TAMKANG JOURNAL OF MATHEMATICS
Volume 37, Number 2, 155-162, Summer 2006

CHARACTERIZATION OF SOME MATRIX CLASSES INVOLVING PARANORMED SEQUENCE SPACES

BINOD CHANDRA TRIPATHY AND MAUSUMI SEN

Abstract. In this article we characterize some matrix classes with one member as $m(p)$ or $m_0(p)$ or $c(p)$ or $c_0(p)$. Some of these results generalize the existing results. Some are new proved in the general setting.

1. Introduction

Throughout the article w, γ, γ_0, c, c_0, ℓ_∞ denote the spaces of *all, summable, summable to zero, convergent, null* and *bounded* sequences respectively. The notion of statistical convergence of sequences was introduced by Fast [3], Schoenberg [12] and Buck [1] independently. Later on the idea was exploited from sequence space point of view and linked with summability by Fridy [4], Šalát [11], Kolk [5], Rath and Tripathy [10], Connor [2], Tripathy([14], [15]) and many others. The basic idea depends on the density of the subsets of N, the set of natural numbers. A subset E of N is said to have density $\delta(E)$ if $\delta(E) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \chi_E(k)$ exists, where χ_E is the characteristic function of E.

A sequence (x_k) is said to be statistically convergent to L if for every $\varepsilon < 0$, $\delta(\{k \in N : |x_k - L| \geq \varepsilon\}) = 0$. We write $x_k \xrightarrow{stat} L$ or stat-lim $x_k = L$.

Tripathy and Sen [17] have generalized the notion on extending it for paranormed sequence spaces. The notion of paranormed sequence space was first studied by Nakano [9] and Simons [13]. Later on it was exploited by Maddox [8], Lascarides and Maddox [7], Lascarides [6], Tripathy [16] and many others. Throughout $p = (p_k) \in \ell_\infty$ denote a non-negative sequence of real numbers. We write $r_k = \frac{1}{p_k}$ for all $k \in N$.

The following known paranormed sequence spaces will be used.

$$c(p) = \{(x_k) \in w : |x_k - L|^{p_k} \to 0, \text{ as } k \to \infty \text{ for some } L\}$$
$$c_0(p) = \{(x_k) \in w : |x_k|^{p_k} \to 0, \text{ as } k \to \infty\}$$
$$\ell_\infty(p) = \{(x_k) \in w : \sup_k |x_k|^{p_k} < \infty\}$$
$$\overline{c}(p) = \{(x_k) \in w : |x_k - L|^{p_k} \xrightarrow{stat} 0, \text{ for some } L\}$$
$$\overline{c}_0(p) = \{(x_k) \in w : |x_k|^{p_k} \xrightarrow{stat} 0\}.$$

Received and revised August 23, 2004.
2000 *Mathematics Subject Classification.* Primary 40C05; Secondary 40A05.
Key words and phrases. Density, paranormed sequence space, statistical convergence.

We write $m(p) = \overline{c}(p) \cap \ell_\infty(p)$ and $m_0(p) = \overline{c}_0(p) \cap \ell_\infty(p)$.

The above spaces are paranormed by $g((x_k)) = \sup_k |x_k|^{\frac{p_k}{M}}$, where $M = \max(1, \sup p_k)$.

2. Preliminaries

The following results will be used for establishing the results of this article.

Lemma 1. (Tripathy and Sen [17], Theorem 2). *The space $m(p)$ is a closed subspace of $\ell_\infty(p)$.*

Lemma 2. (Lascarides [6], Remark, P.494). *Let $p, q \in \ell_\infty$. Then we have $A = (a_{nk}) \in (c_0(p), \ell_\infty(q))$ if and only if there exists an absolute constant $D > 1$ such that*

$$\sup_n \Big\{ \sum_k |a_{nk}| D^{-r_k} \Big\}^{q_n} < \infty. \tag{2.1}$$

In view of the above lemma and using standard techniques we have the following result.

Lemma 3. *Let $(p_k) \in \ell_\infty$. Then $A = (a_{nk}) \in (\ell_\infty, \ell_\infty(p))$ if and only if*

$$\sup_n \Big\{ \sum_k |a_{nk}| \Big\}^{q_n} < \infty. \tag{2.2}$$

Lemma 4. (Lascarides [6], Theorem 9.) *Let $p \in \ell_\infty(p)$. Then $A = (a_{nk}) \in (c(p), c)$ if and only if there exists an absolute constant $D > 1$ such that*

$$\sup_n \sum_k |a_{nk}| D^{-r_k} < \infty, \tag{2.3}$$

$$\lim_{n\to\infty} a_{nk} = \alpha_k \text{ exists for every fixed } k. \tag{2.4}$$

$$\lim_{n\to\infty} \sum_k a_{nk} = \alpha_{nk} \text{ exists.} \tag{2.5}$$

Lemma 5. *Let $0 < \inf p_k \le \sup p_k < \infty$. Then for any linear subspace X of $\ell_\infty(p)$, the following are equivalent:*

$$X \text{ is complete with respect to } g. \tag{2.6}$$

$$\text{If } \sum_k a_{nk} \text{ converges uniformly to } a_n \text{ for each } n \in N \text{ and for each } k \in N,$$
$$a^k = (a_{nk})_{n\in N} \in X, \text{ then } a = (a_n) \in X. \tag{2.7}$$

Proof. (2.6)⇒(2.7). Suppose $\sum_k a_{nk}$ converges uniformly to a_n for each $n \in N$ and $a^k = (a_{nk})_{n\in N} \in X$ for each $k \in N$. Since X is linear, so $s^j = \sum_{k=1}^{j} a^k \in X$, $j \in N$. We have $\|s^j - a\| = \sup\limits_n |\sum_{k>j} a_{nk}|^{\frac{p_k}{M}}$.

Since the convergence of $\sum_k a_{nk}$ is uniform, so given $1 > \varepsilon > 0$, there exists j_0 such that $\|s^j - a\| < \varepsilon^{\frac{h}{M}}$ for all $j > j_0$.
Thus we have $a \in X$, since X is complete.

(2.7)⇒(2.6). Let (x^m), where $x^m = (x_k^m)_{k\in N}$, be a Cauchy sequence in X. Then (x^m) converges (say to x) in $\ell_\infty(p)$, since $\ell_\infty(p)$ is complete. Write $a_{km} = x_k^m - x_k^{m-1}$ $(x_k^0 = 0)$. Then $\sum_m a_{km}$ converges uniformly to x_k and $(a_{km})_{k\in N} = a^m \in X$.

Note 1. Taking $p_n = 1$ for all $n \in N$, one will get Lemma 4 of Rath and Tripathy [10] as particular case.

Lemma 6. *Let $(p_k) \in \ell_\infty$, then $A = (a_{nk}) \in (\gamma, \ell_\infty(p))$ if and only if*

$$T = \sup_n \Big\{ \sum_k |\Delta a_{nk}| \Big\}^{p_n} < \infty, \text{ where } \Delta a_{nk} = a_{nk} - a_{n,k+1}, \text{ for all } k \in N, \quad (2.8)$$

and

$$(a_m) \in \ell_\infty(p). \quad (2.9)$$

Proof. Let $s = (s_k) \in \gamma$ and $S_n = \sum_{k=1}^{n} s_k \to S$ as $n \to \infty$. Then by Abel's summation formula we have

$$A_n s = \sum_{k=1}^{\infty} a_{nk} s_k = S a_{n1} + \sum_{k=1}^{\infty} \Delta a_{nk}(S_k - S). \quad (2.10)$$

The rest of the proof is a routine work in view of Lemma 2 and using standard techniques.

The proof of the following result is a routine work in view of Lemma 6.

Lemma 7. *Let $(p_k) \in \ell_\infty$, then $A = (a_{nk}) \in (\gamma_0, \ell_\infty(p))$ if and only if (2.8) holds.*

3. The Main Results

In this section we establish the results of this article.

Theorem 1. *Let $0 < \inf p_k \le \sup p_k < \infty$. Then $A = (a_{nk}) \in (\gamma, m(p))$ if and only if (2.8) holds and*

$$(a_{nk})_{n\in N} \in m(p), \text{ for every } k \in N. \quad (3.1)$$

Proof. The necessity of (2.8) follows from the inclusion $(\gamma, m(p)) \subset (\gamma, \ell_\infty(p))$ and Lemma 5 and that of (3.1) on considering the sequence $e_k = (0, 0, \cdots 0, 1, 0, \cdots)$ in γ where the only 1 appears at the k-th place.

Sufficiency. Let $s = (s_k) \in \gamma$. We have by (3.1) that $(\Delta a_{nk})_{n\in N} \in m(p)$ for all $k = 1, 2, 3, \cdots$.
Hence we have $Sa_{n1} + \sum_{k\le j_0} \Delta a_{nk}(S_k - S) \in m(p)$, by the linearity.
Next we have

$$\Big|\sum_{k>j_0} \Delta a_{nk}(S_k - S)\Big| \le T^{\frac{1}{h}} \max_{k>j_0} |S_k - S|$$
$$\to 0, \text{ uniformly in } n \text{ as } j_0 \to \infty.$$

Hence by Lemma 1, Lemma 5 and (2.8) we have $As \in m(p)$.
This completes the proof of the theorem.

The proof of the following result is obvious in view of the above result.

Corollary 1. *Let* $0 < \inf p_k \le \sup p_k < \infty$. *Then* $A = (a_{nk}) \in (\gamma_0, m(p))$ *if and only if* (2.8) *holds and* $(a_{nk})_{n\in N} \in m_0(p)$, *for every* $k \in N$.

Following the techniques of Tripathy [15] and the arguments of Theorem 1, we have the following result.

Theorem 2. *Let* $0 < \inf p_k \le \sup p_k < \infty$. *Then* $A = (a_{nk}) \in (\gamma, m(p); P)$ *if and only if* (2.8) *holds and* $(a_{nk} - 1)_{n\in N} \in m_0(p)$, *for all* $k = 1, 2, 3, \cdots$. *In this transformation, the limit is preserved.*

Theorem 3. *Let* $0 < \inf p_k \le \sup p_k < \infty$. *Then* $A = (a_{nk}) \in (\gamma_0, m(p))$ *if and only if* (2.8) *holds and*

$$(\Delta a_{nk})_{n\in N} \in m(p), \text{ for each fixed } k \in N. \tag{3.2}$$

Proof. The necessity of (2.8) follows from the inclusion $(\gamma_0, m(p)) \subset (\gamma_0, \ell_\infty(p))$ and Lemma 7. The necessity of (3.2) follows on considering the series (s_k) whose k-th term is 1 and (k-1)-th term is -1 and rest are zero.
Putting $S = 0$ in (2.10) we have

$$A_n s = \sum_{k=1}^{\infty} \Delta a_{nk} S_k, \quad \text{for all } n = 1, 2, 3, \cdots.$$

Following the techniques of Theorem 1, it can be shown that $As \in m(p)$. This completes the proof of the theorem.

The following result is an easy consequence of the above theorem.

Corollary 2. *Let* $0 < \inf p_k \le \sup p_k < \infty$. *Then* $A = (a_{nk}) \in (\gamma_0, m(p))$ *if and only if* (2.8) *holds and*

$$(\Delta a_{nk})_{n\in N} \in m_0(p), \text{ for all } k = 2, 3, 4, \cdots . \tag{3.3}$$

Note 2. Taking $p_n = 1$ for all $n \in N$ in Theorem 1, Corollary 1, Theorem 2, Theorem 3, and Corollary 2, we have the characterization of the matrix classes (γ, m), (γ, m_0), $(\gamma, m; P)$, (γ_0, m) and (γ_0, m_0) *i.e.* the results of Tripathy [15] as particular cases.

It is well known that $c(p) \subset \ell_\infty$ if $p \in \ell_\infty$. The proofs of the following results are routine works in view of Lemma 5, taking $p_n = 1$ for all $n \in N$ and the technique for establishing the above results.

Proposition 4. *Let* $p \in \ell_\infty$ *and* X *be either* $c(p)$ *or* $c_0(p)$. *Then*

$$A = (a_{nk}) \in (\gamma, X) \text{ if and only if (2.8) holds with } p_n = 1 \text{ for all } n \in N \text{ and } (a_{nk})_{n\in N} \in X \text{ for all } k \in N. \tag{3.4}$$

$$A = (a_{nk}) \in (\gamma_0, X) \text{ if and only if (2.8) holds with } p_n = 1 \text{ for all } n \in N \text{ and } (\Delta a_{nk})_{n\in N} \in X \text{ for all } k \in N. \tag{3.5}$$

Proposition 5. *Let* $p \in \ell_\infty$. *Then* $A = (a_{nk}) \in (\gamma, c(p); P)$ *if and only if* (2.8) *holds and*

$$|a_{nk} - 1|^{p_n} \to 0, \text{ as } n \to \infty \text{ for all } k \in N.$$

Theorem 6. *Let* $0 < \inf p_k \le \sup p_k < \infty$. *Then* $A = (a_{nk}) \in (m(p), c)$ *if and only if* (2.3), (2.4), (2.5) *hold and*

$$\lim_{n\to\infty} \sum_{n\in S} |a_{nk} - \alpha_k| F^{r_k} = 0 \text{ for each } S \subset N \text{ with } \delta(S) = 0 \text{ and for all } F > 1. \tag{3.6}$$

Proof. The necessity of (2.3) is clear in view of the inclusion $(m(p), c) \subset (c(p), c)$. The necessity of (2.4) and (2.5) follow on considering the sequences e_k and $e = (1, 1, 1, \cdots)$ respectively.

Next suppose $A \in (m(p), c)$ but $\lim_{n\to\infty} \sum_{k\in S} |a_{nk} - \alpha_k| F^{r_k} \ne 0$ for some $F > 1$. Let us define the matrix $B = (b_{nk})$ as follows:

$$b_{nk} = \begin{cases} (a_{nk} - \alpha_k) F^{r_k}, & k \in S, \\ 0, & \text{othewise,} \end{cases}$$

for all $n \in N$.

Then

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} |b_{nk}| = \lim_{n\to\infty} \sum_{k\in S} |a_{nk} - \alpha_k| F^{r_k} \ne 0.$$

Thus $B \notin (\ell_\infty, c)$, so there exists $x = (x_k) \in \ell_\infty$ with $\sup_k |x_k| = 1$ such that $Bx \notin c$ *i.e.*

$$\Big(\sum_{k \in S} (a_{nk} - \alpha_k) F^{r_k} x_k \Big) \notin c. \tag{3.7}$$

Define the sequence $y = (y_k)$ as follows:

$$y_k = \begin{cases} x_k F^{r_k}, & k \in S, \\ 0, & \text{othewise,} \end{cases}$$

Then $y \in m(p)$. We have for each $n \in N$,

$$A_n y = \sum_{k=1}^{\infty} a_{nk} y_k = \sum_{k \in S} a_{nk} x_k F^{r_k} = \sum_{k \in S} (a_{nk} - \alpha_k) x_k F^{r_k} + \sum_{k \in S} \alpha_k x_k F^{r_k}.$$

Thus $(A_n y) \notin c$ by (3.7). Hence the necessity of (3.6) follows.

Sufficiency. Let $x = (x_k) \in m(p)$. Then there exists $y = (y_k) \in c(p)$ and $z = (z_k) \in \delta_0(p)$ such that $x_k = y_k + z_k$ for all $k \in N$. By (2.3), (2.4), (2.5) we have $A \in (c(p), c)$. Thus $Ay = (A_n y) \in c$ whenever $y \in c(p)$. Next we have

$$\Big| \sum_{k=1}^{\infty} a_{nk} z_k - \sum_{k=1}^{\infty} \alpha_k z_k \Big| = \Big| \sum_{k \in S} (a_{nk} - \alpha_k) z_k \Big|$$
$$\leq \sum_{k \in S} |a_{nk} - \alpha_k| F^{r_k} \to 0, \text{ as } n \to \infty.$$

Therefore $Az \in c$. Hence $A \in (m(p), c)$. This completes the proof of the theorem.

The proof of the following two results is a routine work in view of the proof of the above result.

Corollary 3. *$A = (a_{nk}) \in (m(p), c; P)$ if and only if the conditions* (2.3)*,* (2.4)*,* (2.5) *with $\alpha = 1$ and* (3.6) *with $\alpha_k = 0$ for each $k = 1, 2, 3, \ldots$ hold.*

Corollary 4. *$A = (a_{nk}) \in (m_0(p), c)$ if and only if* (2.3)*,* (2.4) *and* (3.6) *hold.*

Theorem 7. *Let $p, q \in \ell_\infty$, then $A = (a_{nk}) \in (m_o(p), \ell_\infty(q))$ if and only if* (2.1) *holds and*

$$\sup_n \Big\{ \sum_k |a_{nk}| F^{r_k} \Big\}^{q_n} < \infty \text{ for each } S \subset N \text{ with } \delta(S) = 0 \text{ and for all } F > 1. \tag{3.8}$$

Proof. The necessity of (2.1) follows from the inclusion $(m_0(p), \ell_\infty(q)) \subset (c_0(p), \ell_\infty(q))$. Next let $S \subset N$ be such that $\delta(S) = 0$ and $\sup_n \{\sum_k |a_{nk}| F^{r_k}\}^{q_n} = \infty$ for some $F > 1$. Define a matrix $B = (b_{nk})$ as follows:

$$b_{nk} = \begin{cases} a_{nk} F^{r_k}, & k \in S, \\ 0, & \text{othewise,} \end{cases}$$

for all $n \in N$.

We have

$$\sup_n \Big\{ \sum_{k=1}^{\infty} |b_{nk}| \Big\}^{q_n} = \sup_n \Big\{ \sum_{k\in S} |a_{nk}| F^{r_k} \Big\}^{q_n} = \infty.$$

Hence $B \notin (\ell_\infty, \ell_\infty(q))$. Thus there exists $x = (x_k) \in \ell_\infty$ with $\sup_n |x_k| = 1$ such that

$$\Big(\sum_{k=1}^{\infty} b_{nk} x_k \Big) = \Big(\sum_{k\in S} a_{nk} F^{r_k} x_k \Big) \notin \ell_\infty(q). \tag{3.9}$$

Define the sequence $y = (y_k)$ as follows:

$$y_k = \begin{cases} x_k F^{r_k}, & k \in S, \\ 0, & \text{othewise.} \end{cases}$$

Then clearly $(y_k) \in m_0(p)$. But $(A_n y) \notin \ell_\infty(q)$ by (3.9), as such we arrive at a contradiction. Thus the necessity of (3.8) follows.

Sufficiency. Let $x = (x_k) \in m_0(p)$. Then for a given $0 < \varepsilon < 1$, $\delta(K) = \delta(\{k \in N : |x_k|^{p_k} < \varepsilon\}) = 1$ and $|x_k|^{p_k} < F$ for all $k \in N$. Let $D = \varepsilon^{-1}$, then $D > 1$. If $k \in K$, then $|x_k| < D^{-r_k}$ and for $k \notin K$, we have $|x_k| < F^{r_k}$. We have

$$\begin{aligned} \Big| \sum_{k=1}^{\infty} a_{nk} x_k \Big|^{q_n} &\le C \Big[\Big\{ \sum_{k\in K} |a_{nk}| |x_k| \Big\}^{q_n} + \Big\{ \sum_{k\in K^c} |a_{nk}| |x_k| \Big\}^{q_n} \Big] \\ &\le C \Big[\Big\{ \sum_{k\in K} |a_{nk}| D^{-r_k} \Big\}^{q_n} + \Big\{ \sum_{k\in K^c} |a_{nk}| F^{r_k} \Big\}^{q_n} \Big] \\ &< \infty. \end{aligned}$$

Thus $A \in (m_0(p), \ell_\infty(q))$. This completes the proof of the theorem.

References

[1] R. C. Buck, *Generalized asymptotic density*, Amer. Jour. Math. **75**(1953), 335-346.

[2] J. S. Connor, *The statistical and strong p-Cesàro convergence of sequences*, Analysis **8**(1988), 47-63.

[3] H. Fast, *Sur la convergence statistique*, Colloq. Math. **2**(1951), 241-244.

[4] J. A. Fridy, *Statistical limit points*, Porc. Amer. Math. Sos. **118**(1993), 1187-1192.

[5] E. Kolk, *Matrix summability of statistically convergent sequences*, Analysis **13**(1993), 77-83.

[6] C. G. Lascarides, *A study of certain sequence spaces of Maddox and a generalization of a theorem of Iyer*, Pacific. Jour. Math. **38**(1971), 487-500.

[7] C. G. Lascarides and I. J. Maddox, *Matrix transformation between some classes of sequences*, Proc. Camb. Phil. Soc. **68**(1970), 99-104.

[8] I. J. Maddox, *Paranormed sequence spaces generated by infinite matrices*, Proc. Camb. Phil. Soc. **64**(1968), 335-340.

[9] H. Nakano, *Modulared sequence space*, Proc. Japan Acad. **27**(1951), 508-512.

[10] D. Rath and B. C. Tripathy, *Matrix maps on sequence spaces associated with sets of integers*, Ind. Jour. Pure & Appl. Math. **27**(1996), 197-206.

[11] T. Šalát, *On statistically convergent sequences of real numbers*, Math. Slovaca **30**(1980), 139-150.

[12] I. J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly **66**(1959), 361-375.

[13] S. Simons, *The sequence spaces $\ell(p_v)$ and $m(p_v)$*, Proc. London Math. Soc. **15**(1965), 422-436.

[14] B. C. Tripathy, *Matrix transformations between some classes of sequences*, Jour. Math. Analysis & Appl. **206**(1997), 448-450.

[15] B. C. Tripathy, *Matrix transformations between series and sequences*, Bull. Malaysian Math. Soc. **22**(1998), 17-20.

[16] B. C. Tripathy, *Matrix maps on the power series convergent on the unit disc*, Jour. Analysis **6**(1998), 27-31.

[17] B. C. Tripathy and M. Sen, *On generalized statistically convergent sequences*, Ind. Jour. Pure & Appl. Math. **32**(2001), 1689-1694.

Mathematical Sciences Division, Institute of Advanced Study in Science and Technology, Paschim Boragaon, Garchuk, Guwahati- 781035, Assam, India.

E-mail: tripathybo@yahoo.com

Mathematical Sciences Division, Institute of Advanced Study in Science and Technology, Paschim Boragaon, Garchuk, Guwahati- 781035, Assam, India.