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MONOTONICITY OF SEQUENCES INVOLVING GENERALIZED CONVEXITY FUNCTION AND SEQUENCES

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Abstract. In this paper, by using the theory of generalized convexity functions we introduce and prove monotonicity of sequences of the forms

$$\left\{\left(\prod_{k=1}^{n} f\left(\frac{a_k}{a_n}\right)\right)^{1/n}\right\},\quad \left\{\left(\prod_{k=1}^{n} f\left(\frac{\varphi(k)}{\varphi(n)}\right)\right)^{1/\varphi(n)}\right\},$$

$$\left\{\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{a_n}{a_k}\right)\right\} \quad \text{or} \quad \left\{\frac{1}{\varphi(n)}\sum_{k=1}^{n} f\left(\frac{\varphi(n)}{\varphi(k)}\right)\right\},$$

where f belongs to the classes of AG-convex (concave), HA-convex (concave), or HG-convex (concave) functions defined on suitable intervals, $\{a_n\}$ is a given sequence and φ is a given function that satisfy some preset conditions. As a consequence, we obtain some generalizations of Alzer type inequalities.

1. Introduction

Let f be a real-valued function defined on $[a,b] \subset \mathbb{R}$. The function f is called convex if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y). \tag{1.1}$$

for all $x, y \in [a,b]$ and $\lambda \in [0,1]$. If (1.1) is strict for all $x \neq y$ and $\lambda \in (0,1)$, then f is said to be strictly convex. If the inequality in (1.1) is reversed, then f is said to be concave. If the inequality (1.1) is reversed and strict for all $x \neq y$ and $\lambda \in (0,1)$, then f is said to be strictly concave.

Suppose that I is a subinterval of $(0,\infty)$. A function $f : I \to (0,\infty)$ is called multiplicatively convex if for all $x, y \in I$ and $\lambda \in [0,1]$,

$$f(x^{\lambda} y^{1-\lambda}) \le f(x)^{\lambda} f(y)^{1-\lambda}. \tag{1.2}$$

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If (1.2) is strict for all $x \neq y$ and $\lambda \in (0,1)$, then f is said to be strictly multiplicatively convex. If the inequality in (1.2) is reversed, then f is said to be multiplicatively concave. If inequality (1.2) is reversed and strict for all $x \neq y$ and $\lambda \in (0,1)$, then f is said to be strictly multiplicatively concave.

In [3], F. Qi and B.-N. Guo proved the following theorems:

Theorem 1.1 ([3]). *Let f be an increasing, convex (concave, respectively) function defined on $[0,1]$, $\{a_n\}$ an increasing, positive sequence such that $\{n\big(\frac{a_n}{a_{n+1}}-1\big)\}$ decreases (the sequence $\{n\big(\frac{a_{n+1}}{a_n}-1\big)\}$ increses, respectively), then*

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{a_k}{a_n}\right) \geq \frac{1}{n+1}\sum_{k=1}^{n+1} f\left(\frac{a_k}{a_{n+1}}\right) \geq \int_0^1 f(x)\,dx \tag{1.3}$$

and

Theorem 1.2 ([3]). *Let f be an increasing convex (or concave) positive function defined on $[0,1]$, φ be an increasing convex positive function defined on $(0,\infty)$ such that $\{\varphi(k)\big(\frac{\varphi(k)}{\varphi(k+1)}-1\big)\}$ decreases, then*

$$\frac{1}{\varphi(n)}\sum_{k=1}^{n} f\left(\frac{\varphi(k)}{\varphi(n)}\right) \geq \frac{1}{\varphi(n+1)}\sum_{k=1}^{n+1} f\left(\frac{\varphi(k)}{\varphi(n+1)}\right). \tag{1.4}$$

Jiding Liao and Kaizhong Guan [2] proved the following theorems:

Theorem 1.3 ([2]). *Let f be a positive function defined in $(0,1]$. Suppose that $\{a_n\}$ is an increasing positive sequence such that the sequence $\{\big(\frac{a_{n+1}}{a_n}\big)^n\}$ increases.*

(1) *If f is an increasing and multiplicatively convex (concave) function, then*

$$\left(\prod_{k=1}^{n} f\left(\frac{a_k}{a_n}\right)\right)^{1/n} \geq \left(\prod_{k=1}^{n+1} f\left(\frac{a_k}{a_{n+1}}\right)\right)^{1/(n+1)}. \tag{1.5}$$

(2) *If f is an decreasing and multiplicatively convex (concave) function, then*

$$\left(\prod_{k=1}^{n} f\left(\frac{a_k}{a_n}\right)\right)^{1/n} \leq \left(\prod_{k=1}^{n+1} f\left(\frac{a_k}{a_{n+1}}\right)\right)^{1/(n+1)}. \tag{1.6}$$

and

Theorem 1.4 ([2]). *Let $f:(0,1]\to[1,+\infty)$ be a real-valued function and $\{a_n\}$ an increasing positive sequence such that the sequence $\{\big(\frac{a_{n+1}}{a_n}\big)^{a_n}\}$ increases. Then the following statements are valid.*

(1) *If f is an increasing and multiplicatively convex (concave) function and $\{a_n\}$ is convex sequence, i.e., $a_{n-1}+a_{n+1}\ge 2a_n$, $(n=1,2,\dots)$ where $a_0=0$, then*

$$\left(\prod_{k=1}^{n} f\left(\frac{a_k}{a_n}\right)\right)^{1/a_n} \ge \left(\prod_{k=1}^{n+1} f\left(\frac{a_k}{a_{n+1}}\right)\right)^{1/a_{n+1}}. \tag{1.7}$$

(2) *If f is an decreasing and multiplicatively convex (concave) function and $\{a_n\}$ is concave sequence, i.e., $a_{n-1}+a_{n+1}\le 2a_n$, $(n=1,2,\dots)$ where $a_0=0$, then*

$$\left(\prod_{k=1}^{n} f\left(\frac{a_k}{a_n}\right)\right)^{1/a_n} \le \left(\prod_{k=1}^{n+1} f\left(\frac{a_k}{a_{n+1}}\right)\right)^{1/a_{n+1}}. \tag{1.8}$$

The above results are valid for the convex (concave) function and multiplicatively convex (concave) function. In [1], the authors introduced the class of mean function and generalized convexity. The class related directly to convex (concave) function.

Definition 1.1 ([1])**.** A function $M:(0,\infty)\times(0,\infty)\to(0,\infty)$ is called a *mean function* if

(1) $M(x,y)=M(y,x)$;

(2) $M(x,x)=x$;

(3) $x<M(x,y)<y$, whenever $x<y$;

(4) $M(ax,ay)=aM(x,y)$ for all $a>0$.

Some familiar mean functions such as Arithmetic Mean, Geometric Mean, Harmonic Mean, Logarithmic Mean, Identric Mean and denoted by A,G,H,L,I, respectively. For details concerning mean functions A,G,H,L,I we refer to the papers [1] and [5].

Definition 1.2 ([1])**.** Let $f:I\to(0,\infty)$ be continuous, where I is a subinterval of $(0,\infty)$. Let M and N be any two mean functions. We say f is MN-convex (concave) if

$$f(M(x,y)) \le (\ge)\ N(f(x),f(y)), \tag{1.9}$$

for all $x,y\in I$.

From Definition 1.2, the inequalities (1.1) and (1.2) can be rewritten under the simple forms

$$f(A(x,y))\le A(f(x),f(y)) \quad \text{and} \quad f(G(x,y))\le G(f(x),f(y)).$$

More precisely, f is AA-convex for the first case and GG-convex for the second case.

Our main purpose of this paper is to present some inequalities which are similar to the results in [2] and [3] for some generalized convexity functions such as AG-convex (concave), HA-convex (concave) and HG-convex (concave).

2. The main results

In this section, we investigate the monotonicity of some sequences involving AG, HA, HG- convex (concave) function and convex sequence.

Theorem 2.1. *Let f be an increasing, AG-convex (concave, respectively) function defined on* $(0,1]$.

(1) *If $\{a_n\}$ is an increasing, positive sequence such that $\{n\big(\frac{a_n}{a_{n+1}}-1\big)\}$ decreases (the sequence $\{n\big(\frac{a_{n+1}}{a_n}-1\big)\}$ increses, respectively), then*

$$\left(\prod_{k=1}^{n} f\left(\frac{a_k}{a_n}\right)\right)^{1/n} \geq \left(\prod_{k=1}^{n+1} f\left(\frac{a_k}{a_{n+1}}\right)\right)^{1/(n+1)}. \tag{2.1}$$

(2) *If φ is an increasing convex positive function defined on $(0,\infty)$ such that $\{\varphi(k)\big(\frac{\varphi(k)}{\varphi(k+1)}-1\big)\}$ decreases, then*

$$\left(\prod_{k=1}^{n} f\left(\frac{\varphi(k)}{\varphi(n)}\right)\right)^{1/\varphi(n)} \geq \left(\prod_{k=1}^{n+1} f\left(\frac{\varphi(k)}{\varphi(n+1)}\right)\right)^{1/\varphi(n+1)}. \tag{2.2}$$

Proof. Here we only give the proof of the AG-convex, since that the AG-concave is similar and we omit it.

By Theorem 2.4 in [1], the function f is AG-convex (concave) if and only if $\ln f$ is convex (concave). Obviously, $\ln f$ increases by the increase of f. Hence, applying Theorem 1.1 for $\ln f$, we have

$$\frac{1}{n}\sum_{k=1}^{n} \ln f\left(\frac{a_k}{a_n}\right) \geq \frac{1}{n+1}\sum_{k=1}^{n+1} \ln f\left(\frac{a_k}{a_{n+1}}\right).$$

It is equivalent to

$$\ln \prod_{k=1}^{n} f\left(\frac{a_k}{a_n}\right)^{1/n} \geq \ln \prod_{k=1}^{n+1} f\left(\frac{a_k}{a_{n+1}}\right)^{1/(n+1)} \Leftrightarrow \left(\prod_{k=1}^{n} f\left(\frac{a_k}{a_n}\right)\right)^{1/n} \geq \left(\prod_{k=1}^{n+1} f\left(\frac{a_k}{a_{n+1}}\right)\right)^{1/(n+1)}.$$

So, the proof of (2.1) is complete.

Analogously, if applying Theorem 1.2 for $\ln f$, then

$$\frac{1}{\varphi(n)}\sum_{k=1}^{n} \ln f\left(\frac{\varphi(k)}{\varphi(n)}\right) \geq \frac{1}{\varphi(n+1)}\sum_{k=1}^{n+1} \ln f\left(\frac{\varphi(k)}{\varphi(n+1)}\right).$$

Equivalently,

$$\ln \prod_{k=1}^{n} f\left(\frac{\varphi(k)}{\varphi(n)}\right)^{1/\varphi(n)} \geq \ln \prod_{k=1}^{n+1} f\left(\frac{\varphi(k)}{\varphi(n)}\right)^{1/\varphi(n+1)} \Leftrightarrow \left(\prod_{k=1}^{n} f\left(\frac{\varphi(k)}{\varphi(n)}\right)\right)^{1/\varphi(n)} \geq \left(\prod_{k=1}^{n+1} f\left(\frac{\varphi(k)}{\varphi(n+1)}\right)\right)^{1/\varphi(n+1)}.$$

Hence, the inequality (2.2) is completely proved. □

Theorem 2.2. *Let f be a decreasing, HA-convex (concave, respectively) function defined on* $[1,+\infty)$.

(1) *If* $\{a_n\}$ *an increasing, positive sequence such that* $\{n\big(\frac{a_n}{a_{n+1}}-1\big)\}$ *decreases (the sequence* $\{n\big(\frac{a_{n+1}}{a_n}-1\big)\}$ *increses, respectively), then*

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{a_n}{a_k}\right) \geq \frac{1}{n+1}\sum_{k=1}^{n+1} f\left(\frac{a_{n+1}}{a_k}\right). \tag{2.3}$$

(2) *If φ be an increasing convex positive function defined on* $(0,\infty)$ *such that* $\{\varphi(k)\big(\frac{\varphi(k)}{\varphi(k+1)}-1\big)\}$ *decreases, then*

$$\frac{1}{\varphi(n)}\sum_{k=1}^{n} f\left(\frac{\varphi(n)}{\varphi(k)}\right) \geq \frac{1}{\varphi(n+1)}\sum_{k=1}^{n+1} f\left(\frac{\varphi(n+1)}{\varphi(k)}\right). \tag{2.4}$$

Proof. Here we only give the proof of (2), since that (1) is similar and we omit it.

By Theorem 2.4 in [1], the function f is HA-convex (concave) if and only if $f\big(1/x\big)$ is convex (concave). It's easy to see that $g(x) := f(1/x)$ increases by the decrease of f. Hence, applying Theorem 1.2 for g, we have

$$\frac{1}{\varphi(n)}\sum_{k=1}^{n} g\left(\frac{\varphi(k)}{\varphi(n)}\right) \geq \frac{1}{\varphi(n+1)}\sum_{k=1}^{n+1} g\left(\frac{\varphi(k)}{\varphi(n+1)}\right).$$

Noting that, in the above inequality, $g\big(\frac{\varphi(k)}{\varphi(n)}\big) = f\big(\frac{\varphi(n)}{\varphi(k)}\big)$ for all $k = 1,2,\ldots,n$ and $g\big(\frac{\varphi(k)}{\varphi(n+1)}\big) = f\big(\frac{\varphi(n+1)}{\varphi(k)}\big)$ for all $k = 1,2,\ldots,n+1$, and so the proof of the inequality (2.4) is complete. □

Theorem 2.3. *Let f be a decreasing, HG-convex (concave, respectively) function defined on* $[1,+\infty)$.

(1) *If* $\{a_n\}$ *an increasing, positive sequence such that* $\{n\big(\frac{a_n}{a_{n+1}}-1\big)\}$ *decreases (the sequence* $\{n\big(\frac{a_{n+1}}{a_n}-1\big)\}$ *increses, respectively), then*

$$\left(\prod_{k=1}^{n} f\left(\frac{a_n}{a_k}\right)\right)^{1/n} \geq \left(\prod_{k=1}^{n+1} f\left(\frac{a_{n+1}}{a_k}\right)\right)^{1/(n+1)}. \tag{2.5}$$

(2) *If φ be an increasing convex positive function defined on* $(0,\infty)$ *such that* $\{\varphi(k)\big(\frac{\varphi(k)}{\varphi(k+1)}-1\big)\}$ *decreases, then*

$$\left(\prod_{k=1}^{n} f\left(\frac{\varphi(n)}{\varphi(k)}\right)\right)^{1/\varphi(n)} \geq \left(\prod_{k=1}^{n+1} f\left(\frac{\varphi(n+1)}{\varphi(k)}\right)\right)^{1/\varphi(n+1)}. \tag{2.6}$$

Proof. The proof runs as in the proof of Theorem 2.1. Here, the increase of $\ln f(1/x)$ is deduced from the decrease of f. □

Remark 2.4. In Theorem 1.1, if we replace f increasing with decreasing, then the inequality (1.3) is reversed. That is

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{a_k}{a_n}\right) \le \frac{1}{n+1}\sum_{k=1}^{n+1} f\left(\frac{a_k}{a_{n+1}}\right) \le \int_0^1 f(x)dx \tag{2.7}$$

Indeed, by the decrease of f on $[0,1]$ we have $-f$ is increasing. Therefore, applying directly Theorem 1.1 for this function we obtain the inequality (2.7). This implies the inequality (2.1) is reversed whenever f decreasing and the inequalities (2.3), (2.5) are reversed whenever f increasing.

3. Corollaries

From these theorems, we can obtain many new inequalities related to Alzer's inequality and others or, similar inequalities to those in [3].

Corollary 3.1. *Let φ be an increasing convex positive function defined on $(0,\infty)$ such that $\{\varphi(k)\big(\frac{\varphi(k)}{\varphi(k+1)}-1\big)\}$ decreases, then*

$$\frac{\sqrt[\varphi(n)]{\prod_{k=1}^{n}\varphi(k)}}{\sqrt[\varphi(n+1)]{\prod_{k=1}^{n+1}\varphi(k)}} \ge \frac{\varphi(n)^{n/\varphi(n)}}{\varphi(n+1)^{(n+1)/\varphi(n+1)}}. \tag{3.1}$$

Proof. Taking $f(x) = x$ is an increasing function on $(0,1]$. Moreover, we have $\frac{f'(x)}{f(x)} = \frac{1}{x}$ is a decreasing function on $(0,1]$. By Corollary 2.5 in [1], f is AG-concave. So, applying Theorem 2.1 for this function we get the inequality (3.1). □

Corollary 3.2. *Let $r>0$ and φ be an increasing convex positive function defined on $(0,\infty)$ such that $\{\varphi(k)\big(\frac{\varphi(k)}{\varphi(k+1)}-1\big)\}$ decreases, then*

$$\frac{1}{\varphi(n)}\sum_{k=1}^{n}\frac{\varphi(k)^r}{\varphi(n)^r} \ge \frac{1}{\varphi(n+1)}\sum_{k=1}^{n+1}\frac{\varphi(k)^r}{\varphi(n+1)^r}. \tag{3.2}$$

Proof. Taking $f(x) = 1/x^r$ where $r>0$ for $x\in[1,+\infty)$. Obviously, f is decreasing on $[1,+\infty)$. Moreover, we have

$$g(x) := (x^2 f'(x))' = (-r x^{1-r})' = r(r-1)x^{-r}, \quad \forall x\in(1,+\infty).$$

It's easy to see that $g(x)>0$ whenever $r>1$ and $g(x)<0$ whenever $0<r<1$. So, by Corollary 2.5 in [1], f is HA-convex (concave) whenever $r>1$ ($0<r<1$, respectively). So, applying Theorem 2.2 for this function we get the inequality (3.2). □

If taking $f(x) = x^{1/x}e^{1/x}$ for $x\in[1,+\infty)$, then f is decreasing. And, we have $x^2 f'(x)/f(x) = -\ln x$ is a decreasing function on $(1,+\infty)$. Hence, by Corollary 2.5 in [1], f is HG-concave. By applying direct Theorem 2.3, we obtain

Corollary 3.3. *For all natural number n, the following inequality is valid*

$$\frac{n^{(n+1)/2n}}{(n+1)^{(n+2)/2(n+1)}}e^{1/[2n(n+1)]} \geq \frac{\sqrt[n^2]{\prod_{k=1}^{n} k^k}}{\sqrt[(n+1)^2]{\prod_{k=1}^{n+1} k^k}}. \tag{3.3}$$

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