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CLOSE-TO-CONVEXITY AND STARLIKENESS OF ANALYTIC FUNCTIONS

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Abstract. For functions $f(z) = z^p + a_{n+1}z^{p+1} + \cdots$ defined on the open unit disk, the condition $Re(f'(z)/z^{p-1}) > 0$ is sufficient for close-to-convexity of f. By making use of this result, several sufficient conditions for close-to-convexity are investigated and relevant connections with previously known results are indicated.

1. Introduction

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk and $\mathscr{A}_{p,n}$ be the class of all analytic functions $f : \mathbb{D} \to \mathbb{C}$ of the form $f(z) = z^p + a_{n+p}z^{n+p} + a_{n+p+1}z^{n+p+1} + \ldots$ with $\mathscr{A} := \mathscr{A}_{1,1}$. For studies related to multivalent functions, see [5, 7, 8, 9, 10]. Singh and Singh [16] obtained several interesting conditions for functions $f \in \mathscr{A}$ satisfying inequalities involving $f'(z)$ and $zf''(z)$ to be univalent or starlike in $\mathbb{D}$. Owa *et al.* [11] generalized the results of Singh and Singh [16] and also obtained several sufficient conditions for close-to-convexity, starlikeness and convexity of functions $f \in \mathscr{A}$. In fact, they have proved the following theorems.

Theorem 1.1 ([11], Theorems 1-3)**.** *Let* $0 \le \alpha < 1$ *and* $\beta, \gamma \ge 0$*. If* $f \in \mathscr{A}$*, then*

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{1+3\alpha}{2(1+\alpha)} \implies Re\big(f'(z)\big) > \frac{1+\alpha}{2},$$

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{3+2\alpha}{(2+\alpha)} \implies \big|f'(z) - 1\big| < 1 + \alpha,$$

$$\big|f'(z) - 1\big|^{\beta} |zf''(z)|^{\gamma} < \frac{(1-\alpha)^{\beta+\gamma}}{2^{\beta+2\gamma}} \implies Re\big(f'(z)\big) > \frac{1+\alpha}{2}.$$

Theorem 1.2 ([11], Theorem 4)**.** *Let* $1 < \lambda < 3$*. If* $f \in \mathscr{A}$*, then*

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \begin{cases} \frac{5\lambda-1}{2(\lambda+1)}, & 1 < \lambda \le 2; \\ \frac{\lambda+1}{2(\lambda-1)}, & 2 < \lambda < 3, \end{cases} \implies \frac{zf'(z)}{f(z)} \prec \frac{\lambda(1-z)}{\lambda - z}.$$

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In this present paper, the above results are extended for functions $f \in \mathcal{A}_{p,n}$.

2. Close-to-convexity and starlikeness

For $f \in \mathcal{A}$, the condition $Re f'(z) > 0$ implies close-to-convexity and univalence of f. Similarly, for $f \in \mathcal{A}_{p,1}$, the inequality $Re(f'(z)/z^{p-1}) > 0$ implies p-valency of f. See also [15, 18, 19]. From this result, the functions satisfying the hypothesis of Theorems 2.1–2.4 are p-valent in $\mathbb{D}$. A function $f \in \mathcal{A}_{p,1}$ is close-to-convex if there is a p-valent convex function ϕ such that $Re(f'(z)/\phi(z)) > 0$. Also they are all close-to-convex with respect to $\phi(z) = z^p$.

Theorem 2.1. *If the function $f \in \mathcal{A}_{p,n}$ satisfies the inequality*

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{(2p-n)+\alpha(2p+n)}{2(\alpha+1)}, \quad \text{for } z \in \mathbb{D}, \tag{2.1}$$

then

$$Re\left(\frac{f'(z)}{pz^{p-1}}\right) > \frac{1+\alpha}{2}, \quad \text{for } z \in \mathbb{D}.$$

For the proof of our main results, we need the following lemma.

Lemma 2.2. [6, Lemma 2.2a] *Let $z_0 \in \mathbb{D}$ and $r_0 = |z_0|$. Let $f(z) = a_n z^n + a_{n+1} z^{n+1} + \cdots$ be continuous on $\overline{\mathbb{D}}_{r_0}$ and analytic on $\mathbb{D}_{r_0} \cup \{z_0\}$ with $f(z) \not\equiv 0$ and $n \geq 1$. If*

$$|f(z_0)| = \max\{|f(z)| : z \in \overline{\mathbb{D}}_{r_0}\},$$

then there exists an $m \geq n$ such that

(1) $\dfrac{z_0 f'(z_0)}{f(z_0)} = m$, *and*

(2) $Re \dfrac{z_0 f''(z_0)}{f'(z_0)} + 1 \geq m$.

Proof of Theorem 2.1. Let the function w be defined by

$$\frac{f'(z)}{pz^{p-1}} = \frac{1+\alpha w(z)}{1+w(z)}. \tag{2.2}$$

Then w can be written as

$$w(z) = \frac{1}{\alpha-1}\left[\frac{(n+p)}{p} a_{n+p} z^n - \frac{(n+p)^2}{p^2(1-\alpha)} a_{n+p}^2 z^{2n} + \cdots\right],$$

hence it is analytic in $\mathbb{D}$ with $w(0) = 0$. From (2.2), some computation yields

$$1 + \frac{zf''(z)}{f'(z)} = p + \frac{\alpha z w'(z)}{1+\alpha w(z)} - \frac{zw'(z)}{1+w(z)}. \tag{2.3}$$

Suppose there exists a point $z_0 \in \mathbb{D}$ such that

$$|w(z_0)| = 1 \text{ and } |w(z)| < 1 \text{ when } |z| < |z_0|.$$

Then by applying Lemma 2.2, there exists $m \geq n$ such that

$$z_0 w'(z_0) = m w(z_0), \quad (w(z_0) = e^{i\theta}; \theta \in \mathbb{R}). \tag{2.4}$$

Thus, by using (2.3) and (2.4), it follows that

$$\begin{aligned} Re\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right) &= p + Re\left(\frac{\alpha m w(z_0)}{1 + \alpha w(z_0)}\right) - Re\left(\frac{m w(z_0)}{1 + w(z_0)}\right) \\ &= p + Re\left(\frac{\alpha m e^{i\theta}}{1 + \alpha e^{i\theta}}\right) - Re\left(\frac{m e^{i\theta}}{1 + e^{i\theta}}\right) \\ &= p + \frac{\alpha m(\alpha + \cos\theta)}{1 + \alpha^2 + 2\alpha\cos\theta} - \frac{m}{2} \\ &\leq \frac{(2p - n) + \alpha(2p + n)}{2(\alpha + 1)}, \end{aligned}$$

which contradicts the hypothesis (2.1). It follows that $|w(z)| < 1$, that is,

$$\left| \frac{1 - \frac{f'(z)}{pz^{p-1}}}{\frac{f'(z)}{pz^{p-1}} - \alpha} \right| < 1.$$

This evidently completes the proof of Theorem 2.1. □

Owa [13] shows that a function $f \in \mathcal{A}_{p,1}$ satisfying $Re(1 + zf''(z)/f'(z)) < p + 1/2$ implies f is p-valently starlike. Our next theorem investigates the close-to-convexity of this type of functions. For related results, see [14, 4, 20].

Theorem 2.3. *If the function $f \in \mathcal{A}_{p,n}$ satisfies the inequality*

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{(p+n)\alpha + (2p+n)}{(\alpha + 2)}, \quad \textit{for } z \in \mathbb{D}, \tag{2.5}$$

then

$$\left| \frac{f'(z)}{pz^{p-1}} - 1 \right| < 1 + \alpha, \quad \textit{for } z \in \mathbb{D}.$$

Proof. Consider the function w defined by

$$\frac{f'(z)}{pz^{p-1}} = (1 + \alpha) w(z) + 1. \tag{2.6}$$

It can be checked similarly as above that w is analytic in $\mathbb{D}$ with $w(0) = 0$. From (2.6), some computation yields

$$1 + \frac{zf''(z)}{f'(z)} = p + \frac{(1 + \alpha) z w'(z)}{(1 + \alpha) w(z) + 1}. \tag{2.7}$$

Suppose there exists a point $z_0 \in \mathbb{D}$ such that

$$|w(z_0)| = 1 \text{ and } |w(z)| < 1 \text{ when } |z| < |z_0|.$$

Then by applying Lemma 2.2, there exists $m \geq n$ such that

$$z_0 w'(z_0) = m w(z_0), \quad (w(z_0) = e^{i\theta}; \theta \in \mathbb{R}). \tag{2.8}$$

Thus, by using (2.7) and (2.8), it follows that

$$\begin{aligned} Re\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right) &= p + Re\left(\frac{(1+\alpha) z_0 w'(z_0)}{(1+\alpha) w(z_0) + 1}\right) \\ &= p + Re\left(\frac{(1+\alpha) m e^{i\theta}}{(1+\alpha) e^{i\theta} + 1}\right) \\ &= p + \frac{m(1+\alpha)(1+\alpha+\cos\theta)}{1 + (1+\alpha)^2 + 2(1+\alpha)\cos\theta} \\ &\geq \frac{(p+n)\alpha + (2p+n)}{(\alpha+2)}, \end{aligned}$$

which contradicts the hypothesis (2.5). It follows that $|w(z)| < 1$, that is,

$$\left|\frac{f'(z)}{pz^{p-1}} - 1\right| < 1 + \alpha.$$

This evidently completes the proof of Theorem 2.3. ☐

Owa [12] has also showed that a function $f \in \mathcal{A}$ satisfying $|f'(z)/g'(z) - 1|^{\beta} |zf''(z)/g'(z) - zf'(z)g''(z)/(g'(z))^2|^{\gamma} < (1+\alpha)^{\beta+\alpha}$, for $0 \leq \alpha < 1$, $\beta \geq 0$, $\gamma \geq 0$ and g a convex function, is close-to-convex. Also, see [3]. Our next theorem investigates the close-to-convexity of similar class of functions.

Theorem 2.4. *If $f \in \mathcal{A}_{p,n}$, then for $z \in \mathbb{D}$,*

$$\left|\frac{f'(z)}{pz^{p-1}} - 1\right|^{\beta} \left|\frac{f''(z)}{z^{p-2}} - (p-1)\frac{f'(z)}{z^{p-1}}\right|^{\gamma} < \frac{(pn)^{\gamma}(1-\alpha)^{\beta+\gamma}}{2^{\beta+2\gamma}}, \tag{2.9}$$

implies

$$Re\left(\frac{f'(z)}{pz^{p-1}}\right) > \frac{1+\alpha}{2},$$

and

$$\left|\frac{f'(z)}{pz^{p-1}} - 1\right|^{\beta} \left|\frac{f''(z)}{z^{p-2}} - (p-1)\frac{f'(z)}{z^{p-1}}\right|^{\gamma} < (pn)^{\gamma}|1-\alpha|^{\beta+\gamma}, \tag{2.10}$$

implies

$$\left|\frac{f'(z)}{pz^{p-1}} - 1\right| < 1 - \alpha.$$

Proof. For the function w defined by

$$\frac{f'(z)}{pz^{p-1}} = \frac{1+\alpha w(z)}{1+w(z)}, \tag{2.11}$$

we can rewrite (2.11) to yield

$$\frac{f'(z)}{pz^{p-1}} - 1 = \frac{(\alpha-1)w(z)}{1+w(z)},$$

which leads to

$$\left|\frac{f'(z)}{pz^{p-1}} - 1\right|^{\beta} = \frac{|w(z)|^{\beta}|1-\alpha|^{\beta}}{|1+w(z)|^{\beta}}. \tag{2.12}$$

By some computation, it is evident that

$$\frac{f''(z)}{z^{p-2}} - (p-1)\frac{f'(z)}{z^{p-1}} = \frac{p(\alpha-1)zw'(z)}{(1+w(z))^2}$$

or

$$\left|\frac{f''(z)}{z^{p-2}} - (p-1)\frac{f'(z)}{z^{p-1}}\right|^{\gamma} = \frac{p^{\gamma}|zw'(z)|^{\gamma}|1-\alpha|^{\gamma}}{|1+w(z)|^{2\gamma}}. \tag{2.13}$$

From (2.12) and (2.13), it follows that

$$\left|\frac{f'(z)}{pz^{p-1}} - 1\right|^{\beta}\left|\frac{f''(z)}{z^{p-2}} - (p-1)\frac{f'(z)}{z^{p-1}}\right|^{\gamma} = \frac{p^{\gamma}|w(z)|^{\beta}(1-\alpha)^{\beta+\gamma}|zw'(z)|^{\gamma}}{|1+w(z)|^{\beta+2\gamma}}.$$

Suppose there exists a point $z_0 \in \mathbb{D}$ such that

$$|w(z_0)| = 1 \text{ and } |w(z)| < 1 \text{ when } |z| < |z_0|.$$

Then (2.4) and Lemma 2.2 yield

$$\begin{aligned}\left|\frac{f'(z_0)}{pz_0^{p-1}} - 1\right|^{\beta}\left|\frac{f''(z_0)}{z_0^{p-2}} - (p-1)\frac{f'(z_0)}{z_0^{p-1}}\right|^{\gamma} &= \frac{p^{\gamma}(1-\alpha)^{\beta+\gamma}|w(z_0)|^{\beta}|mw(z_0)|^{\gamma}}{|1+e^{i\theta}|^{\beta+2\gamma}}\\ &= \frac{p^{\gamma}m^{\gamma}(1-\alpha)^{\beta+\gamma}}{(2+2\cos\theta)^{(\beta+2\gamma)/2}}\\ &\geq \frac{p^{\gamma}n^{\gamma}(1-\alpha)^{\beta+\gamma}}{2^{\beta+2\gamma}},\end{aligned}$$

which contradicts the hypothesis (2.9). Hence $|w(z)| < 1$, which implies

$$\left|\frac{1-\frac{f'(z)}{pz^{p-1}}}{\frac{f'(z)}{pz^{p-1}}-\alpha}\right| < 1,$$

or equivalently

$$Re\left(\frac{f'(z)}{pz^{p-1}}\right) > \frac{1+\alpha}{2}.$$

For the second implication in the proof, consider the function w defined by

$$\frac{f'(z)}{pz^{p-1}} = 1 + (1-\alpha)w(z). \tag{2.14}$$

Then

$$\left|\frac{f'(z)}{pz^{p-1}} - 1\right|^{\beta} = |1-\alpha|^{\beta}|w(z)|^{\beta} \tag{2.15}$$

and

$$\left|\frac{f''(z)}{z^{p-2}} - (p-1)\frac{f'(z)}{z^{p-1}}\right|^{\gamma} = p^{\gamma}|zw'(z)|^{\gamma}|1-\alpha|^{\gamma}. \tag{2.16}$$

From (2.15) and (2.16), it is clear that

$$\left|\frac{f'(z)}{pz^{p-1}} - 1\right|^{\beta}\left|\frac{f''(z)}{z^{p-2}} - (p-1)\frac{f'(z)}{z^{p-1}}\right|^{\gamma} = p^{\gamma}|w(z)|^{\beta}|1-\alpha|^{\beta+\gamma}|zw'(z)|^{\gamma}.$$

Suppose there exists a point $z_0 \in \mathbb{D}$ such that

$$|w(z_0)| = 1 \text{ and } |w(z)| < 1 \text{ when } |z| < |z_0|.$$

Then by applying Lemma 2.2 and using (2.4), it follows that

$$\begin{aligned}\left|\frac{f'(z_0)}{pz_0^{p-1}} - 1\right|^{\beta}\left|\frac{f''(z_0)}{z_0^{p-2}} - (p-1)\frac{f'(z_0)}{z_0^{p-1}}\right|^{\gamma} &= p^{\gamma}|w(z_0)|^{\beta}|1-\alpha|^{\beta+\gamma}|z_0w'(z_0)|^{\gamma}\\ &= p^{\gamma}m^{\gamma}|1-\alpha|^{\beta+\gamma}\\ &\geq (pn)^{\gamma}|1-\alpha|^{\beta+\gamma},\end{aligned}$$

which contradicts the hypothesis (2.10). Hence $|w(z)| < 1$ and this implies

$$\left|\frac{f'(z)}{pz^{p-1}} - 1\right| < 1-\alpha.$$

Thus the proof is complete. □

In next theorem, we need the concept of subordination. Let f and g be analytic functions defined on $\mathbb{D}$. Then f is *subordinate* to g, written $f \prec g$, provided there is an analytic function $w : \mathbb{D} \to \mathbb{D}$ with $w(0) = 0$ such that $f = g \circ w$.

Theorem 2.5. *Let λ_1 and λ_2 be given by*

$$\lambda_1 = \frac{n+2}{4p+n-2p},$$
$$\lambda_2 = \frac{n+2}{2-n},$$

and $1 \le \lambda_1 < \lambda < \lambda_2 \le 3$. *If the function* $f \in \mathcal{A}_{p,n}$ *satisfies the inequality*

$$Re\left(1+\frac{zf''(z)}{f'(z)}\right) < \begin{cases} \frac{(4p+n)\lambda-n}{2(\lambda+1)}, & \lambda_1 < \lambda \le \frac{p+n}{p}; \\ \frac{n(\lambda+1)}{2(\lambda-1)}, & \frac{p+n}{p} < \lambda < \lambda_2, \end{cases} \tag{2.17}$$

for $z \in \mathbb{D}$, *then*

$$\frac{1}{p}\frac{zf'(z)}{f(z)} \prec \frac{\lambda(1-z)}{\lambda-z}, \quad \text{for } z \in \mathbb{D}. \tag{2.18}$$

The result is sharp for the function f *given by*

$$f(z) = z^p(\lambda-z)^{p(\lambda-1)}. \tag{2.19}$$

Proof. Let us define w by

$$\frac{1}{p}\frac{zf'(z)}{f(z)} = \frac{\lambda(1-w(z))}{\lambda-w(z)}. \tag{2.20}$$

By doing the logarithmic differentiation on (2.20), we get

$$1+\frac{zf''(z)}{f'(z)} = \frac{p\lambda(1-w(z))}{\lambda-z} - \frac{zw'(z)}{1-w(z)} + \frac{zw'(z)}{\lambda-w(z)}.$$

Assume that there exists a point $z_0 \in \mathbb{D}$ such that $|w(z_0)| = 1$ and $|w(z)| < 1$ when $|z| < |z_0|$. By applying Lemma 2.2 as in Theorem 2.1, it follows that

$$\begin{aligned} Re\left(1+\frac{z_0f''(z_0)}{f'(z_0)}\right) &= Re\left(\frac{p\lambda(1-e^{i\theta})}{\lambda-e^{i\theta}}\right) - Re\left(\frac{me^{i\theta}}{1-e^{i\theta}}\right) + Re\left(\frac{me^{i\theta}}{\lambda-e^{i\theta}}\right) \\ &= \frac{p\lambda(\lambda+1)(1-\cos\theta)}{\lambda^2+1-2\lambda\cos\theta} + \frac{m}{2} + \frac{m(\lambda\cos\theta-1)}{\lambda^2+1-2\lambda\cos\theta} \\ &= \frac{\lambda+1}{2}p + \frac{(\lambda^2-1)[(p+m)-p\lambda]}{2(\lambda^2+1-2\lambda\cos\theta)} \\ &\ge \frac{\lambda+1}{2}p + \frac{(\lambda^2-1)[(p+n)-p\lambda]}{2(\lambda^2+1-2\lambda\cos\theta)}, \end{aligned}$$

which yields the inequality

$$Re\left(1+\frac{z_0f''(z_0)}{f'(z_0)}\right) \ge \begin{cases} \frac{(4p+n)\lambda-n}{2(\lambda+1)}, & \lambda_1 < \lambda \le \frac{p+n}{p}; \\ \frac{n(\lambda+1)}{2(\lambda-1)}, & \frac{p+n}{p} < \lambda < \lambda_2. \end{cases} \tag{2.21}$$

Since (2.21) obviously contradicts hypothesis (2.17), it follows that $|w(z)| < 1$. This proves the subordination (2.18).

Finally, for (2.18) to be sharp, consider

$$\frac{1}{p}\frac{zf'(z)}{f(z)} = \frac{\lambda(1-z)}{\lambda-z}. \tag{2.22}$$

By integrating both sides of the equality and after some arrangement, we get

$$f(z) = z^p (\lambda - z)^{p(\lambda-1)}.$$

This completes the proof. □

Remark 2.1. The subordination (2.18) can be written in equivalent form as

$$\left| \frac{\lambda(zf'(z)/pf(z) - 1)}{zf'(z)/pf(z) - \lambda} \right| < 1,$$

or by further computation, as

$$\left| \frac{1}{p}\frac{zf'(z)}{f(z)} - \frac{\lambda}{\lambda+1} \right| < \frac{\lambda}{\lambda+1}.$$

The last inequality shows that f is starlike in $\mathbb{D}$.

Remark 2.2. When $p = 1$ and $n = 1$, Theorems 2.1–2.5 reduce to Theorems 1.1 and 1.2.

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