EQUALITY OF GRAPHOIDAL AND ACYCLIC GRAPHOIDAL COVERING NUMBER OF A GRAPH

INDRA RAJASINGH AND P. ROUSHINI LEELY PUSHPAM

Abstract. A graphoidal cover of a graph \( G \) is a collection \( \psi \) of (not necessarily open) paths in \( G \) such that every vertex of \( G \) is an internal vertex of at most one path in \( \psi \) and every edge of \( G \) is in exactly one path in \( \psi \). If no member of \( \psi \) is a cycle, then \( \psi \) is called an acyclic graphoidal cover of \( G \). The minimum cardinality of a graphoidal cover is called the graphoidal covering number of \( G \) and is denoted by \( \eta \) and the minimum cardinality of an acyclic graphoidal cover is called an acyclic graphoidal covering number of \( G \) and is denoted by \( \eta_a \). In this paper we characterize the class of graphs for which \( \eta = \eta_a \).

1. Introduction

The concept of graphoidal cover was introduced by B.D. Acharya and E. Sampathkumar in 1987 and a study of the graphoidal covering number was initiated by them [1]. Since then this area of research has been explored by several authors [2, 3, 10, 11, 12, 13, 19]. The concept of an acyclic graphoidal cover was introduced by Suresh Suseela and pursued by Arumugam et al. [14, 15].

By a graph \( G = (V, E) \) we mean a finite, undirected, connected graph without loops or multiple edges. The order and size of \( G \) are denoted by \( p \) and \( q \) respectively. For graph theoretic terminology we refer to Harary [16].

If \( P = (v_0, v_1, \ldots, v_n) \) is a path or a cycle in \( G \), \( v_1, \ldots, v_{n-1} \) are called internal vertices of \( P \). If \( P = (v_0, v_1, \ldots, v_n) \) and \( Q = (v_n = w_0, w_1, \ldots, w_m) \) are two paths in \( G \) then the walk obtained by concatenating \( P \) and \( Q \) at \( v_n \) is denoted by \( P \circ Q \).

Definition 1.1. A graphoidal cover of a graph \( G \) is a set \( \psi \) of (not necessarily open) paths in \( G \) satisfying the following conditions.

(i) Every path in \( \psi \) has at least two vertices.
(ii) Every vertex of \( G \) is an internal vertex of at most one path in \( \psi \).
(iii) Every edge of \( G \) is in exactly one path in \( \psi \).

\( \psi \) is called an acyclic graphoidal cover of \( G \) if no member of \( \psi \) is a cycle in \( G \). The minimum cardinality of (an acyclic) a graphoidal cover of \( G \) is called the (acyclic) graphoidal covering number of \( G \) and is denoted by \( (\eta_a) \) \( \eta \).

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**Definition 1.2.** Let $\psi$ be a collection of internally disjoint paths in $G$. A vertex of $G$ is said to be an interior vertex of $\psi$ if it is an internal vertex of some path in $\psi$. Any vertex which is not an interior vertex of $\psi$ is said to be an exterior vertex of $\psi$.

**Theorem 1.3.** ([18]) For any graphoidal cover $\psi$ of $G$, let $t_\psi$ denote the number of exterior vertices of $\psi$. Let $t = \min t_\psi$, where the minimum is taken over all graphoidal covers of $G$. Then $\eta = q - p + t$ where $p$ and $q$ denote respectively the order and size of $G$.

**Corollary 1.4.** ([18]) For any graph $G$, $\eta \geq q - p$. Moreover the following are equivalent.

(i) $\eta = q - p$.

(ii) There exists a graphoidal cover without exterior vertices.

(iii) There exists a set of internally disjoint and edge disjoint paths without exterior vertices.

**Theorem 1.5.** ([17]) For any graph $G$ with $\delta \geq 3$, $\eta = q - p$.

**Remark 1.6.** Results analogous to Theorem 1.3, Corollary 1.4 and Theorem 1.5 are also true for an acyclic graphoidal covering number of a graph.

**Theorem 1.9.** ([4]) Let $G$ be a 2-connected graph with $p \geq 3$. Then $G \notin \mathcal{F}$ if and only if $G$ is a cycle or a cycle with exactly one chord or a theta graph.

**Theorem 1.10.** ([4]) Let $G$ be a 2-edge connected graph with $\delta = 2$. Then $G \notin \mathcal{F}$ if and only if every block of $G$ is a cycle or a cycle with exactly one chord or a theta graph and at most one block of $G$ is not a cycle.

**Theorem 1.11.** ([6]) Let $G$ be a connected graph with $\delta = 2$ and edge connectivity one. Then $G \notin \mathcal{F}$ if and only if there exists a cut edge $e$ of $G$ such that at least one component of $G - e$ is a graph, all of whose blocks are cycles.
Lemma 1.12. ([6]) Let $G$ be a 2-edge connected graph such that exactly one block of $G$ is either a cycle with exactly one chord or a theta graph and all other blocks are cycles. Let $v \in V(G)$. Then there exists a minimum graphoidal cover $\psi$ of $G$ such that $v$ is the only vertex exterior to $\psi$ and there exists a path in $\psi$ which contains $v$ as an exterior vertex.

Theorem 1.13. ([7]) Let $G$ be a 2-connected graph with $p \geq 3$. Then $G \in \mathcal{F}_a$ if and only if $G \not\in G(f)$.

Theorem 1.14. ([7]) Let $G$ be a graph with $\delta > 1$ and connectivity one. Then $G \not\in F_a$ if and only if at least one end block of $G$ is a member of $G(f)$ with $f$ as a cut vertex of $G$.

Theorem 1.15. ([5]) Let $G \in G(f)$ and $\Delta(G) \geq 3$. Let $v$ be a vertex of $G$ with $\deg v \neq \Delta$. Then there exists a minimum acyclic graphoidal cover $\psi$ of $G$ such that $v$ is the only vertex exterior to $\psi$ and $\eta_a = |\psi| = \Delta - 1$.

2. Main Results

Lemma 2.1. Let $G$ be a graph whose blocks are all cycles. Let $u, v \in V(G)$ with $u \neq v$. Then there exists a graphoidal cover $\psi$ of $G$ such that $u$ and $v$ are the only vertices exterior to $\psi$ and a path $P$ in $\psi$ which contains $u$ and $v$ as end vertices.

Proof. Let $C_1, C_2, \ldots, C_k$ be the blocks of $G$. By hypothesis each $C_i$ is a cycle, $i = 1, 2, \ldots, k$. Let $P$ be a $(u, v)$-path in $G$. Let $S_1, S_2, \ldots, S_s$, $s \in \{1, 2, \ldots, k\}$ be the segments of $P$ that are part of the cycles $C_1, C_2, \ldots, C_s$. Let $R_i = C_i \setminus S_i$, $1 \leq i \leq s$. Then $\psi = \{P, R_1, R_2, \ldots, R_s, C_{s+1}, \ldots, C_k\}$ is a graphoidal cover of $G$ with $u$ and $v$ as the only vertices exterior to $\psi$.

Theorem 2.2. Let $G$ be a 2-edge connected graph. Then either $\eta = q - p$ or $\eta = q - p + 1$.

Proof. Suppose $\eta \neq q - p$. By Theorem 1.10, every block of $G$ is a cycle or a cycle with exactly one chord or a theta graph and at most one block of $G$ is not a cycle. We now prove that $\eta = q - p + 1$ by induction on $m$ where $m$ is the number of blocks of $G$. When $m = 1$, $G$ is either a cycle or a cycle with exactly one chord or a theta graph. Clearly for any minimum graphoidal cover $\psi$ of $G$, exactly one vertex of $G$ is exterior to $\psi$ and hence by Theorem 1.3, $\eta = q - p + 1$.

We now assume that $\eta = q - p + 1$ for all 2-edge connected graphs with $m$ blocks, $m \geq 1$. Let $G$ be a 2-edge connected graph with $m + 1$ blocks. Let $C$ be an end block which is a cycle and $v \in V(C)$ be a cut vertex of $G$. Let $G'$ be the subgraph of $G$ obtained by removing all the vertices of $C - v$. Clearly $G'$ has $m$ blocks. By induction
hypothesis $\eta(G') = q' - p' + 1$ when $p'$ and $q'$ are the order and size of $G'$ respectively.

Now $\eta(G) = \eta(G') + 1$

$= q' - p' + 2$

$= q - p + 1$. This completes the induction and the proof.

**Theorem 2.3.** Let $G$ be a graph with $\kappa' = 1$ and $\delta > 1$, where $\kappa'$ is the edge connectivity of $G$. Let $S = \{e \mid e$ is a cut edge of $G$ and the blocks of at least one component of $G - e$ are all cycles\}. Let $\mathcal{H}_G = \{H_1, H_2, \ldots, H_m\}$, be the collection of all such components. Let $v_i \in V(H_i)$, $1 \leq i \leq m$. Then $\eta = q - p + m$ and there exists a minimum graphoidal cover $\psi$ of $G$ such that $v_i$, $1 \leq i \leq m$ are the only vertices exterior to $\psi$ and there exist paths in $\psi$ which contain $v_i$ as an end vertex, $1 \leq i \leq m$.

**Proof.** Since contracting an edge incident with a cut vertex of degree 2 does not affect the value of $q - p$ and $\eta$, we may assume without loss of generality that any cut vertex has degree at least 3. We observe that if $m = 0$, then the result follows from Theorem 1.11. Hence we assume that $m > 0$ and prove the result by induction on $r$, where $r$ is the number of cut edges of $G$. Since $\kappa' = 1$, $r \geq 1$. Suppose $r = 1$ and let $e = x_1x_2$ be the cut edge of $G$. Let $G_1$ and $G_2$ be the components of $G - e$ with $x_1 \in V(G_1)$ and $x_2 \in V(G_2)$. Clearly $G_1$ and $G_2$ are 2-edge connected graphs. If both $G_1$ and $G_2$ are in $\mathcal{H}_G$ then by Lemma 2.1 there exists a graphoidal cover $\psi_1$ of $G_1$, $i = 1, 2$ and a $(v_i, x_i)$-path $P_i$ in $G_i$ such that $v_i$ and $x_i$ are the only vertices exterior to $\psi_1$, $i = 1, 2$. Then by Theorem 1.3 $\psi = (\psi_1 \setminus \{P_1\}) \cup (\psi_2 \setminus \{P_2\}) \cup \{P_1 \circ e \circ P_2^{-1}\}$ is a minimum graphoidal cover such that $v_1$ and $v_2$ are the only vertices exterior to $\psi$. Thus $\eta = q - p + 2$. Suppose $G_1 \notin \mathcal{H}_G$. For $G_2 \in \mathcal{H}_G$ we define $\psi_2$ as before. If $G_1 \in F$, then there exists a minimum graphoidal cover $\psi_1$ of $G_1$ such that all vertices of $G_1$ are interior to $\psi_1$. Then by Theorem 1.3 $\psi = \psi_1 \cup (\psi_2 \setminus \{P_2\}) \cup (e \circ P_2^{-1})$ is a minimum graphoidal cover of $G$ with $v_2$ as the only vertex exterior to $\psi$, where $e \circ P_2^{-1}$ is a path in $\psi$ which contains $v_2$ as an end vertex. If $G_1 \notin F$, then since $G_1 \notin \mathcal{H}_G$, by Theorem 1.10 exactly one block of $G_1$ is a cycle with exactly one chord or a theta graph and all other blocks are cycles. Hence by Lemma 1.12 there exists a minimum graphoidal over $\psi_1$ of $G_1$ and a path $P_1$ in $\psi_1$ with $x_1$ as its terminus such that $x_1$ is the only vertex exterior to $\psi_1$. Then $\psi = (\psi_1 \setminus \{P_1\}) \cup (\psi_2 \setminus \{P_2\}) \cup \{P_1 \circ e \circ P_2^{-1}\}$ is a minimum graphoidal cover of $G$ such that $v_2$ is the only vertex exterior to $\psi$ and $P_1 \circ e \circ P_2^{-1}$ is a path in $\psi$ which contains $v_2$ as an end vertex. Thus $\eta = q - p + 1$. Hence the result is true for $r = 1$.

We now assume that the result is true for all graphs with at most $r - 1$ cut edges. Let $G$ be a graph with $r$ cut edges satisfying the conditions of the theorem. Let $e = x_1y_1$ be a cut edge of $G$ such that one of the components of $G - e$ is $H_1$. Let $G_1$ be the other component of $G - e$ and let $x_1 \in V(G_1)$ and $y_1 \in V(H_1)$. By Lemma 2.1 there exists a graphoidal cover $\psi_1$ of $H_1$ whose only exterior vertices are $y_1$ and $v_1$ and there exists a $(y_1, v_1)$-path $Q_1$ in $\psi_1$. Clearly $\delta(G_1) > 1$. 


Case(i). \( x_1 \in H \) for some \( H \in \mathcal{H}_G \)

By induction hypothesis there exists a minimum graphoidal cover \( \psi_2 \) of \( G_1 \) whose only exterior vertices are \( v_1, v_2, \ldots, v_m \) and there exists a path \( P_1 \) in \( \psi_2 \) with \( x_1 \) as its terminus and path in \( \psi_2 \) with \( v_1 \) as terminus, \( 2 \leq i \leq m \). Then \( \psi = (\psi_1 \setminus \{Q_1\}) \cup (\psi_2 \setminus \{R_1\}) \cup \{P_1\} \) where \( P_1 = R_1 \circ e \circ Q_1 \) is the required minimum graphoidal cover of \( G \).

Case(ii). \( x_1 \notin H \) for all \( H \in \mathcal{H}_G \)

By induction hypothesis there exists a minimum graphoidal cover \( \psi_2 \) of \( G_1 \) whose only exterior vertices are \( v_2, v_3, \ldots, v_m \) and there exists a path \( \psi_2 \) with \( v_i, 2 \leq i \leq m \) as terminus. Now \( \psi = (\psi_1 \setminus \{Q_1\}) \cup \psi_2 \cup \{e \circ Q_1\} \) is the required minimum graphoidal cover of \( G \). This completes the induction and the proof.

**Theorem 2.4.** Let \( G \) be a connected graph which is not a tree with \( n \) pendant vertices, \( n \geq 1 \) and \( |\mathcal{H}_G| = m, m \geq 0 \). Then \( \eta = q - p + m + n \).

**Proof.** We prove the result by induction on \( n \), the number of pendant vertices in \( G \). Let \( \mathcal{H}_G = \{H_1, H_2, \ldots, H_m\} \). Let \( v_i \in V(H_i), 1 \leq i \leq m \) where \( H_i \in \mathcal{H}_G \). Suppose \( n = 1 \). Let \( v \in V(G) \) be such that \( \deg v = 1 \) and \( P = (v, u_1, u_2, \ldots, u_k, w) \) be a path in \( G \) such that \( \deg u_i = 2 \), for all \( i, 1 \leq i \leq k \) and \( \deg w > 2 \). Such a vertex \( v \) exists because \( G \) is not a path. Let \( G_1 = G \setminus \{v_1, u_1, u_2, \ldots, u_k\} \).

Case(i). \( w \in H \) for some \( H \in \mathcal{H}_G \)

Then by Theorem 2.3, there exists a minimum graphoidal cover \( \psi_1 \) of \( G_1 \) such that \( v_1, v_2, \ldots, v_m, w \) are the only vertices exterior to \( \psi_1 \) and there exists paths in \( \psi_1 \) which contains \( v_i, 1 \leq i \leq m \) and \( w \) as end vertices. Let \( P_1 \) be the path in \( \psi_1 \) with \( w \) as its terminus. Then \( \psi = (\psi_1 \setminus \{P_1\}) \cup \{P_1 \circ P^{-1}\} \) is a minimum graphoidal cover of \( G \) such that \( v_1, v_2, \ldots, v_m, v \) are the only vertices exterior to \( \psi \) and \( \eta = q - p + m + 1 \).

Case(ii). \( v \notin H \) for all \( H \in \mathcal{H}_G \)

By Theorem 2.3, there exists a minimum graphoidal cover \( \psi_1 \) of \( G_1 \) such that \( v_1, v_2, \ldots, v_m \) and \( v \) as the only vertices exterior to \( \psi \). Hence \( \eta = q - p + m + 1 \). Therefore the result is true for \( n = 1 \).

We assume the result is true for \( k \) pendant vertices. Let \( G \) be a graph with \( k + 1 \) pendant vertices and let \( z_1, z_2, \ldots, z_{k+1} \) be the pendant vertices of \( G \). Let \( P = (z_1, u_1, \ldots, u_k, w) \) be a path in \( G \) such that \( \deg u_i = 2 \), for all \( i, 1 \leq i \leq k \) and \( \deg w > 2 \). Let \( G_1 = H \setminus \{z_1, u_1, u_2, \ldots, u_k\} \). If \( w \in H \) for some \( H \in \mathcal{H}_G \), then by induction hypothesis there exists a minimum graphoidal cover \( \psi_1 \) of \( G_1 \) such that \( v_1, v_2, \ldots, v_m, w, z_2, z_3, \ldots, z_{k+1} \) are the only vertices exterior to \( \psi_1 \) and there exists a path \( P_1 \) in \( \psi_1 \) with \( w \) as its terminus. Hence \( \psi = (\psi_1 \setminus \{P_1\}) \cup \{P_1 \circ P^{-1}\} \) is a minimum graphoidal cover of \( G \) with \( z_1, z_2, \ldots, z_{k+1}, v_1, v_2, \ldots, v_m \) as the only vertices exterior to \( \psi \). Thus \( \eta = q - p + m + n \). If \( w \notin H \) for all \( H \in \mathcal{H}_G \), then by induction hypothesis there exists a minimum graphoidal cover \( \psi_1 \) of \( G_1 \) such that \( v_1, v_2, \ldots, v_m, z_2, \ldots, z_{k+1} \) are the only vertices exterior to \( \psi_1 \). Hence \( \psi = \psi_1 \cup \{P\} \) is a minimum graphoidal cover
of $G$ with $v_1, v_2, \ldots, v_m$ and $z_1, z_2, \ldots, z_{k+1}$ as the only vertices exterior to $\psi$. Thus $\eta = q - p + m + n$.

The next theorem determines the acyclic graphoidal covering number of a graph with $\delta = 2$.

**Theorem 2.5.** Let $G$ be a graph with $\delta = 2$. Let $B_1, B_2, \ldots, B_m$, $m \geq 0$ be end blocks of $G$ which are in $G(f_i)$ with $f_i$ as a cut vertex. Let $v_i \in V(B_i)$, $1 \leq i \leq m$ and $v_i$ is not a cut vertex of $G$. Then there exists a minimum acyclic graphoidal cover $\psi$ of $G$ whose only exterior vertices are $v_1, v_2, \ldots, v_m$ and $\eta_a = q - p + m$.

**Proof.** We observe that if $m = 0$, then the result follows from Theorem 1.13 and Theorem 1.14. Hence we assume that $m > 0$ and prove the result by induction on $n$, where $n$ is the number of blocks of $G$. If $n = 1$, the result follows from Theorem 1.15.

We now assume that the result is true for all graphs with at most $n - 1$ blocks. Let $G$ be a graph with $n$ blocks satisfying the conditions of the theorem. Let $\psi_1$ be a minimum acyclic graphoidal cover of $B_1$ whose only exterior vertex is $v_1$. Let $H = G \setminus (V(B_1) \setminus \{f_1\})$.

**Case(i).** $\deg_H f_1 = 1$.

We choose a path $P = (f_1, u_1, \ldots, u_k, w)$, such that $\deg u_i = 2$ for each $i$ and $\deg w > 2$. Let $G_1 = H \setminus \{f_1, u_1, u_2, \ldots, u_k\}$. Clearly $\delta(G_1) > 1$. Let $B$ be a block of $G_1$ containing $w$. If $B$ is an end block of $G_1$ and $B \in G(f)$ with $f$ a cut vertex of $G$, then by induction hypothesis there exists a minimum acyclic graphoidal cover $\psi_2$ of $G_1$ whose only exterior vertices are $w, v_2, \ldots, v_m$. Let $P_1$ be a path in $\psi_2$ having $w$ as its terminus. Then $\psi = (\psi_2 \setminus \{P_1\}) \cup \{P_1 \cup P^{-1}\} \cup \psi_1$ is a minimum acyclic graphoidal cover of $G$ with $v_1, v_2, \ldots, v_m$ as its only exterior vertices.

Otherwise by induction hypothesis there exists a minimum acyclic graphoidal cover $\psi_2$ of $G_1$ whose only exterior vertices are $v_2, v_3, \ldots, v_m$ and $\psi = \psi_2 \cup \{P\} \cup \psi_1$ is a minimum acyclic graphoidal cover of $G$ whose only exterior vertices are $v_1, v_2, \ldots, v_m$.

**Case(ii).** $\deg_H f_1 > 1$.

Let $B$ be a block of $H$ containing $f_1$. If $B$ is an end block of $G$ and $B \in G(f)$ with $f$ as a cut vertex then by induction hypothesis there exists a minimum acyclic graphoidal cover $\psi_2$ of $H$ with $v_2, v_3, \ldots, v_m$ and $f_1$ as its only exterior vertices. Then $\psi = \psi_1 \cup \psi_2$ is the required minimum acyclic graphoidal cover of $G$.

Otherwise by induction hypothesis there exists a minimum acyclic graphoidal cover $\psi_2$ of $H$ with $v_2, v_3, \ldots, v_m$ as its only exterior vertices. Let $P$ be the path in $\psi_2$ having $f_1$ as an internal vertex. Let $x, y$ be the terminal vertices of $P$. Let $P_1$ and $P_2$ be the $(x, f_1)$ and $(f_1, y)$ - sections of $P$ respectively. Then $\psi = \psi_1 \cup (\psi_2 \setminus \{P\}) \cup \{P_1, P_2\}$ is the required minimum acyclic graphoidal cover of $G$. This completes the induction and the proof.

**Theorem 2.6.** Let $G$ be a graph with $n$ pendant vertices, $n \geq 1$ and let $B_1, B_2, \ldots, B_m$, $m \geq 0$ be end blocks of $G$ which are in $G(f_i)$ with $f_i$ as a cut vertex. Then $\eta_a = q - p + m + n$. 
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Proof. Similar to the proof of Theorem 2.3.

We now proceed to the main theorem of characterizing the class of graphs with $\eta = \eta_a$.

Remark 2.7. Since $\eta \leq \eta_a$, $\eta_a = q - p$ implies that $\eta = q - p$. Theorem 1.13 and Theorem 1.14 characterizes the class of all graphs for which $\eta_a = q - p$. Hence for these graphs $\eta = q - p$, in turn $\eta = \eta_a$. Hence we need to consider the case when $\eta_a \neq q - p$.

Theorem 2.8. Let $G$ be a connected graph with $\eta_a \neq q - p$ and $\delta \leq 2$. Then $\eta = \eta_a$ if and only if one of the following holds.

(i) If $G$ has no cut edge, then $G$ is a graph such that an end block of $G$ is either a theta graph or a cycle with exactly one chord whose vertices of degree 3 are not cut vertices and all other blocks are cycles and the block-cut point tree of $G$ is a path (Refer Figure 1).

(ii) If $G$ has a cut edge with $|H_G| = m$ and if $l$ is the number of end blocks in $G(f)$ with $f$ as a cut vertex, then $m = l$, where $H_G$ is as defined in Theorem 2.3 (Refer Figure 2).

Proof. If $G$ is of type (i) by Lemma 1.12 and Theorem 2.3, $\eta = \eta_a = q - p + 1$. If $G$ is of type (ii), let $|H_G| = m$ and let $n$ be the number of pendant vertices in $G$. Then by Theorem 2.4 and Theorem 2.6, $\eta = \eta_a = q - p + m + n$. Hence $\eta = \eta_a$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{figure1.png}
\caption{Figure 1.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{figure2.png}
\caption{Figure 2.}
\end{figure}
Conversely suppose $\eta = \eta_a$. We first prove the theorem when $\delta = 2$. Since $\eta_a \neq q - p$, we have $\eta \neq q - p$. Hence by Theorem 1.9, Theorem 1.10 and Theorem 1.11, $G$ is either (a) a block which is either a cycle or a cycle with exactly one chord or a theta graph, or (b) a graph in which each block is a cycle or a cycle with exactly one chord or a theta graph and at most one block is not a cycle, or (c) a graph which has a cut edge $e$ such that at least one component of $G - e$ is a graph whose blocks are all cycles.

If $G$ is of type (a) or (b) then by Theorem 2.2, $\eta = q - p + 1$. Suppose $G$ is of type (a), a clock. If $G$ is a cycle then clearly $\eta = \eta_a$. Hence $G$ is a cycle with exactly one chord or a theta graph and $G$ reduces to a graph of type (i) given in the theorem. Suppose $G$ is of type (b), not a clock. Let $s$ be the number of end blocks of $G$ which are cycles. If each block of $G$ is a cycle then by Theorem 2.5, $\eta_a = q - p + s$. Since $\eta = \eta_a$, $s = 1$ which is a contradiction to the fact that $G$ is not a block. Hence there exists a block $B$ in $G$ which is not a cycle. We claim that $s = 1$. Suppose $s > 1$. Then by Theorem 2.5,

$$\eta_a = \begin{cases} q - p + s + 1, & \text{if } B \text{ is an end block of } G \text{ and a vertex of degree } 3 \text{ is a cut vertex of } G, \\ q - p + s, & \text{otherwise.} \end{cases}$$

Hence $\eta_a - \eta = s$ or $s - 1$ and $s > 1$ which is a contradiction to the fact that $\eta = \eta_a$. Hence $s = 1$ and this proves that the block-cut point tree of $G$ is a path. If a vertex of degree 3 in $B$ is a cut vertex then again $\eta_a = q - p + 2$ which is a contradiction. Hence vertices of degree 3 in $B$ are not cut vertices and $G$ reduces to a graph of type (i) given in the theorem.

If $G$ is of type (c), let $l$ be the number of end blocks in $G$ which are in $G(f)$ with $f$ as a cut vertex and let $|\mathcal{H}_G| = m$. Then by Theorem 2.3 and Theorem 2.5, $\eta = q - p + m$, $\eta_a = q - p + l$. Since $\eta = \eta_a$, $m = l$. Thus $G$ reduces to a graph of type (ii) given in the theorem.

Now let $\delta = 1$. Let $n$ be the number of pendant vertices of $G$. We define $l, m$ as before. By Theorem 2.4 and Theorem 2.6, $\eta = q - p + m + n$ and $\eta_a = q - p + l + n$. Since $\eta = \eta_a$, $l = m$. Thus $G$ reduces to a graph of type (ii) given in the theorem.

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Department of Mathematics, Loyola College, Chennai - 600 034, India.

Department Mathematics, D. B. Jain College, Chennai - 600 096, India.