TAMKANG JOURNAL OF MATHEMATICS
Volume 46, Number 1, 85-90, March 2015
doi:10.5556/j.tkjm.46.2015.1601
Available online at http://journals.math.tku.edu.tw/

CLASS OF BOUNDED OPERATORS ASSOCIATED WITH AN ATOMIC SYSTEM

P. SAM JOHNSON AND G. RAMU

Abstract. K-frames, more general than the ordinary frames, have been introduced by Laura Găvruţa in Hilbert spaces to study atomic systems with respect to a bounded linear operator. Using the frame operator, we find a class of bounded linear operators in which a given Bessel sequence is an atomic system for every member in the class.

1. Introduction

Frames in Hilbert spaces were introduced by J. Duffin and A.C. Schaffer [1] in 1952, in the context of nonharmonic Fourier series. After a couple of years, in 1986, frames were brought to life by Daubechies, Grossmann and Meyer [2]. Now frames play an important role not only in the theoretics but also in many kinds of applications, and have been widely applied in signal processing [3], sampling theory [4], coding and communications [5] and so on. The notion of K-frames has been recently introduced by Laura Găvruţa to study the atomic systems with respect to a bounded linear operator K in Hilbert spaces. It is known that K-frames are more general than ordinary frames, and many properties for ordinary frames may not hold for K-frames. Several methods to construct K-frames and the stability of perturbations for the K-frames have been discussed in [6]. In this paper, we construct a frame sequence for the closed subspace $R(K)$ (the range of K) from an atomic system for a closed range operator K. In the end, we find a class of bounded linear operators in which a given Bessel sequence is an atomic system for every member in the class.

Throughout the paper, H is a separable Hilbert space. We denote by $\mathscr{B}(H)$ the space of all bounded linear operators on H. For $T \in \mathscr{B}(H)$, we denote by $R(T)$ the range of T and $N(T)$ the null space of T.

Received February 17, 2014, accepted April 22, 2014.
2010 *Mathematics Subject Classification.* 42C15.
Key words and phrases. Bessel sequences, frames, atomic systems, K-Frames.
Corresponding author: G. Ramu.

2. Notations and preliminaries

Definition 2.1. A family $\{f_i\}_{i=1}^{\infty}$ of vectors in H is called a *Bessel sequence* if there exists a constant $B > 0$ such that

$$\sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \text{ for all } f \in H. \tag{2.1}$$

For a Bessel sequence $\{f_i\}_{i=1}^{\infty}$, an operator $T : \ell_2 \to H$ defined by $T(\{c_i\}_{i=1}^{\infty}) = \sum_{i=1}^{\infty} c_i f_i$, is bounded. T is called the *pre-frame operator* or the *synthesis operator*. The adjoint of T, $T^* : H \to \ell_2$ defined by $T^* f = \{\langle f, f_i\rangle\}_{i=1}^{\infty}$ is called the *analysis operator*. By composing T and T^*, we obtain the *frame operator*

$$Sf = TT^* f = \sum_{i=1}^{\infty} \langle f, f_i\rangle f_i, \text{ for } f \in H. \tag{2.2}$$

Moreover, for each $f \in H$, $\langle Sf, f\rangle = \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2$, S is a bounded positive self-adjoint operator and by Lemma A.6.7 in [7], S has a unique positive square root, denoted by $S^{1/2}$.

Definition 2.2. A Bessel sequence $\{f_i\}_{i=1}^{\infty}$ is a *frame* for H if there is a constant $A > 0$ such that

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2, \text{ for all } f \in H. \tag{2.3}$$

A and B are called the *lower and upper frame bounds* for the frame, they are not unique.

Definition 2.3 ([8])**.** Let $K \in \mathscr{B}(H)$. A sequence $\{f_i\}_{i=1}^{\infty}$ in H is called an atomic system for K, if the following conditions are satisfied :

1. $\{f_i\}_{i=1}^{\infty}$ is a Bessel sequence;
2. there exixts $c > 0$ such that for every $f \in H$ there exists $a_f = \{a_i\}_{i=1}^{\infty} \in \ell_2$ such that $\|a_f\|_{\ell_2} \leq c\|f\|$ and $Kf = \sum_{i=1}^{\infty} a_i f_i$.

Every operator $K \in \mathscr{B}(H)$ has an atomic system. One may ask whether every Bessel sequence $\{f_i\}_{i=1}^{\infty}$ has an operator K which makes $\{f_i\}_{i=1}^{\infty}$ an atomic system for K. The answer is in the affirmative by the following proposition.

Proposition 2.4. *Let $\{f_i\}_{i=1}^{\infty}$ be a Bessel sequence in H. Then $\{f_i\}_{i=1}^{\infty}$ is an atomic system for the frame operator S.*

Proof. Since $\{f_i\}_{i=1}^{\infty}$ is a Bessel sequence in H, the frame operator S defined as in (2.2), is bounded on H. Let $a_f = \{a_i\}_{i=1}^{\infty} = \{\langle f, f_i\rangle\}_{i=1}^{\infty} \in \ell_2$. Now

$$\|a_f\|_{\ell_2}^2 = \|\{\langle f, f_i\rangle\}_{i=1}^{\infty}\|_{\ell_2}^2 = \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \leq B\|f\|^2.$$

As $\|a_f\|_{\ell_2} \leq \sqrt{B}\|f\|$ for each $f \in H$ and $\{f_i\}_{i=1}^{\infty}$ is a Bessel sequence, $\{f_i\}_{i=1}^{\infty}$ is an atomic system for the frame operator S. □

Definition 2.5. [8] Let $K \in \mathscr{B}(H)$. A sequence $\{f_i\}_{i=1}^{\infty}$ in H is called a *K-frame* for H if there exist constants $A, B > 0$ such that

$$A\|K^* f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \text{ for all } f \in H.$$

We call A, B the *lower and upper frame bounds* for the K-frame $\{f_i\}_{i=1}^{\infty}$ respectively.

Definition 2.6. Let $\{f_i\}_{i=1}^{\infty}$ be a sequence in H. We say that $\{f_i\}_{i=1}^{\infty}$ is a *frame sequence* if it is a frame for the closed subspace $\overline{span}\{f_i\}_{i=1}^{\infty}$ of H.

Definition 2.7 ([9])**.** Let H be a Hilbert space, and suppose that $E \in \mathscr{B}(H)$ has a closed range. Then there exists an operator $E^{\dagger} \in \mathscr{B}(H)$ for which

$$N(E^{\dagger}) = R(E)^{\perp}, \quad R(E^{\dagger}) = N(E)^{\perp}, \quad EE^{\dagger} y = y, \quad y \in R(E).$$

We call the operator $E^{\dagger}$ the *pseudo-inverse* of E. This operator is uniquely determined by these properties. In fact, if E is invertible, then we have $E^{-1} = E^{\dagger}$.

Definition 2.8 ([10])**.** Assume that $S, K \in \mathscr{B}(H)$. Then S *majorizes* K if there exists $M > 0$ such that $\|Kx\| \leq M\|Sx\|$ for all $x \in H$.

Theorem 2.9 (Douglas' majorization theorem [10])**.** *Let H be a Hilbert space and $S, K \in \mathscr{B}(H)$. Then the following are equivalent:*

1. $R(K) \subseteq R(S)$;
2. $KK^* \leq \lambda^2 SS^*$ *for some* $\lambda \geq 0$ *(i.e.,* S^* *majorizes* K^**)* ;
3. $K = SU$ *for some* $U \in \mathscr{B}(H)$.

3. Main results

Theorem 3.1 ([8])**.** *Let $\{f_i\}_{i=1}^{\infty}$ be a sequence in H and $K \in \mathscr{B}(H)$. Then the following statements are equivalent:*

1. $\{f_i\}_{i=1}^{\infty}$ *is an atomic system for K;*
2. $\{f_i\}_{i=1}^{\infty}$ *is a K-frame for H;*
3. *there exists a Bessel sequence $\{g_i\}_{i=1}^{\infty}$ such that $Kf = \sum_{i=1}^{\infty} \langle f, g_i\rangle f_i$.*

Theorem 3.2. *[6] Let $\{f_i\}_{i=1}^{\infty}$ be a Bessel sequence in H and $K \in \mathscr{B}(H)$. Then $\{f_i\}_{i=1}^{\infty}$ is a K-frame for H if and only if there exists $A > 0$ such that $S \geq AKK^*$, where S is the frame operator for $\{f_i\}_{i=1}^{\infty}$.*

Each atomic system is associated with a bounded operator on H. We analyse a class of operators in $\mathscr{B}(H)$ associated with a given atomic system.

Theorem 3.3. *Let $K_1, K_2 \in \mathscr{B}(H)$. If $\{f_i\}_{i=1}^{\infty}$ is an atomic system for K_1 and K_2, and α, β are scalars, then $\{f_i\}_{i=1}^{\infty}$ is an atomic system for $\alpha K_1 + \beta K_2$ and $K_1 K_2$.*

Proof. It is given that $\{f_i\}_{i=1}^{\infty}$ is an atomic system for K_1 and K_2, then there are positive constants $A_n, B_n > 0$ $(n = 1, 2)$ such that

$$A_n \|K_n^* f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \leq B_n \|f\|^2, \text{ for all } f \in H. \tag{3.1}$$

By simple calculations, we have

$$\frac{A_1 A_2}{A_2|\alpha|^2 + A_1|\beta|^2} \|(\alpha K_1 + \beta K_2)^* f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2.$$

Hence $\{f_i\}_{i=1}^{\infty}$ satisfies the lower frame condition. And from inequalities (3.1), we get

$$\sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \leq \left(\frac{B_1 + B_2}{2}\right) \|f\|^2, \text{ for all } f \in H.$$

Therefore $\{f_i\}_{i=1}^{\infty}$ is an atomic system for $\alpha K_1 + \beta K_2$.
Now for each $f \in H$, we have $\|(K_1K_2)^* f\|^2 = \|K_2{}^* K_1{}^* f\|^2 \leq \|K_2{}^*\|^2 \|K_1{}^* f\|^2$. Since $\{f_i\}_{i=1}^{\infty}$ is an atomic system for K_1,

$$\frac{\|(K_1K_2)^* f\|^2}{\|K_2{}^*\|^2} \leq \|K_1{}^* f\|^2 \leq \frac{1}{A_1} \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \leq \frac{B_1}{A_1} \|f\|^2.$$

This implies that $\frac{A_1}{\|K_2{}^*\|^2} \|(K_1K_2)^* f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \leq B_1 \|f\|^2$, for all $f \in H$. Therefore $\{f_i\}_{i=1}^{\infty}$ is an atomic system for K_1K_2. □

Corollary 3.4. *If $\{f_i\}_{i=1}^{\infty}$ is an atomic system for $\mathscr{A}$, where $\mathscr{A} \subseteq \mathscr{B}(H)$, then $\{f_i\}_{i=1}^{\infty}$ is an atomic system for any operator in the subalgebra generated by $\mathscr{A}$.*

Corollary 3.5. *If $\{f_i\}_{i=1}^{\infty}$ is an atomic system for a normal operator K, then $\{f_i\}_{i=1}^{\infty}$ is an atomic system for any operator in the subalgebra generated by K and K^*.*

Theorem 3.6. *Let $\{f_i\}_{i=1}^{\infty}$ be an atomic system for a closed range operator K (i.e., K has a closed range). Then there exists a Bessel sequence $\{g_i\}_{i=1}^{\infty}$ such that $\{(K^{\dagger}|_{R(K)})^* g_i\}_{i=1}^{\infty}$ is a frame sequence for $R(K)$.*

Proof. As $\{f_i\}_{i=1}^{\infty}$ is an atomic system, by Theorem 3.1, there exists a Bessel sequence $\{g_i\}_{i=1}^{\infty}$ such that $Kf = \sum_{i=1}^{\infty} \langle f, g_i\rangle f_i$. Since $\{g_i\}_{i=1}^{\infty}$ is a Bessel sequence, there exists $B > 0$ such that

$$\sum_{i=1}^{\infty} |\langle f, g_i\rangle|^2 \leq B\|f\|^2, \text{ for every } f \in H.$$

Hence

$$\sum_{i=1}^{\infty} |\langle f, K^{\dagger^*} g_i\rangle|^2 \le D\|f\|^2, \text{ where } D = B\|K^\dagger\|^2.$$

Using the definition of pseudo-inverse and (3) of Theorem 3.1, for any $f \in R(K)$,

$$f = KK^\dagger f = \sum_{i=1}^{\infty} \langle K^\dagger f, g_i\rangle f_i = \sum_{i=1}^{\infty} \langle f, K^{\dagger^*} g_i\rangle f_i.$$

Also

$$\|f\|^4 = |\langle f, f\rangle|^2 = \left|\langle f, \sum_{i=1}^{\infty} \langle f, K^{\dagger^*} g_i\rangle f_i\rangle\right|^2 \le \sum_{i=1}^{\infty} |\langle f, K^{\dagger^*} g_i\rangle|^2 B\|f\|^2.$$

Therefore $\frac{1}{B}\|f\|^2 \le \sum_{i=1}^{\infty} |\langle f, K^{\dagger^*} g_i\rangle|^2$, for all $f \in R(K)$. Thus $\{(K^\dagger|_{R(K)})^* g_i\}_{i=1}^{\infty}$ is a frame sequence for $R(K)$. □

The following example illustrates that a Bessel sequence $\{f_i\}_{i=1}^{\infty}$ is an atomic system for an operator K but it is not the same for other operator L.

Example 3.7. Let $H = \mathbb{C}^3$ and $\{e_1, e_2, e_3\}$ be an orthonormal basis for H. Define $K : H \to H$ by $Ke_1 = e_1$, $Ke_2 = e_1$, $Ke_3 = e_2$. Then $\{f_i\}_{i=1}^{3} = \{e_1, e_1, e_2\}$ is a K-frame for H. The frame operator is $S = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and its square root is $S^{1/2} = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Let $L = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ and $f = e_3 \in H$. Then $\sum_{i=1}^{3} |\langle f, f_i\rangle|^2 = 0$ and $\|L^* f\|^2 = 4$. Hence $\{f_i\}_{i=1}^{3}$ is not a L-frame for H.

Theorem 3.8. *Let $\{f_i\}_{i=1}^{\infty}$ be a Bessel sequence in H. Then $\{f_i\}_{i=1}^{\infty}$ is a K-frame for H if and only if $K = S^{1/2}T$, for some $T \in \mathcal{B}(H)$.*

Proof. Suppose $\{f_i\}_{i=1}^{\infty}$ is a K-frame, by Theorem 3.2, there exists $A > 0$ such that

$$AKK^* \le S^{1/2}S^{1/2^*}.$$

Then by definition of inner product, for each $f \in H$, $\|K^* f\|^2 \le A^{-1}\|S^{1/2} f\|^2$. Therefore $S^{1/2}$ majorizes K^*. By Douglas' majorization theorem, $K = S^{1/2}T$, for some $T \in \mathcal{B}(H)$.

On the other hand, let $K = S^{1/2}T$ for some $T \in \mathcal{B}(H)$. Then by Douglas' majorization theorem, $S^{1/2}$ majorizes K^*. Then there is a positive A such that

$$\|K^* f\| \le A\|S^{1/2} f\|, \text{ for all } f \in H$$

which implies that $KK^* \le A^2 S$. Hence by Theorem 3.2, $\{f_i\}_{i=1}^{\infty}$ is a K-frame for H. □

Remark 3.9. In the above example, the operator L is not of the form $S^{1/2}T$, for any operator $T \in \mathcal{B}(H)$, because L has a column which is not a linear combination of columns of $S^{1/2}$.

Acknowledgement

The present work of first author was partially supported by National Board for Higher Mathematics (NBHM), Ministry of Atomic Energy, Government of India (Reference No.2/48 (16)/2012/NBHM(R.P.)/R&D 11/9133). The authors would like to thank the anonymous referee for valuable comments and suggestions to improve the quality of the paper.

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Department of Mathematical and Computational Sciences, National Institute of Technology Karnataka, Surathkal, Mangalore 575 025, India.

E-mail: sam@nitk.ac.in

Department of Mathematical and Computational Sciences, National Institute of Technology Karnataka, Surathkal, Mangalore 575 025, India.

E-mail: ramu.geddavalasa@gmail.com