyponormal case was made by T. Furuta [5], who obtained the following result.

Theorem 2.14. *If A and B^* are hyponormal operators in $\mathbf{B}(\mathscr{H})$, then for any $X \in \mathscr{C}_2$, $AX - XB \in \mathscr{C}_2$ implies $A^*X - XB^* \in \mathscr{C}_2$.*

In the following theorem, we relax the hypotheses on A and B^* in Theorem 2.14 to p-hyponormality.

Theorem 2.15. *Let $A, B \in \mathbf{B}(\mathscr{H})$ be such that A, B^* are p-hyponormal. Then for any $X \in \mathscr{C}_2$, $AX - XB \in \mathscr{C}_2$ implies $A^*X - XB^* \in \mathscr{C}_2$.*

Proof. Let A, B^* be p-hyponormal for $p \geq \frac{1}{2}$ and let $U|B|$ be the the polar decomposition of B. Then it follows from [2] that the Aluthge transform $\widetilde{B^*}$ of B is hyponormal and satisfies

$$|\widetilde{B}| \leq |B| \leq |\widetilde{B^*}| \tag{1.1}$$

and

$$AY - Y\widetilde{B} \in \mathscr{C}_2 \tag{1.2}$$

, where $Y = XU|B|^{\frac{1}{2}}$. Using the decomposition $\mathscr{H} = \ker Y^{\perp} \oplus \ker Y$, we see that $A, \widetilde{B}, Y$ are of the form

$$A = \begin{pmatrix} A_1 & T \\ 0 & A_2 \end{pmatrix}, \widetilde{B} = \begin{pmatrix} B_1 & 0 \\ S & B_2 \end{pmatrix}, Y = \begin{pmatrix} Y_1 & 0 \\ 0 & 0 \end{pmatrix},$$

where, A_1 is p-hyponormal, B_1^* is hyponormal and Y_1 is one-one mapping with dense range. It follows from equation 1.2 that

$$A_1 Y_1 - Y_1 B_1 \in \mathscr{C}_2. \tag{1.3}$$

Hence A_1, B_1 are normal by [11, Theorem 10], so that $T = 0$ by [15, Lemma 12] and $S = 0$ by [6]. Thus $|B| = |B_1| \oplus J$, for some positive operator J, by equation 1.1 and [15, Lemma13] that $U = \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix}$. Let $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ be a 2×2 matrix representation of X with respect to the decomposition $\mathscr{H} = \ker Y^{\perp} \oplus \ker Y$. Then, $Y = XU|B|^{\frac{1}{2}}$ implies that $Y_1 = X_{11} U_{11} |B_1|^{\frac{1}{2}}$ and hence $\ker B_1 \subset \ker Y_1 = \{0\}$. This shows that B_1 is one-one (hence, it has dense range), so that $U_{12} = 0$ and $B = B_1 \oplus B_3$, for some co-p-hyponormal operator B_3 by [15, Lemma 13]. Since,

$$\begin{pmatrix} Y_1 & 0 \\ 0 & 0 \end{pmatrix} = Y = XU|B|^{\frac{1}{2}} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} U_{11}|B_1|^{\frac{1}{2}} & 0 \\ 0 & U_{22}|A_{33}|^{\frac{1}{2}} \end{pmatrix},$$

we have the following statements.
$X_{12} U_{22} |B_3|^{\frac{1}{2}} = 0$; hence $X_{12} B_3 = 0$ because $B_3 = U_{22}|B_3|$.
$X_{12} U_{11} |B_1|^{\frac{1}{2}} = 0$; hence $X_{12} = 0$ because $U_{11}|B_1|^{\frac{1}{2}}$ has dense range.
$X_{22} U_{22} |B_3|^{\frac{1}{2}} = 0$; hence $X_{22} B_3 = 0$.
The assumption $AX - XB \in \mathscr{C}_2$ imply that
$A_1 X_{11} - X_{11} B_1 \in \mathscr{C}_2$, $X_{12} B_3 = A_1 X_{12} = 0$ and $X_{22} B_3 = A_2 X_{22} = 0$. Since A_1 and B_1 are normal we have $A_1^* X_{11} - X_{11} B_1^* \in \mathscr{C}_2$, by Fuglede-Putnam Theorem. The p-hyponormality of B_3^* shows that $ranB_3^* \subset \overline{B_3}$. Also we have $\ker A_2 \subset \ker A_2^*$ from the fact A_2 is p-hyponormal. Hence, we also have $X_{12} B_3^* = A_1^* X_{12} = 0$ and $X_{22} B_3^* = A_2^* X_{22} = 0$. This implies that $AX - XB \in \mathscr{C}_2$.

Next, we prove the case where $0 < p \le \frac{1}{2}$. Let Y be as above. Then $\widetilde{B^*}$ is $p + \frac{1}{2}$-hyponormal and satisfies $AX - X\widetilde{B} \in \mathscr{C}_2$. Use the same argument as above. We obtain $\widetilde{B} = B_1 \oplus B_2$ on $\mathscr{H} = \ker Y^{\perp} \oplus \ker Y$ and $A = A_1 \oplus A_2$, where B_1 is an injective normal operator and A_1 is also normal. Hence, we have $B = B_1 \oplus B_3$ for some p-hyponormal B_3^*. Again using the same argument as above we obtain the result.

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