CLOSE-TO-CONVEXITY AND STARLIKENESS OF ANALYTIC FUNCTIONS

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Abstract. For functions \( f(z) = z^p + a_{n+1}z^{n+1} + \cdots \) defined on the open unit disk, the condition \( \text{Re}(f'(z)/z^p-1) > 0 \) is sufficient for close-to-convexity of \( f \). By making use of this result, several sufficient conditions for close-to-convexity are investigated and relevant connections with previously known results are indicated.

1. Introduction

Let \( D := \{ z \in \mathbb{C} : |z| < 1 \} \) be the open unit disk and \( A_{p,n} \) be the class of all analytic functions \( f : D \to \mathbb{C} \) of the form \( f(z) = z^p + a_{n+1}z^{n+1} + \cdots \) with \( A := A_{1,1} \). For studies related to multivalent functions, see [5, 7, 8, 9, 10]. Singh and Singh [16] obtained several interesting conditions for functions \( f \in A \) satisfying inequalities involving \( f'(z) \) and \( zf''(z) \) to be univalent or starlike in \( D \). Owa et al. [11] generalized the results of Singh and Singh [16] and also obtained several sufficient conditions for close-to-convexity, starlikeness and convexity of functions \( f \in A \). In fact, they have proved the following theorems.

Theorem 1.1 ([11], Theorems 1-3). Let \( 0 \leq \alpha < 1 \) and \( \beta, \gamma \geq 0 \). If \( f \in A \), then

\[
\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{1 + 3\alpha}{2(1 + \alpha)} \implies \text{Re}(f'(z)) > \frac{1 + \alpha}{2},
\]

\[
\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{3 + 2\alpha}{2 + \alpha} \implies |f'(z) - 1| < 1 + \alpha,
\]

\[
|f'(z) - 1|^\beta |zf''(z)|^\gamma < \frac{(1 - \alpha)^{\beta + \gamma}}{2^{\beta + 2\gamma}} \implies \text{Re}(f'(z)) > \frac{1 + \alpha}{2}.
\]

Theorem 1.2 ([11], Theorem 4). Let \( 1 < \lambda < 3 \). If \( f \in A \), then

\[
\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \begin{cases} 
\frac{5\lambda - 1}{2(\lambda + 1)}, & 1 < \lambda \leq 2; \\
\frac{\lambda + 1}{2(\lambda - 1)}, & 2 < \lambda < 3,
\end{cases} \quad \text{implies} \quad \frac{zf'(z)}{f(z)} < \frac{\lambda(1 - z)}{\lambda - z}.
\]
In this present paper, the above results are extended for functions \( f \in A_{p,n} \).

2. Close-to-convexity and starlikeness

For \( f \in A \), the condition \( \text{Re} f'(z) > 0 \) implies close-to-convexity and univalence of \( f \). Similarly, for \( f \in A_{p,1} \), the inequality \( \text{Re}(f'(z)/z^{p-1}) > 0 \) implies \( p \)-valency of \( f \). See also [15, 18, 19]. From this result, the functions satisfying the hypothesis of Theorems 2.1–2.4 are \( p \)-valent in \( D \). A function \( f \in A_{p,1} \) is close-to-convex if there is a \( p \)-valent convex function \( \phi \) such that \( \text{Re}(f'(z)/\phi(z)) > 0 \). Also they are all close-to-convex with respect to \( \phi(z) = z^p \).

**Theorem 2.1.** If the function \( f \in A_{p,n} \) satisfies the inequality

\[
\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \frac{(2p-n)+\alpha(2p+n)}{2(\alpha+1)}, \quad \text{for } z \in D,
\]

then

\[
\text{Re} \left( \frac{f'(z)}{pz^{p-1}} \right) > \frac{1+\alpha}{2}, \quad \text{for } z \in D.
\]

For the proof of our main results, we need the following lemma.

**Lemma 2.2.** [6, Lemma 2.2a] Let \( z_0 \in D \) and \( r_0 = |z_0| \). Let \( f(z) = a_n z^n + a_{n+1} z^{n+1} + \cdots \) be continuous on \( \overline{D}_{r_0} \) and analytic on \( D_{r_0} \cup \{z_0\} \) with \( f(z) \neq 0 \) and \( n \geq 1 \). If

\[
|f(z_0)| = \max \{|f(z)| : z \in \overline{D}_{r_0}\},
\]

then there exists an \( m \geq n \) such that

1. \( \frac{z_0 f'(z_0)}{f(z_0)} = m \), and

2. \( \text{Re} \frac{z_0 f''(z_0)}{f'(z_0)} + 1 \geq m \).

**Proof of Theorem 2.1.** Let the function \( w \) be defined by

\[
f'(z) = \frac{1+aw(z)}{1+w(z)}.
\]

Then \( w \) can be written as

\[
w(z) = \frac{1}{\alpha-1} \left[ \frac{(n+p)}{p} a_{n+p} z^n - \frac{(n+p)^2}{p^2(1-\alpha)} a_{n+p}^2 z^{2n} + \cdots \right],
\]

hence it is analytic in \( D \) with \( w(0) = 0 \). From (2.2), some computation yields

\[
1 + \frac{zf''(z)}{f'(z)} = p + \frac{aw'(z)}{1+aw(z)} - \frac{zw'(z)}{1+w(z)}.
\]
Suppose there exists a point \( z_0 \in \mathbb{D} \) such that
\[
|w(z_0)| = 1 \quad \text{and} \quad |w(z)| < 1 \quad \text{when} \quad |z| < |z_0|.
\]
Then by applying Lemma 2.2, there exists \( m \geq n \) such that
\[
z_0 w'(z_0) = mw(z_0), \quad (w(z_0) = e^{i\theta}; \theta \in \mathbb{R}). \tag{2.4}
\]
Thus, by using (2.3) and (2.4), it follows that
\[
\Re\left(1 + \frac{z f''(z_0)}{f'(z_0)}\right) = p + \Re\left(\frac{amw(z_0)}{1 + aw(z_0)}\right) - \Re\left(\frac{mw(z_0)}{1 + w(z_0)}\right) \\
= p + \Re\left(\frac{ame^{i\theta}}{1 + e^{i\theta}}\right) - \Re\left(\frac{me^{i\theta}}{1 + e^{i\theta}}\right) \\
= p + \frac{am(a + \cos\theta)}{1 + a^2 + 2a \cos\theta} - \frac{m}{2} \\
\leq \frac{(2p - n) + a(2p + n)}{2(a + 1)},
\]
which contradicts the hypothesis (2.1). It follows that \( |w(z)| < 1 \), that is,
\[
\left|\frac{1 - \frac{f'(z)}{pz^p - 1}}{\frac{f'(z)}{pz^p - 1} - \alpha}\right| < 1.
\]
This evidently completes the proof of Theorem 2.1.

Owa [13] shows that a function \( f \in \mathcal{A}_{p,1} \) satisfying \( \Re(1 + zf''(z)/f'(z)) < p + 1/2 \) implies \( f \) is \( p \)-valently starlike. Our next theorem investigates the close-to-convexity of this type of functions. For related results, see [14, 4, 20].

**Theorem 2.3.** If the function \( f \in \mathcal{A}_{p,n} \) satisfies the inequality
\[
\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{(p + n)\alpha + (2p + n)}{(\alpha + 2)}, \quad \text{for} \quad z \in \mathbb{D}, \tag{2.5}
\]
then
\[
\left|\frac{f'(z)}{pz^p - 1} - 1\right| < 1 + \alpha, \quad \text{for} \quad z \in \mathbb{D}.
\]

**Proof.** Consider the function \( w \) defined by
\[
\frac{f'(z)}{pz^p - 1} = (1 + \alpha)w(z) + 1. \tag{2.6}
\]
It can be checked similarly as above that \( w \) is analytic in \( \mathbb{D} \) with \( w(0) = 0 \). From (2.6), some computation yields
\[
1 + \frac{zf''(z)}{f'(z)} = p + \frac{(1 + \alpha)zw'(z)}{(1 + \alpha)w(z) + 1}. \tag{2.7}
\]
Suppose there exists a point \( z_0 \in \mathbb{D} \) such that
\[
|w(z_0)| = 1 \text{ and } |w(z)| < 1 \text{ when } |z| < |z_0|.
\]

Then by applying Lemma 2.2, there exists \( m \geq n \) such that
\[
z_0 w'(z_0) = mw(z_0), \quad (w(z_0) = e^{i\theta}; \theta \in \mathbb{R}).
\]

Thus, by using (2.7) and (2.8), it follows that
\[
Re \left( 1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right) = p + Re \left( \frac{(1 + \alpha) z_0 w'(z_0)}{(1 + \alpha) w(z_0) + 1} \right)
\]
\[
= p + Re \left( \frac{(1 + \alpha) m e^{i\theta}}{(1 + \alpha) e^{i\theta} + 1} \right)
\]
\[
= p + \frac{m(1 + \alpha)(1 + \alpha + \cos \theta)}{1 + (1 + \alpha)^2 + 2(1 + \alpha) \cos \theta}
\]
\[
\geq \frac{m(1 + \alpha)(1 + \alpha + \cos \theta)}{1 + (1 + \alpha)^2 + 2(1 + \alpha) \cos \theta}
\]

which contradicts the hypothesis (2.5). It follows that \( |w(z)| < 1 \), that is,
\[
\left| \frac{f'(z)}{pz^{p-1} - 1} \right| < 1 + \alpha.
\]

This evidently completes the proof of Theorem 2.3. \( \square \)

Owa [12] has also showed that a function \( f \in \mathcal{A} \) satisfying
\[
|f'(z)/g'(z) - 1| \beta |z f''(z)/g'(z) - z f'(z)g''(z)/(g'(z))^2| \gamma < (1 + \alpha)^{\beta + \alpha}, \text{ for } 0 \leq \alpha < 1, \beta \geq 0, \gamma \geq 0 \text{ and } g \text{ a convex function, is close-to-convex. Also, see [3].}
\]

Our next theorem investigates the close-to-convexity of similar class of functions.

**Theorem 2.4.** If \( f \in \mathcal{A}_{p,n} \), then for \( z \in \mathbb{D} \),
\[
\left| \frac{f'(z)}{pz^{p-1} - 1} \right| \beta \left| \frac{f''(z)}{z^{p-2}} - (p - 1) \frac{f'(z)}{z^{p-1}} \right| \gamma < \left( pn \right)^{\gamma} (1 - \alpha)^{\beta + \gamma},
\]

implies
\[
Re \left( \frac{f'(z)}{pz^{p-1} - 1} \right) > \frac{1 + \alpha}{2},
\]

and
\[
\left| \frac{f'(z)}{pz^{p-1} - 1} \right| \beta \left| \frac{f''(z)}{z^{p-2}} - (p - 1) \frac{f'(z)}{z^{p-1}} \right| \gamma < (pn)^{\gamma} |1 - \alpha|^{\beta + \gamma},
\]

implies
\[
\left| \frac{f'(z)}{pz^{p-1} - 1} \right| < 1 - \alpha.
\]
Proof. For the function \( w \) defined by

\[
\frac{f'(z)}{pz^{p-1}} = \frac{1 + aw(z)}{1 + w(z)}, \tag{2.11}
\]

we can rewrite (2.11) to yield

\[
\frac{f'(z)}{pz^{p-1}} - 1 = \frac{(\alpha - 1)w(z)}{1 + w(z)},
\]

which leads to

\[
\left| \frac{f'(z)}{pz^{p-1}} - 1 \right|^{\beta} = \frac{|w(z)|^{\beta}|1 - \alpha|^{\beta}}{|1 + w(z)|^{\beta}}. \tag{2.12}
\]

By some computation, it is evident that

\[
\frac{f''(z)}{zp^{-2}} - (p - 1)\frac{f'(z)}{zp^{-1}} = \frac{p(\alpha - 1)zw'(z)}{(1 + w(z))^2}
\]

or

\[
\left| \frac{f''(z)}{zp^{-2}} - (p - 1)\frac{f'(z)}{zp^{-1}} \right|^{\gamma} = \frac{p^{\gamma}|zw'(z)|^{\gamma}|1 - \alpha|^{\gamma}}{|1 + w(z)|^{2\gamma}}. \tag{2.13}
\]

From (2.12) and (2.13), it follows that

\[
\left| \frac{f'(z)}{pz^{p-1}} - 1 \right|^{\beta} \left| \frac{f''(z)}{zp^{-2}} - (p - 1)\frac{f'(z)}{zp^{-1}} \right|^{\gamma} = \frac{p^{\gamma}|w(z)|^{\beta}(1 - \alpha)^{\beta+\gamma}|zw'(z)|^{\gamma}}{|1 + w(z)|^{\beta+2\gamma}}.
\]

Suppose there exists a point \( z_0 \in \mathbb{D} \) such that

\[
|w(z_0)| = 1 \text{ and } |w(z)| < 1 \text{ when } |z| < |z_0|.
\]

Then (2.4) and Lemma 2.2 yield

\[
\left| \frac{f'(z_0)}{pz_0^{p-1}} - 1 \right|^{\beta} \left| \frac{f''(z_0)}{z_0^{p-2}} - (p - 1)\frac{f'(z_0)}{z_0^{p-1}} \right|^{\gamma} = \frac{p^{\gamma}(1 - \alpha)^{\beta+\gamma}|w(z_0)|^{\beta}|m(w(z_0))|^{\gamma}}{|1 + e^{i\theta}|^{\beta+2\gamma}}
\]

\[
\geq \frac{p^{\gamma}n^{\gamma}(1 - \alpha)^{\beta+\gamma}}{2^{\beta+2\gamma}},
\]

which contradicts the hypothesis (2.9). Hence \( |w(z)| < 1 \), which implies

\[
\left| \frac{1 - \frac{f'(z)}{pz^{p-1}}}{\frac{f'(z)}{pz^{p-1}} - \alpha} \right| < 1,
\]

or equivalently

\[
\text{Re} \left( \frac{f'(z)}{pz^{p-1}} \right) > \frac{1 + \alpha}{2}.
\]
For the second implication in the proof, consider the function \(w\) defined by

\[
\frac{f'(z)}{pz^{p-1}} = 1 + (1 - \alpha)w(z).
\]

(2.14)

Then

\[
\left| \frac{f'(z)}{pz^{p-1}} - 1 \right|^\beta = |1 - \alpha|^\beta |w(z)|^\beta
\]

(2.15)

and

\[
\left| \frac{f''(z)}{z^{p-2}} - (p-1)\frac{f'(z)}{z^{p-1}} \right|^{\gamma} = p^\gamma |zw'(z)|^{\gamma}|1 - \alpha|^{\gamma}.
\]

(2.16)

From (2.15) and (2.16), it is clear that

\[
\left| \frac{f'(z)}{pz^{p-1}} - 1 \right|^\beta \left| \frac{f''(z)}{z^{p-2}} - (p-1)\frac{f'(z)}{z^{p-1}} \right|^{\gamma} = p^\gamma |w(z)|^{\beta}|1 - \alpha|^\beta |zw'(z)|^{\gamma}.
\]

Suppose there exists a point \(z_0 \in D\) such that

\[
|w(z_0)| = 1 \quad \text{and} \quad |w(z)| < 1 \quad \text{when} \quad |z| < |z_0|.
\]

Then by applying Lemma 2.2 and using (2.4), it follows that

\[
\left| \frac{f'(z_0)}{pz_0^{p-1}} - 1 \right|^\beta \left| \frac{f''(z_0)}{z_0^{p-2}} - (p-1)\frac{f'(z_0)}{z_0^{p-1}} \right|^{\gamma} = p^\gamma |w(z_0)|^{\beta}|1 - \alpha|^\beta |zw'(z_0)|^{\gamma}
\]

\[
= p^\gamma m^\gamma |1 - \alpha|^\beta + \gamma
\]

\[
\geq (pn)^\gamma |1 - \alpha|^\beta + \gamma,
\]

which contradicts the hypothesis (2.10). Hence \(|w(z)| < 1\) and this implies

\[
\left| \frac{f'(z)}{pz^{p-1}} - 1 \right| < 1 - \alpha.
\]

Thus the proof is complete.

In next theorem, we need the concept of subordination. Let \(f\) and \(g\) be analytic functions defined on \(D\). Then \(f\) is subordinate to \(g\), written \(f \prec g\), provided there is an analytic function \(w : D \rightarrow D\) with \(w(0) = 0\) such that \(f = g \circ w\).

**Theorem 2.5.** Let \(\lambda_1\) and \(\lambda_2\) be given by

\[
\lambda_1 = \frac{n + 2}{4p + n - 2p},
\]

\[
\lambda_2 = \frac{n + 2}{2 - n},
\]

...
and $1 \leq \lambda_1 < \lambda < \lambda_2 \leq 3$. If the function $f \in \mathcal{A}_{p,n}$ satisfies the inequality

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \begin{cases} \frac{(4p+n)\lambda-n}{2(\lambda+1)}, & \lambda_1 < \lambda \leq \frac{p+n}{p}; \\ \frac{n(\lambda+1)}{2(\lambda-1)}, & \frac{p+n}{p} < \lambda < \lambda_2, \end{cases} \quad (2.17)$$

for $z \in \mathbb{D}$, then

$$\frac{1}{p} \frac{zf'(z)}{f(z)} < \frac{\lambda(1-z)}{\lambda-z}, \quad \text{for } z \in \mathbb{D}. \quad (2.18)$$

The result is sharp for the function $f$ given by

$$f(z) = z^p (\lambda - z)^{p(\lambda-1)}. \quad (2.19)$$

**Proof.** Let us define $w$ by

$$\frac{1}{p} \frac{zf'(z)}{f(z)} = \frac{\lambda(1-w(z))}{\lambda - w(z)}. \quad (2.20)$$

By doing the logarithmic differentiation on (2.20), we get

$$1 + \frac{zf''(z)}{f'(z)} = \frac{p\lambda(1-w(z))}{\lambda - z} - \frac{zw'(z)}{1 - w(z)} + \frac{zw'(z)}{\lambda - w(z)}. $$

Assume that there exists a point $z_0 \in \mathbb{D}$ such that $|w(z_0)| = 1$ and $|w(z)| < 1$ when $|z| < |z_0|$. By applying Lemma 2.2 as in Theorem 2.1, it follows that

$$\Re\left(1 + \frac{z_0f''(z_0)}{f'(z_0)}\right) = \Re\left(p\frac{\lambda(1-e^{i\theta})}{\lambda - e^{i\theta}}\right) - \Re\left(m\frac{e^{i\theta}}{1 - e^{i\theta}}\right) + \Re\left(e^{i\theta}\lambda\right)$$

$$= \frac{p\lambda(\lambda+1)(1-\cos\theta)}{\lambda^2 + 1 - 2\lambda \cos \theta} + \frac{m}{2} + \frac{m(\lambda \cos \theta - 1)}{\lambda^2 + 1 - 2\lambda \cos \theta}$$

$$= \frac{\lambda^2 + 1 - 2\lambda \cos \theta}{2(\lambda^2 + 1 - 2\lambda \cos \theta)} \left(\frac{(p+\lambda)}{2} - \frac{\lambda^2 - 1}{2(p+n) - p}\right)$$

which yields the inequality

$$\Re\left(1 + \frac{z_0f''(z_0)}{f'(z_0)}\right) \geq \begin{cases} \frac{(4p+n)\lambda-n}{2(\lambda+1)}, & \lambda_1 < \lambda \leq \frac{p+n}{p}; \\ \frac{n(\lambda+1)}{2(\lambda-1)}, & \frac{p+n}{p} < \lambda < \lambda_2. \end{cases} \quad (2.21)$$

Since (2.21) obviously contradicts hypothesis (2.17), it follows that $|w(z)| < 1$. This proves the subordination (2.18).

Finally, for (2.18) to be sharp, consider

$$\frac{1}{p} \frac{zf'(z)}{f(z)} = \frac{\lambda(1-z)}{\lambda-z}. \quad (2.22)$$
By integrating both sides of the equality and after some arrangement, we get

\[ f(z) = z^p (\lambda - z)^{p(\lambda - 1)}. \]

This completes the proof.

**Remark 2.1.** The subordination (2.18) can be written in equivalent form as

\[ \frac{\lambda (zf'(z)/pf(z) - 1)}{zf'(z)/pf(z) - \lambda} < 1, \]

or by further computation, as

\[ \frac{1}{p} \frac{zf'(z)}{f(z)} - \frac{\lambda}{\lambda + 1} < \frac{\lambda}{\lambda + 1}. \]

The last inequality shows that \( f \) is starlike in \( D \).

**Remark 2.2.** When \( p = 1 \) and \( n = 1 \), Theorems 2.1–2.5 reduce to Theorems 1.1 and 1.2.

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