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ELLIPTICALLY SYMMETRIC BESSEL DISTRIBUTION: THE DISTRIBUTION OF XY

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Abstract. The exact distribution of the product XY is derived when (X, Y) has the elliptically symmetric Bessel distribution.

1. Introduction

For a bivariate random vector (X, Y), the distribution of the product $\mid XY \mid$ is of interest in problems in biological and physical sciences, econometrics, and classification. As an example in Physics, Sornette (1998) mentions:

> "... To mimic system size limitation, Takayasu, Sato, and Takayasu introduced a threshold x_c ... and found a stretched exponential truncating the power-law pdf beyond x_c. Frisch and Sornette recently developed a theory of extreme deviations generalizing the central limit theorem which, when applied to multiplication of random variables, predicts the generic presence of stretched exponential pdfs. The problem thus boils down to determining the tail of the pdf for a product of random variables ..."

The distribution of XY has been studied by several authors especially when X and Y are independent random variables and come from the same family. For instance, see Sakamoto (1943) for uniform family, Harter (1951) and Wallgren (1980) for Student's t family, Springer and Thompson (1970) for normal family, Stuart (1962) and Podolski (1972) for gamma family, Steece (1976), Bhargava and Khatri (1981) and Tang and Gupta (1984) for beta family and Abu-Salih (1983) for power function family (see also Rathie and Rohrer (1987) for a comprehensive review of known results). However, there is relatively little work of this kind when X and Y are correlated random variables. The only work known to the authors is that by Malik and Trudel (1986) for exponential family and that by Garg et al. (2002) for Dirichlet family.

In this paper, we study the distribution of the product XY when (X, Y) has the elliptically symmetric Bessel distribution given by the joint pdf

$$f_{X,Y}(x,y) = \frac{\left\{x^2+y^2-2\rho xy\right\}^{a/2}}{2^{a+1}\pi b^{a+2}\Gamma(a+1)\left(1-\rho^2\right)^{(a+1)/2}} K_a\left(\frac{\sqrt{x^2+y^2-2\rho xy}}{b\sqrt{1-\rho^2}}\right) \tag{1}$$

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for $-\infty < x < \infty$, $-\infty < y < \infty$, $a > -1$, $b > 0$ and $-1 < \rho < 1$, where $K_a(\cdot)$ is the modified Bessel function of the third kind defined by

$$K_a(x) = \frac{\pi \left\{ I_{-a}(x) - I_a(x) \right\}}{2 \sin(a\pi)},$$

and

$$I_a(x) = \sum_{k=0}^{\infty} (x/2)^{2k+a} k! \Gamma(k + a + 1).$$

When $a = 0$ and $b = \sigma/\sqrt{2}$, (1) reduces to the elliptically symmetric Laplace distribution. The parameter ρ is the correlation coefficient between the x and y components. For details on properties of these distributions see Jensen (1985) and Fang et al. (1990).

The aim of this paper is to calculate the distribution of the product XY when (X, Y) has the joint pdf (1). The calculations of this paper involve the hypergeometric functions defined by

$$G\left(\alpha; \beta, \gamma; x\right) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\beta)_k \, (\gamma)_k} \frac{x^k}{k!}$$

and

$$H\left(\alpha, \beta; \gamma, \delta, \eta; x\right) = \sum_{k=0}^{\infty} \frac{(\alpha)_k \, (\beta)_k}{(\gamma)_k \, (\delta)_k \, (\eta)_k} \frac{x^k}{k!},$$

where $(c)_k = c(c + 1) \cdots (c + k - 1)$ denotes the ascending factorial. We also need the following lemmas.

Lemma 1. (Equation (2.16.2.3), Prudnikov et al. (1986), volume 2) *For $a > 0$, $\beta > 0$ and $\alpha > \nu$,*

$$\int_0^a x^{\alpha-1} (a - x)^{\beta-1} K_\nu(cx) dx = A(\nu) + A(-\nu),$$

where

$$A(\nu) = 2^{\nu-1} a^{\alpha+\beta-\nu-1} c^{-\nu} \Gamma(\nu) B(\beta, \alpha - \nu) \times H\left(\frac{\alpha - \nu}{2}, \frac{1 + \alpha - \nu}{2}; 1 - \nu, \frac{\alpha + \beta - \nu}{2}, \frac{1 + \alpha + \beta - \nu}{2}; \frac{a^2 c^2}{4} \right).$$

Lemma 2. (Equation (2.16.2.4), Prudnikov et al. (1986), volume 2) *For $a > 0$, $\beta > 0$ and $c > 0$,*

$$\int_a^\infty x^{\alpha-1} (a - x)^{\beta-1} K_\nu(cx) dx = A(\nu) + A(-\nu) + B(\nu) - C(\nu),$$

where

$$A(\nu) = 2^{\nu-1} a^{\alpha+\beta-\nu-1} c^{-\nu} \Gamma(\nu) B(\beta, 1-\alpha-\beta+\nu) \times H\left(\frac{\alpha-\nu}{2}, \frac{1+\alpha-\nu}{2}; 1-\nu, \frac{\alpha+\beta-\nu}{2}, \frac{1+\alpha+\beta-\nu}{2}; \frac{a^2c^2}{4}\right),$$

$$B(\nu) = 2^{\alpha+\beta-3} c^{1-\alpha-\beta} \Gamma\left(\frac{\alpha+\beta+\nu-1}{2}\right) \Gamma\left(\frac{\alpha+\beta-\nu-1}{2}\right) \times H\left(\frac{1-\beta}{2}, 1-\frac{\beta}{2}; \frac{1}{2}, \frac{3-\alpha-\beta-\nu}{2}, \frac{3+\nu-\alpha-\beta}{2}; \frac{a^2c^2}{4}\right),$$

and

$$C(\nu) = 2^{\alpha+\beta-4} a c^{2-\alpha-\beta} (\beta-1) \Gamma\left(\frac{\alpha+\beta+\nu}{2}-1\right) \Gamma\left(\frac{\alpha+\beta-\nu}{2}-1\right) \times H\left(1-\frac{\beta}{2}, \frac{3-\beta}{2}; \frac{3}{2}, 2+\frac{\nu-\alpha-\beta}{2}, 2-\frac{\nu+\alpha+\beta}{2}; \frac{a^2c^2}{4}\right).$$

Lemma 3. (Equation (2.16.2.2), Prudnikov et al. (1986), volume 2) *For $c > 0$ and $\alpha > \nu$,*

$$\int_0^\infty x^{\alpha-1} K_\nu(cx) dx = 2^{\alpha-2} c^{-\alpha} \Gamma\left(\frac{\alpha+\nu}{2}\right) \Gamma\left(\frac{\alpha-\nu}{2}\right).$$

Further properties of the above special functions can be found in Prudnikov et al. (1986) and Gradshteyn and Ryzhik (2000).

2. PDF

Theorem 1 and Corollary 1 derive explicit expressions for the pdf of $Z = XY$ in terms of the hypergeometric functions. Theorem 1 considers the case $\rho \neq 0$ while the case $\rho = 0$ is considered by Corollary 1.

Theorem 1. *Suppose X and Y are jointly distributed according to* (1) *with $\rho \neq 0$. Then, the pdf of $Z = XY$ can be expressed as*

$$f_Z(z) = \frac{1}{2^a \pi b^{a+2} \Gamma(a+1) \left(1-\rho^2\right)^{(a+1)/2}} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{-1/2}{j} \binom{-1-2j}{k} \left(-4z^2\right)^j \times \left\{(2\rho z)^{-(1+2j+k)} g_1(j,k,z) + (2\rho z)^k g_2(j,k,z)\right\} \quad (2)$$

for $-\infty < z < \infty$ and $A \leq B$, and as

$$f_Z(z) = \frac{1}{2^a \pi b^{a+2} \Gamma(a+1) \left(1-\rho^2\right)^{(a+1)/2}} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{-1/2}{j} \binom{-1-2j}{k} \left(-4z^2\right)^j \times (2\rho z)^k g_3(j,k,z) \quad (3)$$

for $-\infty < z < \infty$ *and* $A > B$, *where* $A = 2 \mid z \mid -2\rho z$, $B = 2 \mid \rho z \mid$,

$$
\begin{aligned}
& g_1(j,k,z) \\
&= \frac{2^{a-1} b^a \left(1-\rho^2\right)^{a/2} \Gamma(a) B^{4(k+1)}}{k+1} G\left(k+1; 1-a, k+2; \frac{B^4}{4b^2\left(1-\rho^2\right)}\right) \\
&\quad + \frac{\Gamma(-a) B^{4(a+k+1)}}{2^{a+1} b^a \left(1-\rho^2\right)^{a/2} (a+k+1)} G\left(a+k+1; 1+a, a+k+2; \frac{B^4}{4b^2\left(1-\rho^2\right)}\right) \\
&\quad - \frac{2^{a-1} b^a \left(1-\rho^2\right)^{a/2} \Gamma(a) A^{4(k+1)}}{k+1} G\left(k+1; 1-a, k+2; \frac{A^4}{4b^2\left(1-\rho^2\right)}\right) \\
&\quad - \frac{\Gamma(-a) A^{4(a+k+1)}}{2^{a+1} b^a \left(1-\rho^2\right)^{a/2} (a+k+1)} G\left(a+k+1; 1+a, a+k+2; \frac{A^4}{4b^2\left(1-\rho^2\right)}\right), \qquad (4)
\end{aligned}
$$

$$
\begin{aligned}
& g_2(j,k,z) \\
&= \frac{2^{a-1} b^a \left(1-\rho^2\right)^{a/2} \Gamma(a)}{(2j+k) B^{4(2j+k)}} G\left(-2j-k; 1-a, 1-2j-k; \frac{B^4}{4b^2\left(1-\rho^2\right)}\right) \\
&\quad + \frac{\Gamma(-a) B^{4(a-2j-k)}}{2^{a+1} b^a \left(1-\rho^2\right)^{a/2} (2j+k-a)} G\left(a-2j-k; 1+a, 1+a-2j-k; \frac{B^4}{4b^2\left(1-\rho^2\right)}\right) \qquad (5)
\end{aligned}
$$

and

$$
\begin{aligned}
& g_3(j,k,z) \\
&= \frac{2^{a-1} b^a \left(1-\rho^2\right)^{a/2} \Gamma(a)}{(2j+k) A^{4(2j+k)}} G\left(-2j-k; 1-a, 1-2j-k; \frac{A^4}{4b^2\left(1-\rho^2\right)}\right) \\
&\quad + \frac{\Gamma(-a) A^{4(a-2j-k)}}{2^{a+1} b^a \left(1-\rho^2\right)^{a/2} (2j+k-a)} G\left(a-2j-k; 1+a, 1+a-2j-k; \frac{A^4}{4b^2\left(1-\rho^2\right)}\right). \qquad (6)
\end{aligned}
$$

Proof. Set $(X, Y) = (X, Z/X)$. Under this transformation, the Jacobian is $1/ \mid X \mid$ and so one can express the joint pdf of (X, Z) as

$$
f_{X,Z}(x,z) = \frac{\left\{x^2 + z^2/x^2 - 2\rho z\right\}^{a/2}}{2^{a+1} \pi b^{a+2} \Gamma(a+1) \left(1-\rho^2\right)^{(a+1)/2} \mid x \mid} K_a\left(\frac{\sqrt{x^2 + z^2/x^2 - 2\rho z}}{b\sqrt{1-\rho^2}}\right). \qquad (7)
$$

Note that (7) is an even function with respect to x. So, we only need to investigate the behavior for $x > 0$. Set $u(x) = x^2 + z^2/x^2 - 2\rho z$ for $x > 0$. Note that

$$
\frac{du(x)}{dx} = \frac{2\left(x^4 - z^2\right)}{x^3}.
$$

It follows that $u(x)$ decreases over $(0, \sqrt{\mid z \mid})$, increases over $(\sqrt{\mid z \mid}, \infty)$ and $u(\sqrt{\mid z \mid}) = A$. Expressing x in terms of u, one can write

$$\frac{1}{\mid x \mid} \frac{dx}{du(x)} = \pm \frac{1}{2\sqrt{(u+2\rho z)^2 - 4z^2}}$$

which, by using

$$(1+w)^{-c} = \sum_{m=0}^{\infty} \binom{-c}{m} w^m, \tag{8}$$

can be expanded as

$$\frac{1}{\mid x \mid} \frac{dx}{du(x)} = \pm \frac{1}{2} \sum_{j=0}^{\infty} \binom{-1/2}{j} (-1)^j \left(4z^2\right)^j (u+2\rho z)^{-(1+2j)}. \tag{9}$$

The last term within the sum of (9) can be expanded further by using (8). One has to consider the cases $u < B$ and $u > B$: if $u < B$ then

$$\frac{1}{\mid x \mid} \frac{dx}{du(x)} = \pm \frac{1}{2} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{-1/2}{j} \binom{-(1+2j)}{k} (-1)^j \left(4z^2\right)^j (2\rho z)^{-(1+2j+k)} u^k \tag{10}$$

and if $u > B$ then

$$\frac{1}{\mid x \mid} \frac{dx}{du(x)} = \pm \frac{1}{2} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{-1/2}{j} \binom{-(1+2j)}{k} (-1)^j \left(4z^2\right)^j (2\rho z)^k u^{-(1+2j+k)}. \tag{11}$$

Combining (7), (10) and (11), the marginal pdf of Z can be written as

$$\begin{aligned} f_Z(z) = & \frac{1}{2^a \pi b^{a+2} \Gamma(a+1) \left(1-\rho^2\right)^{(a+1)/2}} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{-1/2}{j} \binom{-(1+2j)}{k} \left(-4z^2\right)^j \\ & \times \left\{ (2\rho z)^{-(1+2j+k)} I_1 + (2\rho z)^k I_2 \right\} \end{aligned} \tag{12}$$

if $A \leq B$ and as

$$f_Z(z) = \frac{1}{2^a \pi b^{a+2} \Gamma(a+1) \left(1-\rho^2\right)^{(a+1)/2}} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{-1/2}{j} \binom{-(1+2j)}{k} \left(-4z^2\right)^j (2\rho z)^k I_3 \tag{13}$$

if $A > B$, where

$$\begin{aligned} I_1 &= \int_A^B u^{a/2+k} K_a \left(\frac{\sqrt{u}}{b\sqrt{1-\rho^2}} \right) du \\ &= 2 \int_{A^2}^{B^2} z^{a+2k+1} K_a \left(\frac{z}{b\sqrt{1-\rho^2}} \right) dz, \end{aligned} \tag{14}$$

$$
\begin{aligned}
I_2 &= \int_B^\infty u^{a/2-2j-k-1} K_a \left(\frac{\sqrt{u}}{b\sqrt{1-\rho^2}} \right) du \\
&= 2 \int_{B^2}^\infty z^{a-4j-2k-1} K_a \left(\frac{z}{b\sqrt{1-\rho^2}} \right) dz \qquad (15)
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 &= \int_A^\infty u^{a/2-2j-k-1} K_a \left(\frac{\sqrt{u}}{b\sqrt{1-\rho^2}} \right) du \\
&= 2 \int_{A^2}^\infty z^{a-4j-2k-1} K_a \left(\frac{z}{b\sqrt{1-\rho^2}} \right) dz. \qquad (16)
\end{aligned}
$$

Direct application of Lemmas 1 and 2 shows that (14)–(16) reduce to (4)–(6), respectively. The results of the theorem in (2) and (3) follow from (12) and (13), respectively.

Corollary 1. *Suppose X and Y are jointly distributed according to* (1) *with $\rho = 0$. Then, the pdf of $Z = XY$ can be expressed as*

$$
f_Z(z) = \frac{1}{2^a \pi b^{a+2} \Gamma(a+1)} \sum_{j=0}^\infty \binom{-1/2}{j} \left(-4z^2\right)^j g(j,z) \qquad (17)
$$

for $-\infty < z < \infty$, where $A = 2 \mid z \mid$ and

$$
\begin{aligned}
g(j,z) &= \frac{2^{a-1} b^a \Gamma(a)}{2j A^{8j}} G\left(-2j; 1-a, 1-2j; \frac{A^4}{4b^2}\right) \\
&\quad + \frac{\Gamma(-a) A^{4(a-2j)}}{2^{a+1} b^a (2j-a)} G\left(a-2j; 1+a, 1+a-2j; \frac{A^4}{4b^2}\right). \qquad (18)
\end{aligned}
$$

Proof. Follows easily from the proof of Theorem 1: combining (7) and (9) for $\rho = 0$, the marginal pdf of Z can be written as

$$
f_Z(z) = \frac{1}{2^a \pi b^{a+2} \Gamma(a+1)} \sum_{j=0}^\infty \binom{-1/2}{j} \left(-4z^2\right)^j I_3, \qquad (19)
$$

where I_3 is given by (16) or equivalently (6) for $k = 0$. The result of the corollary follows by noting that (6) for $k = 0$ reduces to (18).

Figure 1 below illustrates possible shapes of the pdf (2)–(3) for a range of values of a and ρ. Note the diminishing scale of the densities with increasing values of a. Some numerical investigation suggests that $f(z) \to \infty$ as $z \to 0^\pm$ and that the areas under the curves are equal to 1.

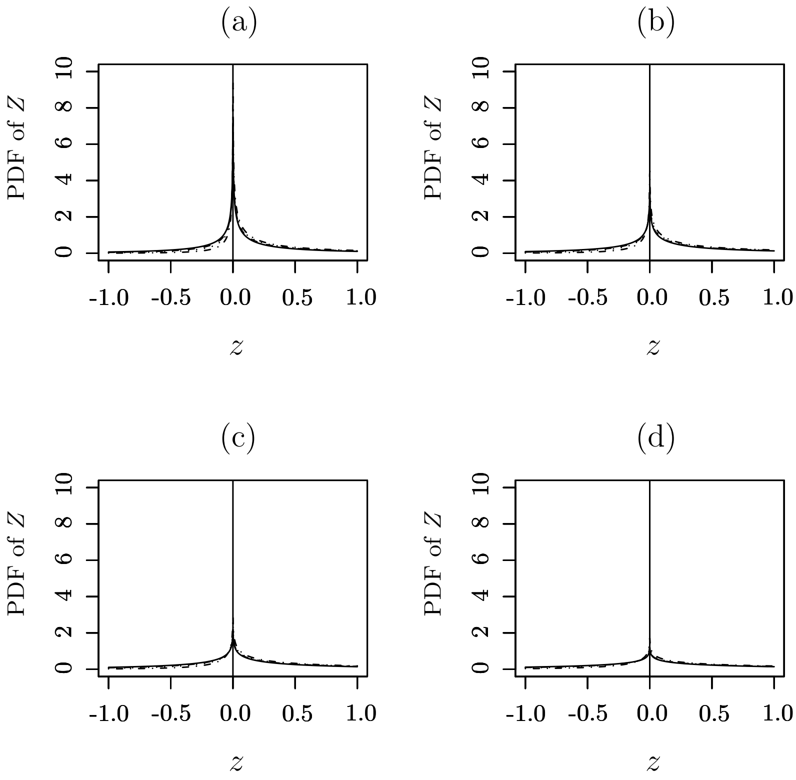

Figure 1. Plots of the pdf (2)–(3) for $b = 1$ and (a): $a = 0$; (b): $a = 0.5$; (c): $a = 1$; and, (d): $a = 2$. In each plot, there are four curves: the unbroken curve ($\rho = 0.2$), the curve of dashes ($\rho = 0.4$), the curve of dots ($\rho = 0.6$), and the curve of dashes and dots ($\rho = 0.8$).

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