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EQUALITY OF GRAPHOIDAL AND ACYCLIC GRAPHOIDAL COVERING NUMBER OF A GRAPH

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Abstract. A *graphoidal cover* of a graph G is a collection ψ of (not necessarily open) paths in G such that every vertex of G is an internal vertex of at most one path in ψ ad every edge of G is in exactly one path in ψ. If no member of ψ is a cycle, then ψ is called an *acyclic graphoidal cover* of G. The minimum cardinality of a graphoidal cover is called the *graphoidal covering number* of G and is denoted by η and the minimum cardinality of an acyclic graphoidal cover is called an *acyclic graphoidal covering number* of G and is denoted by η_a. In this paper we characterize the class of graphs for which $\eta = \eta_a$.

1. Introduction

The concept of graphoidal cover was introduced by B.D. Acharya and E. Sampathkumar in 1987 and a study of the graphoidal covering number was initiated by them [1]. Since then this area of research has been explored by several authors [2, 3, 10, 11, 12, 13, 19]. The concept of an acyclic graphoidal cover was introduced by Suresh Suseela and pursued by Arumugam *et al.*, [14, 15].

By a *graph* $G = (V, E)$ we mean a finite, undirected, connected graph without loops or nultiple edges. The order and size of G are denoted by p and q respectively. For graph theroetic terminology we refer to Harary [16].

If $P = (v_0, v_1, \ldots, v_n)$ is a path or a cycle in $G, v_1, \ldots, v_{n-1}$ are called *internal vertices* of P. If $P = (v_0, v_1, \ldots, v_n)$ and $Q = (v_n = w_0, w_1, \ldots, w_m)$ are two paths in G then the walk obtained by concatenating P and Q at v_n is denoted by $P \ o \ Q$.

Definition 1.1. A *graphoidal cover* of a graph G is a set ψ of (not necessarily open) paths in G satisfying the following conditions.

(i) Every path in ψ has at least two vertices.
(ii) Every vertex of G is an internal vertex of at most one path in ψ.
(iii) Every edge of G is in exactly one path in ψ.

ψ is called an *acyclic graphoidal cover* of G if no member of ψ is a cycle in G. The minimum cardinality of (an acyclic) a graphoidal cover of G is called the (*acyclic*) *graphoidal covering number* of G and is denoted by (η_a) η.

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Definition 1.2. Let ψ be a collection of internally disjoint paths in G. A vertex of G is said to be an *interior vertex* of ψ if it is an internal vertex of some path in ψ. Any vertex which is not an interior vertex of ψ is said to be an *exterior vertex* of ψ.

Theorem 1.3.([18]) *For any graphoidal cover ψ of G, let t_ψ denote the number of exterior vertices of ψ. Let $t = \min t_\psi$, where the minimum is taken over all graphoidal covers of G. Then $\eta = q - p + t$ where p and q denote respectively the order and size of G.*

Corollary 1.4.([18]) *For any graph G, $\eta \geq q - p$. Moreover the following are equivalent.*

(i) $\eta = q - p$.
(ii) *There exists a graphoidal cover without exterior vertices.*
(iii) *There exists a set of internally disjoint and edge disjoint paths without exterior vertices.*

Theorem 1.5.([17]) *For any graph G with $\delta \geq 3$, $\eta = q - p$*

Remark 1.6. Results analogous to Theorem 1.3, Corollary 1.4 and Theorem 1.5 are also true for an acyclic graphoidal covering number of a graph.

Theorem 1.7.([15]) *For any graph with $\delta \geq 3$, $\eta = \eta_a$.*

For graphs with $\delta \leq 2$ it is not necessary that $\eta = \eta_a = q - p$. Hence the problem of characterizing graphs with $\delta \leq 2$ satisfying $\eta = \eta_a$ is challenging. In this paper we solve this problem. For our further discussion we confine ourselves to graphs with $\delta \leq 2$. We need the following definition and theorems.

Definition 1.8.([7]) Let $\mathcal{G}(f)$ denote the collection of all blocks whose edge set can be decomposed into a cycle C and a collection $\wp$ of internally disjoint paths such that each path P in $\wp$ has $f \in V(C)$ as its origin and $|V(P) \cap V(C)| \leq 2$ (The collection $\wp$ may be empty in which case the corresponding member of $\mathcal{G}(f)$ is a cycle). We observe that if $G \in \mathcal{G}(f)$ and G is not a cycle, then $\deg f = |\wp| + 2 = \Delta$ and there is at most one vertex $v \neq f$ with $\deg v = \Delta$.

Let $\mathcal{F}$ and $\mathcal{F}_a$ denote respectively the class of all graphs G with $\eta = q - p$ and $\eta_a = q - p$.

Theorem 1.9.([4]) *Let G be a 2-connected graph with $p \geq 3$. Then $G \notin \mathcal{F}$ if and only if G is a cycle or a cycle with exactly one chord or a theta graph.*

Theorem 1.10.([4]) *Let G be a 2-edge connected graph with $\delta = 2$. Then $G \notin \mathcal{F}$ if and only if every block of G is a cycle or a cycle with exactly one chord or a theta graph and at most one block of G is not a cycle.*

Theorem 1.11.([6]) *Let G be a connected graph with $\delta = 2$ and edge connectivity one. Then $G \notin \mathcal{F}$ if and only if there exists a cut edge e of G such that at least one component of $G - e$ is a graph, all of whose blocks are cycles.*

Lemma 1.12.([6]) *Let G be a 2-edge connected graph such that exactly one block of G is either a cycle with exactly one chord or a theta graph and all other blocks are cycles. Let $v \in V(G)$. Then there exists a minimum graphoidal cover ψ of G such that v is the only vertex exterior to ψ and there exists a path in ψ which contains v as an exterior vertex.*

Theorem 1.13.([7]) *Let G be a 2-connected graph with $p \geq 3$. Then $G \in \mathcal{F}_a$ if and only if $G \notin \mathcal{G}(f)$.*

Theorem 1.14.([7]) *Let G be a graph with $\delta > 1$ and connectivity one. Then $G \notin \mathcal{F}_a$ if and only if at least one end block of G is a member of $\mathcal{G}(f)$ with f as a cut vertex of $\mathcal{G}$.*

Theorem 1.15.([5]) *Let $G \in \mathcal{G}(f)$ and $\Delta(G) \geq 3$. Let v be a vertex of G with $\deg v \neq \Delta$. Then there exists a minimum acyclic graphoidal cover ψ of G such that v is the only vertex exterior to ψ and $\eta_a = |\psi| = \Delta - 1$.*

2. Main Results

Lemma 2.1. *Let G be a graph whose blocks are all cycles. Let $u, v \in V(G)$ with $u \neq v$. Then there exists a graphoidal over ψ of G such that u and v are the only vertices exterior to ψ and a path P in ψ which contains u and v as end vertices.*

Proof. Let $C_1, C_2, \ldots, C_k$ be the blocks of G. By hypothesis each C_i is a cycle, $i = 1, 2, \ldots, k$. Let P be a (u, v) - path in G. Let $S_1, S_2, \ldots, S_s$, $s \in \{1, 2, \ldots, k\}$ be the segments of P that are part of the cycles $C_1, C_2, \ldots, C_s$. Let $R_i = C_i \backslash S_i$, $1 \leq i \leq s$. Then $\psi = \{P, R_1, R_2, \ldots, R_s, C_{s+1}, \ldots, C_k\}$ is a graphoidal cover of G with u and v as the only vertices exterior to ψ.

Theorem 2.2. *Let G be a 2-edge connected graph. Then either $\eta = q - p$ or $\eta = q - p + 1$.*

Proof. Suppose $\eta \neq q - p$. By Theorem 1.10, every block of G is a cycle or a cycle with exactly one chord or a theta graph and at most one block of G is not a cycle. We now prove that $\eta = q - p + 1$ by induction on m where m is the number of blocks of G. When $m = 1$, G is either a cycle or a cycle with exactly one chord or a theta graph. Clearly for any minimum graphoidal cover ψ of G, exactly one vertex of G is exterior to ψ and hence by Theorem 1.3, $\eta = q - p + 1$.

We now assume that $\eta = q - p + 1$ for all 2-edge connected graphs with m blocks, $m \geq 1$. Let G be a 2-edge connected graph with $m + 1$ blocks. Let C be an end block which is a cycle and $v \in V(C)$ be a cut vertex of G. Let G' be the subgraph of G obtained by removing all the vertices of $C - v$. Clearly G' has m blocks. By induction hypothesis $\eta(G') = q' - p' + 1$ when p' and q' are the order and size of G' respectively.

$$\begin{aligned}\text{Now}\quad \eta(G) &= \eta(G') + 1\\ &= q' - p' + 2\\ &= q - p + 1.\end{aligned}$$ This completes the induction and the proof.

Theorem 2.3. *Let G be a graph with $\kappa' = 1$ and $\delta > 1$, where κ' is the edge connectivity of G. Let $S = \{e \mid e$ is a cut edge of G and the bolcks of at least one component of G-e are all cycles$\}$. Let $\mathcal{H}_G = \{H_1, H_2, \ldots, H_m\}$, be the collection of all such components. Let $v_i \in V(H_i)$, $1 \leq i \leq m$. Then $\eta = q - p + m$ and there exists a minimum graphoidal cover ψ of G such that v_i, $1 \leq i \leq m$ are the only vertices exterior to ψ and there exist paths in ψ which contain v_i as an end vertex, $1 \leq i \leq m$.*

Proof. Since contracting an edge incident with a cut vertex of degree 2 does not affect the value of $q - p$ and η, we may assume without loss of generality that any cut vertex has degree at least 3. We observe that if $m = 0$, then the result follows from Theorem 1.11. Hence we assume that $m > 0$ and prove the result by induction on r, where r is the number of cut edges of G. Since $\kappa' = 1$, $r \geq 1$. Suppose $r = 1$ and let $e = x_1x_2$ be the cut edge of G. Let G_1 and G_2 be the components of G-e with $x_1 \in V(G_1)$ and $x_2 \in V(G_2)$. Clearly G_1 and G_2 are 2-edge connected graphs. If both G_1 and G_2 are in $\mathcal{H}_G$ then by Lemma 2.1 there exists a graphoidal cover ψ_i of G_i, $i = 1, 2$ and a (v_i, x_i)-path P_i in G_i such that v_i and x_i are the only vertices exterior to ψ_i, $i = 1, 2$. Then by Theorem 1.3 $\psi = (\psi_1 \backslash \{P_1\}) \cup (\psi_2 \backslash \{P_2\}) \cup \{P_1 \ o \ e \ o \ P_2^{-1}\}$ is a minimum graphoidal cover such that v_1 and v_2 are the only vertices exterior to ψ. Thus $\eta = q - p + 2$. Suppose $G_1 \notin \mathcal{H}_G$. For $G_2 \in \mathcal{H}_G$ we define ψ_2 as before. If $G_1 \in \mathcal{F}$, then there exists a minimum graphoidal cover ψ_1 of G_1 such that all vertices of G_1 are interior to ψ_1. Then by Theorem 1.3 $\psi = \psi_1 \cup (\psi_2 \backslash \{P_2\}) \cup (e \ o \ P_2^{-1})$ is a minimum graphoidal cover of G with v_2 as the only vertex exterior to ψ, where $e \ o \ P_2^{-1}$ is a path in ψ which contains v_2 as an end vertex. If $G_1 \notin \mathcal{F}$, then since $G_1 \notin \mathcal{H}_G$, by Theorem 1.10 exactly one block of G_1 is a cycle with exactly one chord or a theta graph and all other blocks are cycles. Hence by Lemma 1.12 there exists a minimum graphoidal over ψ_1 of G_1 and a path P_1 in ψ_1 with x_1 as its terminus such that x_1 is the only vertex exterior to ψ_1. Then $\psi = (\psi_1 \backslash \{P_1\}) \cup (\psi_2 \backslash \{P_2\}) \cup \{P_1 \ o \ e \ o \ P_2^{-1}\}$ is a minimum graphoidal cover of G such that v_2 is the only vertex exterior to ψ and $P_1 \ o \ e \ o \ P_2^{-1}$ is a path in ψ which contains v_2 as an end vertex. Thus $\eta = q - p + 1$. Hence the result is true for $r = 1$.

We now assume that the result is true for all graphs with at most $r - 1$ cut edges. Let G be a graph with r cut edges satisfying the conditions of the theorem. Let $e = x_1y_1$ be a cut edge of G such that one of the components of $G - e$ is H_1. Let G_1 be the other component of $G - e$ and let $x_1 \in V(G_1)$ and $y_1 \in V(H_1)$. By Lemma 2.1 there exists a graphoidal cover ψ_1 of H_1 whose only exterior vertices are y_1 and v_1 and there exists a (y_1, v_1)-path Q_1 in ψ_1. Clearly $\delta(G_1) > 1$.

Case(i). $x_1 \in H$ for some $H \in \mathcal{H}_{G_1}$

By induction hypothesis there exists a minimum graphoidal cover ψ_2 of G_1 whose only exterior vertices are $x_1, v_2, \ldots, v_m$ and there exists a path R_1 in ψ_2 with x_1 as its terminus and path in ψ_2 with v_1 as terminus, $2 \le i \le m$. Then $\psi = (\psi_1 \backslash \{Q_1\}) \cup (\psi_2 \backslash \{R_1\}) \cup \{P_1\}$ where $P_1 = R_1 \; o \; e \; o \; Q_1$ is the required minimum graphoidal cover of G.

Case(ii). $x_1 \notin H$ for all $H \in \mathcal{H}_{G_1}$

By induction hypothesis there exists a minimum graphoidal cover ψ_2 of G_1 whose only exterior vertices are $v_2, v_3, \ldots, v_m$ and there exists a path ψ_2 with v_i, $2 \le i \le m$ as terminus. Now $\psi = (\psi_1 \backslash \{Q_1\}) \cup \psi_2 \cup \{e \; o \; Q_1\}$ is the required minimum graphoidal cover of G. This completes the induction and the proof.

Theorem 2.4. *Let G be a connected graph which is not a tree with n pendant vertices, $n \ge 1$ and $|\mathcal{H}_G| = m$, $m \ge 0$. Then $\eta = q - p + m + n$.*

Proof. We prove the result by induction on n, the number of pendant vertices in G. Let $\mathcal{H}_G = \{H_1, H_2, \ldots, H_m\}$. Let $v_i \in V(H_i)$, $1 \le i \le m$ where $H_i \in \mathcal{H}_G$. Suppose $n = 1$. Let $v \in V(G)$ be such that $\deg v = 1$ and $P = (v, u_1, u_2, \ldots, u_k, w)$ be a path in G such that $\deg u_i = 2$, for all i, $1 \le i \le k$ and $\deg w > 2$. Such a vertex w exists because G is not a path. Let $G_1 = G \backslash \{v, u_1, u_2, \ldots, u_k\}$.

Case(i). $w \in H$ for some $H \in \mathcal{H}_{G_1}$

Then by Theorem 2.3, there exists a minimum graphoidal cover ψ_1 of G_1 such that $v_1, v_2, \ldots, v_m, w$ are the only vertices exterior to ψ_1 and there exists paths in ψ_1 which contains v_i, $1 \le i \le m$ and w as end vertices. Let P_1 be the path in ψ_1 with w as its terminus. Then $\psi = (\psi_1 \backslash \{P_1\}) \cup \{P_1 \; o \; P^{-1}\}$ is a minimum graphoidal cover of G such that $v_1, v_2, \ldots, v_m, v$ are the only vertices exterior to ψ and $\eta = q - p + m + 1$.

Case(ii). $w \notin H$ for all $H \in \mathcal{H}_{G_1}$

By Theorem 2.3, there exists a minimum graphoidal cover ψ_1 of G_1 such that $v_1, v_2, \ldots, v_m$ are the only vertices exterior to ψ_1. Hence $\psi = \psi_1 \cup \{P\}$ is a minimum graphoidal cover with $v_1, v_2, \ldots, v_m$ and v as the only vertices exterior to ψ. Hence $\eta = q - p + m + 1$. Therefore the result is true for $n = 1$.

We assume the result is true for k pendant vertices. Let G be a graph with $k + 1$ pendant vertices and let $z_1, z_2, \ldots, z_{k+1}$ be the pendant vertices of G. Let $P = (z_1, u_1, \ldots, u_k, w)$ be a path in G such that $\deg u_i = 2$, for all i, $1 \le i \le k$ and $\deg w > 2$. Let $G_1 = H \backslash \{z_1, u_1, u_2, \ldots, u_k\}$. If $w \in H$ for some $H \in \mathcal{H}_{G_1}$, then by induction hypothesis there exists a minimum graphoidal cover ψ_1 of G_1 such that $v_1, v_2, \ldots, v_m, w, z_2, z_3, \ldots, z_{k+1}$ are the only vertices exterior to ψ_1 and there exists a path P_1 in ψ_1 with w as its terminus. Hence $\psi = (\psi_1 \backslash \{P_1\}) \cup \{P_1 \; o \; P^{-1}\}$ is a minimum graphoidal cover of G with $z_1, z_2, \ldots, z_{k+1}, v_1, v_2, \ldots, v_m$ as the only vertices exterior to ψ. Thus $\eta = q - p + n + m$. If $w \notin H$ for all $H \in \mathcal{H}_G$, then by induction hypothesis there exists a minimum graphoidal cover ψ_1 of G_1 such that $v_1, v_2, \ldots, v_m, z_2, \ldots, z_{k+1}$ are the only vertices exterior to ψ_1. Hence $\psi = \psi_1 \cup \{P\}$ is a minimum graphoidal cover of G with $v_1, v_2, \ldots, v_m$ and $z_1, z_2, \ldots, z_{k+1}$ as the only vertices exterior to ψ. Thus $\eta = q - p + m + n$.

The next theorem determines the acyclic graphoidal covering number of a graph with $\delta = 2$.

Theorem 2.5. *Let G be a graph with $\delta = 2$. Let $B_1, B_2, \ldots, B_m$, $m \geq 0$ be end blocks of G which are in $\mathcal{G}(f_i)$ with f_i as a cut vertex. Let $v_i \in V(B_i)$, $1 \leq i \leq m$ and v_i is not a cut vertex of G. Then there exists a minimum acyclic graphoidal cover ψ of G whose only exterior vertices are $v_1, v_2, \ldots, v_m$ and $\eta_a = q - p + m$.*

Proof. We observe that if $m = 0$, then the result follows from Theorem 1.13 and Theorem 1.14. Hence we assume that $m > 0$ and prove the result by induction on n, where n is the number of blocks of G. If $n = 1$, the result follows from Theorem 1.15. We now assume that the result is true for all graphs with at most $n - 1$ blocks. Let G be a graph with n blocks satisfying the conditions of the theorem. Let ψ_1 be a minimum acyclic graphoidal cover of B_1 whose only exterior vertex is v_1. Let $H = G \backslash (V(B_1) \backslash \{f_1\})$.

Case(i). $\deg_H f_1 = 1$.

We choose a path $P = (f_1, u_1, \ldots, u_k, w)$, such that $\deg u_i = 2$ for each i and $\deg w > 2$. Let $G_1 = H \backslash \{f_1, u_1, u_2, \ldots, u_k\}$. Clearly $\delta(G_1) > 1$. Let B be a block of G_1 containing w. If B is an end block of G_1 and $B \in \mathcal{G}(f)$ with f a cut vertex of G, then by induction hypothesis there exists a minimum acyclic graphoidal cover ψ_2 of G_1 whose only exterior vertices are $w, v_2, \ldots, v_m$. Let P_1 be a path in ψ_2 having w as its terminus. Then $\psi = (\psi_2 \backslash \{P_1\}) \cup \{P_1 \; o \; P^{-1}\} \cup \psi_1$ is a minimum acyclic graphoidal cover of G with $v_1, v_2, \ldots, v_m$ as its only exterior vertices.

Otherwise by induction hypothesis there exists a minimum acyclic graphoidal cover ψ_2 of G_1 whose only exterior vertices are $v_2, v_3, \ldots, v_m$ and $\psi = \psi_2 \cup \{P\} \cup \psi_1$ is a minimum acyclic graphoidal cover of G whose only exterior vertices are $v_1, v_2, \ldots, v_m$.

Case(ii). $\deg_H f_1 > 1$.

Let B be a block of H containing f_1. If B is an end block of G and $B \in \mathcal{G}(f)$ with f as a cut vertex then by induction hypothesis there exists a minimum acyclic graphoidal cover ψ_2 of H with $v_2, v_3, \ldots, v_m$ and f_1 as its only exterior vertices. Then $\psi = \psi_1 \cup \psi_2$ is the required minimum acyclic graphoidal cover of G.

Otherwise by induction hypothesis there exists a minimum acyclic graphoidal cover ψ_2 of H with $v_2, v_3, \ldots, v_m$ as its only exterior vertices. Let P be the path in ψ_2 having f_1 as an internal vertex. Let x, y be the terminal vertices of P. Let P_1 and P_2 be the (x, f_1) and (f_1, y) - sections of P respectively. Then $\psi = \psi_1 \cup (\psi_2 \backslash \{P\}) \cup \{P_1, P_2\}$ is the required minimum acyclic graphoidal cover of G. This completes the induction and the proof.

Theorem 2.6. *Let G be a graph with n pendant vertices, $n \geq 1$ and let $B_1, B_2, \ldots,$ B_m, $m \geq 0$ be end blocks of G which are in $\mathcal{G}(f_i)$ with f_i as a cut vertex. Then $\eta_a = q - p + m + n$.*

Proof. Similar to the proof of Theorem 2.3.

We now proceed to the main theorem of characterizing the class of graphs with $\eta = \eta_a$.

Remark 2.7. Since $\eta \leq \eta_a$, $\eta_a = q - p$ implies that $\eta = q - p$. Theorem 1.13 and Theorem 1.14 characterizes the class of all graphs for which $\eta_a = q - p$. Hence for these graphs $\eta = q - p$, in turn $\eta = \eta_a$. Hence we need to consider the case when $\eta_a \neq q - p$.

Theorem 2.8. *Let G be a connected graph with $\eta_a \neq q - p$ and $\delta \leq 2$. Then $\eta = \eta_a$ if and only if one of the following holds.*

(i) *If G has no cut edge, then G is a graph such that an end block of G is either a theta graph or a cycle with exactly one chord whose vertices of degree* 3 *are not cut vertices and all other blocks are cycles and the block-cut point tree of G is a path* (*Refer Figure* 1).

Figure 1.

(ii) *If G has a cut edge with $|\mathcal{H}_G| = m$ and if l is the number of end blocks in $\mathcal{G}(f)$ with f as a cut vertex, then $m = l$, where $\mathcal{H}_G$ is as defined in Theorem* 2.3 (*Refer Figure* 2).

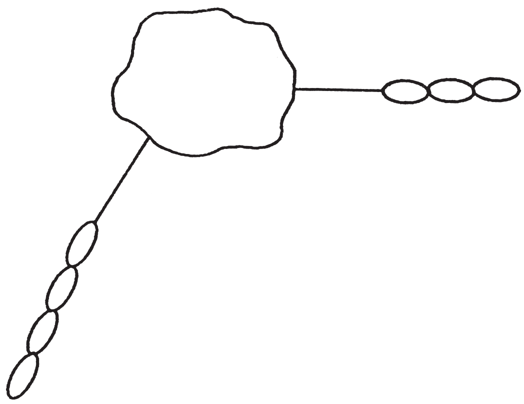

Figure 2.

Proof. If G is of type (i) by Lemma 1.12 and Theorem 2.3, $\eta = \eta_a = q - p + 1$. If G is of type (ii), let $|\mathcal{H}_G| = m$ and let n be the number of pendant vertices in G. Then by Theorem 2.4 and Theorem 2.6, $\eta = \eta_a = q - p + m + n$. Hence $\eta = \eta_a$.

Conversely suppose $\eta = \eta_a$. We first prove the theorem when $\delta = 2$. Since $\eta_a \neq q-p$, we have $\eta \neq q-p$. Hence by Theorem 1.9, Theorem 1.10 and Theorem 1.11, G is either

(a) a block which is either a cycle or a cycle with exactly one chord or a theta graph, or
(b) a graph in which each block is a cycle or a cycle with exactly one chord or a theta graph and at most one block is not a cycle, or
(c) a graph which has a cut edge e such that at least one component of $G-e$ is a graph whose blocks are all cycles.

If G is of type (a) or (b) then by Theorem 2.2, $\eta = q-p+1$. Suppose G is of type (a), a clock. If G is a cycle then clearly $\eta \neq \eta_a$. Hence G is a cycle with exactly one chord or a theta graph and G reduces to a graph of type (i) given in the theorem. Suppose G is of type (b), not a clock. Let s be the number of end blocks of G which are cycles. If each block of G is a cycle then by Theorem 2.5, $\eta_a = q-p+s$. Since $\eta = \eta_a$, $s = 1$ which is a contradiction to the fact that G is not a block. Hence there exists a block B in G which is not a cycle. We claim that $s = 1$. Suppose $s > 1$. Then by Theorem 2.5,

$$\eta_a = \begin{cases} q-p+s+1, & \text{if } B \text{ is an end block of } G \text{ and a vertex of} \\ & \text{degree 3 is a cut vertex of } G, \\ q-p+s, & \text{otherwise.} \end{cases}$$

Hence $\eta_a - \eta = s$ or $s-1$ and $s > 1$ which is a contradiction to the fact that $\eta = \eta_a$. Hence $s = 1$ and this proves that the block-cut point tree of G is a path. If a vertex of degree 3 in B is a cut vertex then again $\eta_a = q-p+2$ which is a contradiction. Hence vertices of degree 3 in B are not cut vertices and G reduces to a graph of type (i) given in the theorem.

If G is of type (c), let l be the number of end blocks in G which are in $\mathcal{G}(f)$ with f as a cut vertex and let $|\mathcal{H}_G| = m$. Then by Theorem 2.3 and Theorem 2.5, $\eta = q-p+m$, $\eta_a = q-p+l$. Since $\eta = \eta_a$, $m = l$. Thus G reduces to a graph of type (ii) given in the theorem.

Now let $\delta = 1$. Let n be the number of pendant vertices of G. We define l, m as before. By Theorem 2.4 and Theorem 2.6, $\eta = q-p+m+n$ and $\eta_a = q-p+l+n$. Since $\eta = \eta_a$, $l = m$. Thus G reduces to a graph of type (ii) given in the theorem.

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