APPLICATIONS OF OSTROWSKI'S VERSION OF THE GRÜSS INEQUALITY FOR TRAPEZOID TYPE RULES

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Abstract. Some applications of the Ostrowski inequality and a perturbed version of it for integral inequalities of the trapezoid type are given.

1. Introduction

In [1], A. Ostrowski proved the following inequality of the Grüss type,

$$
\left| \frac{1}{b-a} \int_a^b f(x) g(x) \, dx - \frac{1}{b-a} \int_a^b f(x) \, dx \cdot \frac{1}{b-a} \int_a^b g(x) \, dx \right|
$$
$$
\leq \frac{1}{8} (b-a)(M-m) \|f'\|_{[a,b],\infty} \tag{1.1}
$$

provided g is integrable on $[a, b]$ and satisfies the condition,

$$
-\infty < m \leq g(x) \leq M < \infty \text{ for a.e. } x \in [a, b] \tag{1.2}
$$

and f is absolutely continuous on $[a, b]$ with $f' \in L_\infty [a, b]$.

The constant $\frac{1}{8}$ is the best possible in (1.1) in the sense that it cannot be replaced by a smaller one.

In this paper we present some applications of (1.1) as well as a perturbed version of it that can also be applied to create some useful integral inequalities.

2. Integral Inequalities

The following trapezoid type result for n-time differentiable functions has been obtained in [2].

Lemma 1. *Let* $f : [a, b] \to \mathbb{R}$ *be a mapping such that the derivative* $f^{(n-1)}$ $(n \geq 1)$ *is absolutely continuous on* $[a, b]$, *then for any* $x \in [a, b]$,

$$\int_a^b f(t)\, dt = \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b) \right]$$

$$+ \frac{1}{n!} \int_a^b (x-t)^n f^{(n)}(t)\, dt. \tag{2.1}$$

Some useful particular cases follow.

(1) For $n = 1$, we can retrieve the identity (see for example [2]),

$$\int_a^b f(t)\, dt = (x-a) f(a) + (b-x) f(b) + \int_a^b (x-t) f'(t)\, dt, \tag{2.2}$$

for each $x \in [a, b]$.

(2) For $n = 2$, we have (see for example [2]),

$$\int_a^b f(t)\, dt = (x-a) f(a) + (b-x) f(b)$$

$$+ \frac{1}{2} \left[(x-a)^2 f'(a) + (b-x)^2 f'(b) \right] + \frac{1}{2} \int_a^b (x-t)^2 f''(t)\, dt. \tag{2.3}$$

If in (2.2) we choose $x = \frac{a+b}{2}$, then we get the trapezoid identity,

$$\int_a^b f(t)\, dt = \frac{f(b) + f(a)}{2} (b-a) + \int_a^b \left(\frac{a+b}{2} - t \right) f'(t)\, dt, \tag{2.4}$$

while the same choice of x will produce, in (2.3), the following perturbed version of the trapezoid identity,

$$\int_a^b f(t)\, dt = \frac{f(b) + f(a)}{2} (b-a) + \frac{(b-a)^2}{8} [f'(a) - f'(b)]$$

$$+ \frac{1}{2} \int_a^b \left(t - \frac{a+b}{2} \right)^2 f''(t)\, dt. \tag{2.5}$$

Consider now the following results.

Theorem 1. *Let* $f : [a, b] \to \mathbb{R}$ *be a mapping such that the derivative* $f^{(n-1)}$ $(n \geq 1)$ *is absolutely continuous on* $[a, b]$ *and there exist real numbers* γ, Γ *such that* $-\infty < \gamma \leq$

$f^{(n)}(x) \leq \Gamma < \infty$ *for a.e.* $x \in [a, b]$, *then,*

$$\left| \int_a^b f(t)\, dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b) \right] \right.$$
$$\left. - \frac{(x-a)^{n+1} + (-1)^n (b-x)^{n+1}}{(n+1)!} \left[f^{(n-1)}; a, b \right] \right|$$
$$\leq \frac{1}{8(n-1)!} (b-a)^2 (\Gamma - \gamma) \left[\frac{1}{2} (b-a) + \left| x - \frac{a+b}{2} \right| \right]^{n-1} \tag{2.6}$$

for any $x \in [a, b]$, *where* $[h; a, b] = \frac{h(b) - h(a)}{b - a}$ *is the divided difference of* h *on* $[a, b]$.

Proof. If we use Ostrowski's inequality (1.1) we may write,

$$\left| \frac{1}{b-a} \int_a^b (x-t)^n f^{(n)}(t)\, dt - \frac{1}{b-a} \int_a^b (x-t)^n\, dt \cdot \frac{1}{b-a} \int_a^b f^{(n)}(t)\, dt \right|$$
$$\leq \frac{1}{8}(b-a)(\Gamma - \gamma)\, n \sup_{t \in [a,b]} |x-t|^{n-1}$$
$$= \frac{n}{8}(b-a)(\Gamma - \gamma) \left[\max(x-a, b-x) \right]^{n-1}$$
$$= \frac{n}{8}(b-a)(\Gamma - \gamma) \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]^{n-1}$$

giving

$$\left| \int_a^b (x-t)^n f^{(n)}(t)\, dt - \frac{(x-a)^{n+1} + (-1)^n (b-x)^{n+1}}{n+1} \left[f^{(n-1)}; a, b \right] \right|$$
$$\leq \frac{n}{8}(b-a)^2 (\Gamma - \gamma) \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]^{n-1}. \tag{2.7}$$

If we divide this by $n!$ and use the representation (2.1), we obtain the desired inequality (2.6).

Remark 1. If $n = 1$ in (2.6), we deduce,

$$\left| \int_a^b f(t)\, dt - (x-a) f(a) - (b-x) f(b) - (b-a) \left(x - \frac{a+b}{2} \right) [f; a, b] \right|$$
$$\leq \frac{1}{8}(b-a)^2 (\Gamma - \gamma),$$

which implies the trapezoid inequality,

$$\left| \int_a^b f(t)\, dt - \frac{f(b) + f(a)}{2}(b-a) \right| \leq \frac{1}{8}(b-a)^2 (\Gamma - \gamma). \tag{2.8}$$

It has been shown in various papers that $\frac{1}{8}$ is a sharp constant (see [4], and [3]).

Remark 2. Further work has yet to be done on comparing, for any $x \in [a,b]$, the bounds provided by (2.6) and the bound,

$$\frac{\Gamma - \gamma}{2} \cdot \frac{1}{n!} I(x,n),$$

where

$$I(x,n) := \frac{1}{(n+1)\sqrt{2n+1}} \left\{ n^2 (b-a) \left[(x-a)^{2n+1} + (b-x)^{2n+1} \right] \right.$$
$$\left. + (2n+1)(x-a)(b-x) \left[(x-a)^n + (b-x)^n \right]^2 \right\}^{\frac{1}{2}}$$

has been obtained in [2].

The Ostrowski inequality (1.1) can also be applied in the following way.

Theorem 2. *Let* $f : [a,b] \to \mathbb{R}$ *be a mapping such that the derivative* $f^{(n)}$ *is absolutely continuous and* $f^{(n+1)} \in L_\infty [a,b]$, *then,*

$$\left| \int_a^b f(t)\, dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b) \right] \right.$$
$$\left. - \frac{(x-a)^{n+1} + (-1)^n (b-x)^{n+1}}{(n+1)!} \left[f^{(n-1)}; a,b \right] \right|$$
$$\leq \begin{cases} \dfrac{1}{8n!} (b-a)^2 \left[\dfrac{1}{2}(b-a) + \left| x - \dfrac{a+b}{2} \right| \right]^n \left\| f^{(n+1)} \right\|_{[a,b],\infty} & \text{if } n \text{ is even,} \\[4mm] \dfrac{1}{8n!} (b-a)^2 \left[(x-a)^n + (b-x)^n \right] \left\| f^{(n+1)} \right\|_{[a,b],\infty} & \text{if } n \text{ is odd.} \end{cases} \tag{2.9}$$

Proof. For $n = 2k$, consider $h_{2k}(t) = (x-t)^{2k}$. It is obvious that

$$\inf_{t \in [a,b]} h_{2k}(t) = 0,$$

$$\sup_{t \in [a,b]} h_{2k}(t) = \max \left[(x-a)^{2k}, (b-x)^{2k} \right] = \left[\max (x-a, b-x) \right]^{2k}$$
$$= \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]^{2k}.$$

If we now apply Ostrowski's inequality (1.1) for h_{2k} and $f^{(2k)}$, we deduce

$$\left| \frac{1}{b-a} \int_a^b (x-t)^{2k} f^{(2k)}(t)\, dt - \frac{1}{b-a} \int_a^b (x-t)^{2k}\, dt \cdot \frac{1}{b-a} \int_a^b f^{(2k)}(t)\, dt \right|$$

$$\leq \frac{1}{8}(b-a) \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]^{2k} \left\| f^{(2k+1)} \right\|_{[a,b],\infty},$$

by a similar argument to that in Theorem 1, proving the first part of (2.9).

For $n = 2k+1$, consider $h_{2k+1}(t) = (x-t)^{2k+1}$, then

$$h'_{2k+1}(t) = -(2k+1)(x-t)^{2k},$$

showing that h_{2k+1} is decreasing on $[a,b]$, and thus,

$$\inf_{t \in [a,b]} h_{2k+1}(t) = h_{2k+1}(b) = (x-b)^{2k+1} = -(b-x)^{2k+1}$$

and

$$\sup_{t \in [a,b]} h_{2k+1}(t) = h_{2k+1}(a) = (x-a)^{2k+1}.$$

Now apply Ostrowski's inequality (1.1) for h_{2k+1} and $f^{(2k+1)}$ we get,

$$\left| \frac{1}{b-a} \int_a^b (x-t)^{2k+1} f^{(2k+1)}(t)\, dt \right.$$

$$\left. - \frac{1}{b-a} \int_a^b (x-t)^{2k+1}\, dt \cdot \frac{1}{b-a} \int_a^b f^{(2k+1)}(t)\, dt \right|$$

$$\leq \frac{1}{8}(b-a) \left[(x-a)^{2k+1} + (b-x)^{2k+1} \right] \left\| f^{(2k+1)} \right\|_{[a,b],\infty},$$

giving, by a similar procedure to that in Theorem 1, the second part of (2.9).

3. A Perturbed Version

The following result holds.

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$ such that the derivative $f' : [a,b] \to \mathbb{R}$ satisfies the condition:*

$$-\infty < \gamma \leq f'(x) \leq \Gamma < \infty \text{ for a.e. } x \in [a,b]. \tag{3.1}$$

If $g : [a,b] \to \mathbb{R}$ is such that,

$$-\infty < m \leq g(x) \leq M < \infty \text{ for a.e. } x \in [a,b], \tag{3.2}$$

then,

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) \, dx - \frac{1}{b-a} \int_a^b f(x) \, dx \cdot \frac{1}{b-a} \int_a^b g(x) \, dx \right.$$

$$\left. - \frac{\gamma + \Gamma}{2} \cdot \frac{1}{b-a} \int_a^b \left(x - \frac{a+b}{2} \right) g(x) \, dx \right| \leq \frac{1}{16} (b-a)(M-m)(\Gamma - \gamma). \quad (3.3)$$

The constant $\frac{1}{16}$ is the best possible.

Proof. We know that,

$$\left| \frac{1}{b-a} \int_a^b h(x) g(x) \, dx - \frac{1}{b-a} \int_a^b h(x) \, dx \cdot \frac{1}{b-a} \int_a^b g(x) \, dx \right|$$

$$\leq \frac{1}{8} (b-a)(M-m) \|h'\|_{[a,b],\infty}, \quad (3.4)$$

provided $-\infty < m \leq g(x) \leq M < \infty$ for a.e. $x \in [a, b]$.

Choose $h(x) = f(x) - \frac{\gamma + \Gamma}{2} (x - \alpha)$, $\alpha \in \mathbb{R}$, then,

$$h'(x) = f'(x) - \frac{\gamma + \Gamma}{2}$$

and since

$$|h'(x)| \leq \frac{\Gamma - \gamma}{2},$$

for a.e. $x \in [a, b]$, we have,

$$\left| \frac{1}{b-a} \int_a^b \left[f(x) - \frac{\gamma + \Gamma}{2} (x - \alpha) \right] g(x) \, dx \right.$$

$$\left. - \frac{1}{b-a} \int_a^b \left[f(x) - \frac{\gamma + \Gamma}{2} (x - \alpha) \right] dx \cdot \frac{1}{b-a} \int_a^b g(x) \, dx \right|$$

$$\leq \frac{1}{16} (b-a)(M-m)(\Gamma - \gamma). \quad (3.5)$$

However,

$$\frac{1}{b-a} \int_a^b \left[f(x) - \frac{\gamma + \Gamma}{2} (x - \alpha) \right] g(x) \, dx$$

$$= \frac{1}{b-a} \int_a^b f(x) g(x) \, dx - \frac{\gamma + \Gamma}{2} \cdot \frac{1}{b-a} \int_a^b (x - \alpha) g(x) \, dx$$

and

$$\frac{1}{b-a}\int_a^b \left[f\left(x\right) - \frac{\gamma+\Gamma}{2}\left(x-\alpha\right)\right] dx \cdot \frac{1}{b-a}\int_a^b g\left(x\right) dx$$

$$= \frac{1}{b-a}\int_a^b f\left(x\right) dx \cdot \frac{1}{b-a}\int_a^b g\left(x\right) dx$$

$$- \frac{\gamma+\Gamma}{2}\cdot \frac{1}{b-a}\int_a^b \left(x-\alpha\right) dx \cdot \frac{1}{b-a}\int_a^b g\left(x\right) dx.$$

By (3.5) we deduce,

$$\left| \frac{1}{b-a}\int_a^b f\left(x\right) g\left(x\right) dx - \frac{\gamma+\Gamma}{2}\cdot \frac{1}{b-a}\int_a^b \left(x-\alpha\right) g\left(x\right) dx \right.$$

$$\left. - \frac{1}{b-a}\int_a^b f\left(x\right) dx \cdot \frac{1}{b-a}\int_a^b g\left(x\right) dx + \frac{\gamma+\Gamma}{2}\frac{1}{b-a}\int_a^b \left(x-\alpha\right) dx \cdot \frac{1}{b-a}\int_a^b g\left(x\right) dx \right|$$

$$\leq \frac{1}{16}\left(b-a\right)\left(M-m\right)\left(\Gamma-\gamma\right). \tag{3.6}$$

Now, observe that,

$$\frac{\gamma+\Gamma}{2}\cdot \frac{1}{b-a}\int_a^b \left(x-\alpha\right) g\left(x\right) dx$$

$$- \frac{\gamma+\Gamma}{2}\cdot \frac{1}{b-a}\int_a^b \left(x-\alpha\right) dx \cdot \frac{1}{b-a}\int_a^b g\left(x\right) dx$$

$$= \frac{\gamma+\Gamma}{2}\cdot \frac{1}{\left(b-a\right)^2}\left\{ \left(b-a\right)\left[\int_a^b xg\left(x\right) dx - \alpha \int_a^b g\left(x\right) dx \right. \right.$$

$$\left. \left. - \left[\int_a^b x dx - \alpha \left(b-a\right)\right] \int_a^b g\left(x\right) dx \right] \right\}$$

$$= \frac{\gamma+\Gamma}{2}\cdot \frac{1}{\left(b-a\right)^2}\left[\left(b-a\right)\int_a^b xg\left(x\right) dx - \int_a^b x dx \cdot \int_a^b g\left(x\right) dx \right]$$

$$= \frac{\gamma+\Gamma}{2}\left[\frac{1}{b-a}\int_a^b xg\left(x\right) dx - \frac{a+b}{2}\cdot \frac{1}{b-a}\int_a^b g\left(x\right) dx \right]$$

and by (3.6) we deduce the desired result.

The fact that $\frac{1}{16}$ is the best constant, follows by Ostrowski's result on choosing $\gamma = -\|f'\|_{[a,b],\infty}$, $\Gamma = \|f'\|_{[a,b],\infty}$. We omit the details.

In what follows, an application of the above perturbed version of Ostrowski's inequality is given.

Using Lemma 1, we have the identity, (see also [2]),

$$\int_a^b f(t)\, dt = \sum_{k=0}^{n-1} \frac{1}{(k+1)!2^{k+1}} (b-a)^{k+1} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right]$$

$$+ \frac{1}{n!} \int_a^b \left(\frac{a+b}{2} - t \right)^n f^{(n)}(t)\, dt. \tag{3.7}$$

We can further state the following result.

Theorem 4. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that the derivative $f^{(n-1)}$ $(n \geq 1)$ is absolutely continuous on $[a,b]$ and there exists the real numbers m, M such that $-\infty < m \leq f^{(n)}(x) \leq M < \infty$ for a.e., $x \in [a,b]$. We then have the inequality:*

$$\left| \int_a^b f(t)\, dt - \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] - \frac{(b-a)^{n+1}}{2^n (n+1)!} \left[f^{(n-1)}; b, a \right] \right|$$

$$\leq \frac{(b-a)^{n+1}}{(n-1)!2^{n+2}} (M - m) \tag{3.8}$$

if n is even and

$$\left| \int_a^b f(t)\, dt - \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right.$$

$$\left. + \frac{(b-a)^n}{(n-1)!2^{n-1}} \left[\left[f^{(n-2)}; b, a \right] - \frac{f^{(n-1)}(b) + f^{(n-1)}(a)}{2} \right] \right|$$

$$\leq \frac{(b-a)^{n+1}}{(n-1)!2^{n+3}} (M - m) \tag{3.9}$$

if n is odd.

Proof. The proof is by application of Theorem 3 for $g \to f^{(n)}$ and $f \to \left(\frac{a+b}{2} - \cdot \right)^n$. We first observe that,

$$\int_a^b \left(\frac{a+b}{2} - t \right)^n dt = \frac{(b-a)^{n+1} [1 + (-1)^n]}{2^{n+1}(n+1)},$$

$$\frac{1}{b-a} \int_a^b f^{(n)}(t)\, dt = \left[f^{(n-1)}; b, a \right],$$

$$\gamma = \inf_{t \in [a,b]} \left[\frac{d}{dt} \left(\frac{a+b}{2} - t \right)^n \right] = \begin{cases} -n \left(\dfrac{b-a}{2} \right)^{n-1} & \text{if } n \text{ is even,} \\[2em] -n \left(\dfrac{b-a}{2} \right)^{n-1} & \text{if } n \text{ is odd,} \end{cases}$$

$$\Gamma = \sup_{t \in [a,b]} \left[\frac{d}{dt} \left(\frac{a+b}{2} - t \right)^n \right] = \begin{cases} n \left(\dfrac{b-a}{2} \right)^{n-1} & \text{if } n \text{ is even,} \\[2em] 0 & \text{if } n \text{ is odd,} \end{cases}$$

and then,

$$\frac{\gamma + \Gamma}{2} = \begin{cases} 0 & \text{if } n \text{ is even,} \\[2em] -n \left(\dfrac{b-a}{2} \right)^{n-1} & \text{if } n \text{ is odd,} \end{cases}$$

and

$$\Gamma - \gamma = \begin{cases} 2n \left(\dfrac{b-a}{2} \right)^{n-1} & \text{if } n \text{ is even,} \\[2em] n \left(\dfrac{b-a}{2} \right)^{n-1} & \text{if } n \text{ is odd.} \end{cases}$$

Also,

$$\int_a^b \left(\frac{a+b}{2} - t \right) f^{(n)}(t) \, dt$$

$$= f^{(n-1)}(t) \left(\frac{a+b}{2} - t \right) \Big|_a^b + \int_a^b f^{(n-1)}(t) \, dt$$

$$= -(b-a) \frac{f^{(n-1)}(b) + f^{(n-1)}(a)}{2} + \left[f^{(n-2)}; b, a \right] (b-a).$$

Now, when n is even, we have,

$$\left| \int_a^b \left(\frac{a+b}{2} - t \right)^n f^{(n)}(t) \, dt - \int_a^b \left(\frac{a+b}{2} - t \right)^n dt \cdot \frac{1}{b-a} \int_a^b f^{(n)}(t) \, dt \right|$$

$$\leq \frac{1}{16} (b-a)^2 \left[2n \left(\frac{b-a}{2} \right)^{n-1} \right] (M - m),$$

i.e.,

$$\left| \int_a^b \left(\frac{a+b}{2} - t \right)^n f^{(n)}(t) \, dt - \frac{(b-a)^{n+1}}{2^n (n+1)!} \cdot \left[f^{(n-1)}; b, a \right] \right| \leq \frac{n (b-a)^{n+1} (M - m)}{2^{n+2}}.$$

Dividing by $n!$ and using (3.7), we get (3.8).

Now, when n is odd, we have,

$$\left| \int_a^b \left(\frac{a+b}{2} - t \right)^n f^{(n)}(t)\, dt - \int_a^b \left(\frac{a+b}{2} - t \right)^n dt \cdot \frac{1}{b-a} \int_a^b f^{(n)}(t)\, dt \right.$$

$$\left. + n \left(\frac{b-a}{2} \right)^{n-1} \left[(b-a) \left[f^{(n-2)}; b, a \right] - (b-a) \frac{f^{(n-1)}(b) + f^{(n-1)}(a)}{2} \right] \right|$$

$$\leq \frac{1}{16} (b-a)^2 (M-m) n \left(\frac{b-a}{2} \right)^{n-1},$$

i.e.,

$$\left| \int_a^b \left(\frac{a+b}{2} - t \right)^n f^{(n)}(t)\, dt + \frac{n(b-a)^n}{2^{n-1}} \left[\left[f^{(n-2)}; b, a \right] - \frac{f^{(n-1)}(b) + f^{(n-1)}(a)}{2} \right] \right|$$

$$\leq \frac{n(b-a)^{n+1}(M-m)}{2^{n+3}},$$

and dividing by $n!$ and using (3.7) we have (3.9).

The theorem is thus proved.

The second approach is incorporated in the following theorem.

Theorem 5. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that the derivative $f^{(n)}$ is absolutely continuous on $[a,b]$ and assume that there exist constants $\phi, \Phi \in \mathbb{R}$ such that $-\infty < \phi \leq f^{(n+1)}(x) \leq \Phi < \infty$ for a.e. $x \in [a,b]$, then,*

$$\left| \int_a^b f(t)\, dt - \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)! 2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right.$$

$$\left. - \frac{[1 + (-1)^n](b-a)^{n+1}}{2^{n+1}(n+1)!} \left[f^{(n-1)}; b, a \right] + \frac{\phi + \Phi}{2} \cdot \frac{[(-1)^n + 1](b-a)^{n+2}}{2^{n+2}(n+2) n!} \right|$$

$$\leq \begin{cases} \dfrac{(b-a)^{n+2}}{2^{n+4} n!} (\Phi - \phi) \text{ if } n \text{ is even,} \\[4mm] \dfrac{(b-a)^{n+2}}{2^{n+3} n!} (\Phi - \phi) \text{ if } n \text{ is odd.} \end{cases} \qquad (3.10)$$

Proof. We apply Theorem 3 for the choices $g \to \left(\frac{a+b}{2} - \cdot \right)^n$ and $f \to f^{(n)}$.

We observe that,

$$\int_a^b \left(t - \frac{a+b}{2} \right) \left(\frac{a+b}{2} - t \right)^n dt = - \int_a^b \left(\frac{a+b}{2} - t \right)^{n+1} dt$$

$$= - \frac{(b-a)^{n+2} \left[(-1)^{n+1} + 1 \right]}{2^{n+2}(n+2)}.$$

By (3.3) we then get,

$$\left| \int_a^b \left(\frac{a+b}{2} - t \right)^n f^{(n)}(t)\, dt - \int_a^b \left(\frac{a+b}{2} - t \right)^n dt \cdot \frac{1}{b-a} \int_a^b f^{(n)}(t)\, dt \right.$$

$$\left. - \frac{\phi + \Phi}{2} \int_a^b \left(t - \frac{a+b}{2} \right) \left(\frac{a+b}{2} - t \right)^n dt \right|$$

$$\leq \frac{1}{16} (b-a)^2 (\Phi - \phi) \begin{cases} \dfrac{(b-a)^n}{2^n} & \text{if } n \text{ is even,} \\[2ex] \dfrac{(b-a)^n}{2^{n-1}} & \text{if } n \text{ is odd,} \end{cases}$$

that is,

$$\left| \int_a^b \left(\frac{a+b}{2} - t \right)^n f^{(n)}(t)\, dt - \frac{(b-a)^{n+1} \left[1 + (-1)^n \right]}{2^{n+1}(n+1)} \left[f^{(n-1)}; b, a \right] \right.$$

$$\left. + \frac{\phi + \Phi}{2} \cdot \frac{(b-a)^{n+2} \left[(-1)^{n+1} + 1 \right]}{2^{n+2}(n+2)} \right| \leq \begin{cases} \dfrac{(b-a)^{n+2}}{2^{n+4}} (\Phi - \phi) & \text{if } n \text{ is even,} \\[2ex] \dfrac{(b-a)^{n+2}}{2^{n+3}} (\Phi - \phi) & \text{if } n \text{ is odd,} \end{cases}$$

and the inequality (3.10) is proved.

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