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# ON THE HADAMARD TYPE INEQUALITIES INVOLVING PRODUCT OF TWO CONVEX FUNCTIONS ON THE CO-ORDINATES

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**Abstract**. In this paper some Hadamard-type inequalities for product of convex functions of 2−variables on the co-ordinates are given.

## 1. Introduction

The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function defined on the interval $I$ of $\mathbb{R}$, the set of real numbers and $a, b \in I$ with $a < b$, is well known in the literature as Hadamard's inequality.

For some recent results related to this classic inequality, see [1], [7], [12], [13], and [15].

In [2], Hudzik and Maligranda considered, among others, the class of functions which are $s-$convex in the second sense. This class is defined by Breckner as following in [18]:

**Definition 1.** A function $f : [0,\infty) \to \mathbb{R}$ is said to be $s-$convex in the second sense if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds for all $x, y \in [0,\infty)$, $\lambda \in [0,1]$ and for some fixed $s \in (0,1]$.

The class of $s-$convex functions in the second sense is usually denoted with $K_s^2$. It is clear that if we choose $s = 1$, we have ordinary convexity of functions defined on $[0,\infty)$.

In [16], Kırmacı *et al.* proved the following inequalities related to product of convex functions. These are given in the next theorems.

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**Theorem 1.** *Let $f,g:[a,b]\to\mathbb{R}$, $a,b\in[0,\infty)$, $a<b$, be functions such that $g$ and $fg$ are in $L^1([a,b])$. If $f$ is convex and nonnegative on $[a,b]$, and if $g$ is $s-$convex on $[a,b]$ for some fixed $s\in(0,1)$, then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \le \frac{1}{s+2}M(a,b)+\frac{1}{(s+1)(s+2)}N(a,b) \tag{1.2}$$

*where*

$$M(a,b)=f(a)g(a)+f(b)g(b) \text{ and } N(a,b)=f(a)g(b)+f(b)g(a).$$

**Theorem 2.** *Let $f,g:[a,b]\to\mathbb{R}$, $a,b\in[0,\infty)$, $a<b$, be functions such that $g$ and $fg$ are in $L^1([a,b])$. If $f$ is $s_1-$convex and $g$ is $s_2-$convex on $[a,b]$ for some fixed $s_1,s_2\in(0,1)$, then*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)g(x)dx &\le \frac{1}{s_1+s_2+1}M(a,b)+B(s_1+1,s_2+1)N(a,b)\\ &= \frac{1}{s_1+s_2+1}\left[M(a,b)+s_1s_2\frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+s_2+1)}N(a,b)\right] \end{aligned} \tag{1.3}$$

*where*

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},\quad x>0,y>0$$

*and*

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt,\ x>0.$$

**Theorem 3.** *Let $f,g:[a,b]\to\mathbb{R}$, $a,b\in[0,\infty)$, $a<b$, be functions such that $g$ and $fg$ are in $L^1([a,b])$. If $f$ is convex and nonnegative on $[a,b]$, and if $g$ is $s-$convex on $[a,b]$ for some fixed $s\in(0,1)$, then*

$$\begin{aligned}&2^s f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)g(x)dx\\ &\le \frac{1}{(s+1)(s+2)}M(a,b)+\frac{1}{s+2}N(a,b).\end{aligned} \tag{1.4}$$

For similar results, see the papers [2] and [14].

In [13], Dragomir defined convex functions on the co-ordinates as follows and proved Lemma 1 related to this definiton:

**Definition 2.** Let us consider the bidimensional interval $\Delta=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$, $c<d$. A function $f:\Delta\to\mathbb{R}$ will be called convex on the co-ordinates if the partial mappings $f_y:[a,b]\to\mathbb{R}$, $f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}$, $f_x(v)=f(x,v)$ are convex where defined for all $y\in[c,d]$ and $x\in[a,b]$. Recall that the mapping $f:\Delta\to\mathbb{R}$ is convex on $\Delta$ if the following inequality holds,

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)\le\lambda f(x,y)+(1-\lambda)f(z,w)$$

for all $(x,y),(z,w)\in\Delta$ and $\lambda\in[0,1]$.

**Lemma 1.** *Every convex mapping $f:\Delta\to\mathbb{R}$ is convex on the co-ordinates, but converse is not general true.*

A formal definition for co-ordinated convex functions may be stated as follow [see [17]]:

**Definition 3.** A function $f:\Delta\to\mathbb{R}$ is said to be convex on the co-ordinates on $\Delta$ if the following inequality:

$$f(tx+(1-t)y, su+(1-s)w)\leq tsf(x,u)+t(1-s)f(x,w)+s(1-t)f(y,u)+(1-t)(1-s)f(y,w)$$

holds for all $t,s\in[0,1]$ and $(x,u),(x,w),(y,u),(y,w)\in\Delta$.

In [13], Dragomir established the following inequalities:

**Theorem 4.** *Suppose that $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ is convex on the co-ordinates on $\Delta$. Then one has the inequalities:*

$$\begin{aligned} f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dx\,dy \\ &\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned} \tag{1.5}$$

Similar results, refinements and generalizations can be found in [4], [5], [6], [8], [9], [10], [11] and [17].

In [6], Alomari and Darus defined $s-$convexity on $\Delta$ as follows:

**Definition 4.** Consider the bidimensional interval $\Delta=[a,b]\times[c,d]$ in $[0,\infty)^2$ with $a<b$ and $c<d$. The mapping $f:\Delta\to\mathbb{R}$ is $s-$convex on $\Delta$ if

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)\leq\lambda^s f(x,y)+(1-\lambda)^s f(z,w)$$

holds for all $(x,y)$, $(z,w)\in\Delta$ with $\lambda\in[0,1]$ and for some fixed $s\in(0,1]$.

In [6], Alomari and Darus proved the following lemma:

**Lemma 2.** *Every $s-$convex mappings $f:\Delta:=[a,b]\times[c,d]\subset[0,\infty)^2\to[0,\infty)$ is $s-$convex on the co-ordinates, but converse is not true in general.*

In [3], Latif and Alomari established Hadamard-type inequalities for product of two convex functions on the co-ordinates as follow:

**Theorem 5.** *Let $f, g : \Delta = [a,b] \times [c,d] \subset \mathbb{R}^2 \to [0,\infty)$ be convex functions on the co-ordinates on $\Delta$ with $a < b$ and $c < d$. Then*

$$\begin{aligned}&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)\,dxdy\\ &\leq \frac{1}{9}L(a,b,c,d)+\frac{1}{18}M(a,b,c,d)+\frac{1}{36}N(a,b,c,d)\end{aligned} \tag{1.6}$$

*where*

$$\begin{aligned}L(a,b,c,d) &= f(a,c)g(a,c)+f(b,c)g(b,c)+f(a,d)g(a,d)+f(b,d)g(b,d)\\ M(a,b,c,d) &= f(a,c)g(a,d)+f(a,d)g(a,c)+f(b,c)g(b,d)+f(b,d)g(b,c)\\ &\quad +f(b,c)g(a,c)+f(b,d)g(a,d)+f(a,c)g(b,c)+f(a,d)g(b,d)\\ N(a,b,c,d) &= f(b,c)g(a,d)+f(b,d)g(a,c)+f(a,c)g(b,d)+f(a,d)g(b,c).\end{aligned}$$

**Theorem 6.** *Let $f, g : \Delta \subset \mathbb{R}^2 \to [0,\infty)$ be convex functions on the co-ordinates on $\Delta$ with $a < b$ and $c < d$. Then*

$$\begin{aligned}4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)\,dxdy\\ &\quad +\frac{5}{36}L(a,b,c,d)+\frac{7}{36}M(a,b,c,d)+\frac{2}{9}N(a,b,c,d)\end{aligned} \tag{1.7}$$

*where $L(a,b,c,d)$, $M(a,b,c,d)$, $N(a,b,c,d)$ are as in Theorem 5.*

The main purpose of this paper is to establish new inequalities like (1.6) and (1.7), but now for product of convex functions and $s-$convex functions of $2-$variables on the co-ordinates which are generalizations of the inequalities (1.6) and (1.7).

## 2. Main results

**Theorem 7.** *Let $f : \Delta \subset [0,\infty)^2 \to [0,\infty)$ be convex function on the co-ordinates and $g : \Delta \subset [0,\infty)^2 \to [0,\infty)$ be $s-$convex function on the co-ordinates with $a < b$, $c < d$ and $f_x(y)g_x(y)$, $f_y(x)g_y(x) \in L^1[\Delta]$ for some fixed $s \in (0,1]$. Then one has the inequality:*

$$\begin{aligned}&\frac{1}{(d-c)(b-a)}\int_a^b\int_c^d f(x,y)g(x,y)\,dxdy\\ &\leq \frac{1}{(s+2)^2}L(a,b,c,d)+\frac{1}{(s+1)(s+2)^2}M(a,b,c,d)+\frac{1}{(s+1)^2(s+2)^2}N(a,b,c,d)\end{aligned} \tag{2.1}$$

*where $L(a,b,c,d)$, $M(a,b,c,d)$, $N(a,b,c,d)$ are as in Theorem 5.*

**Proof.** Since $f$ is co-ordinated convex and $g$ is co-ordinated $s-$convex, from Lemma 1 and Lemma 2, the partial mappings

$$f_y : [a,b] \to [0,\infty),\ f_y(x) = f(x,y),\ y \in [c,d]$$

$$f_x : [c,d] \to [0,\infty),\ f_x(y) = f(x,y),\ x \in [a,b]$$

are convex on $[a,b]$ and $[c,d]$, respectively, where $x \in [a,b]$, $y \in [c,d]$. Similarly;

$$g_y : [a,b] \to [0,\infty),\ g_y(x) = g(x,y),\ y \in [c,d]$$
$$g_x : [c,d] \to [0,\infty),\ g_x(y) = g(x,y),\ x \in [a,b]$$

are $s-$convex on $[a,b]$ and $[c,d]$, respectively, where $x \in [a,b]$, $y \in [c,d]$.

Using (1.2), we can write

$$\begin{aligned}\frac{1}{d-c}\int_c^d f_x(y)g_x(y)dy &\le \frac{1}{s+2}\left[f_x(c)g_x(c)+f_x(d)g_x(d)\right]\\ &\quad+\frac{1}{(s+1)(s+2)}\left[f_x(c)g_x(d)+f_x(d)g_x(c)\right].\end{aligned}$$

That is

$$\begin{aligned}\frac{1}{d-c}\int_c^d f(x,y)g(x,y)dy &\le \frac{1}{s+2}\left[f(x,c)g(x,c)+f(x,d)g(x,d)\right]\\ &\quad+\frac{1}{(s+1)(s+2)}\left[f(x,c)g(x,d)+f(x,d)g(x,c)\right].\end{aligned}$$

Dividing both sides by $(b-a)$ and integrating over $[a,b]$, we get

$$\begin{aligned}&\frac{1}{(d-c)(b-a)}\int_a^b\int_c^d f(x,y)g(x,y)dxdy\\ &\le \frac{1}{(b-a)(s+2)}\left[\int_a^b f(x,c)g(x,c)dx+\int_a^b f(x,d)g(x,d)dx\right]\\ &\quad+\frac{1}{(s+1)(s+2)}\left[\frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx+\frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx\right].\end{aligned}\tag{2.2}$$

By applying (1.2) to each term of right hand side of above inequality, we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx &\le \frac{1}{s+2}\left[f(a,c)g(a,c)+f(b,c)g(b,c)\right]\\ &\quad+\frac{1}{(s+1)(s+2)}\left[f(a,c)g(b,c)+f(b,c)g(a,c)\right],\end{aligned}$$

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx &\le \frac{1}{s+2}\left[f(a,d)g(a,d)+f(b,d)g(b,d)\right]\\ &\quad+\frac{1}{(s+1)(s+2)}\left[f(a,d)g(b,d)+f(b,d)g(a,d)\right],\end{aligned}$$

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx &\le \frac{1}{s+2}\left[f(a,c)g(a,d)+f(b,c)g(b,d)\right]\\ &\quad+\frac{1}{(s+1)(s+2)}\left[f(a,c)g(b,d)+f(b,c)g(a,d)\right],\end{aligned}$$

and

$$\frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx \leq \frac{1}{s+2}\left[f(a,d)g(a,c)+f(b,d)g(b,c)\right] + \frac{1}{(s+1)(s+2)}\left[f(a,d)g(b,c)+f(b,d)g(a,c)\right].$$

Using these inequalities in (2.2), we obtain

$$\begin{aligned}
&\frac{1}{(d-c)(b-a)}\int_a^b\int_c^d f(x,y)g(x,y)dxdy \\
&\leq \frac{1}{(s+2)^2}\left(f(a,c)g(a,c)+f(b,c)g(b,c)+f(a,d)g(a,d)+f(b,d)g(b,d)\right) \\
&\quad+\frac{1}{(s+1)(s+2)^2}\left(f(a,c)g(b,c)+f(b,c)g(a,c)+f(a,d)g(b,d)+f(b,d)g(a,d)\right) \\
&\quad+\frac{1}{(s+1)(s+2)^2}\left(f(a,c)g(a,d)+f(b,c)g(b,d)+f(a,d)g(a,c)+f(b,d)g(b,c)\right) \\
&\quad+\frac{1}{(s+1)^2(s+2)^2}\left(f(a,c)g(b,d)+f(b,c)g(a,d)+f(a,d)g(b,c)+f(b,d)g(a,c)\right).
\end{aligned}$$

Which completes the proof. We can find the same result by using $f_y(x)$ and $g_y(x)$. □

**Theorem 8.** *Let $f:\Delta\subset[0,\infty)^2\to[0,\infty)$ be $s_1-$convex function on the co-ordinates and $g:\Delta\subset[0,\infty)^2\to[0,\infty)$ be $s_2-$convex function on the co-ordinates with $a<b$, $c<d$ and $f_x(y)g_x(y)$, $f_y(x)g_y(x)\in L^1[\Delta]$ for some fixed $s_1,s_2\in(0,1]$. Then one has the inequality:*

$$\begin{aligned}
&\frac{1}{(d-c)(b-a)}\int_a^b\int_c^d f(x,y)g(x,y)dxdy \\
&\leq \frac{1}{(s_1+s_2+1)^2}L(a,b,c,d)+\frac{B(s_1+1,s_2+1)}{s_1+s_2+1}M(a,b,c,d)+\left[B(s_1+1,s_2+1)\right]^2N(a,b,c,d) \\
&= \frac{1}{(s_1+s_2+1)^2}\left[L(a,b,c,d)+\frac{s_1s_2\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+s_2+1)}M(a,b,c,d)+\left[\frac{s_1s_2\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+s_2+1)}\right]^2N(a,b,c,d)\right] \quad (2.3)
\end{aligned}$$

*where $L(a,b,c,d)$, $M(a,b,c,d)$, $N(a,b,c,d)$ are as in Theorem 5.*

**Proof.** Since $f$ is co-ordinated $s_1-$convex and $g$ is co-ordinated $s_2-$convex, by using (1.3), we get

$$\frac{1}{d-c}\int_c^d f_x(y)g_x(y)dy \leq \frac{1}{s_1+s_2+1}\left[f_x(c)g_x(c)+f_x(d)g_x(d)\right] + B(s_1+1,s_2+1)\left[f_x(c)g_x(d)+f_x(d)g_x(c)\right].$$

Therefore

$$\frac{1}{d-c}\int_c^d f(x,y)g(x,y)dy \leq \frac{1}{s_1+s_2+1}\left[f(x,c)g(x,c)+f(x,d)g(x,d)\right] + B(s_1+1,s_2+1)\left[f(x,c)g(x,d)+f(x,d)g(x,c)\right].$$

Dividing both sides of the above inequality by $(b-a)$ and integrating over $[a,b]$, we have

$$\begin{aligned} &\frac{1}{(d-c)(b-a)}\int_a^b\int_c^d f(x,y)g(x,y)dxdy \\ &\leq \frac{1}{s_1+s_2+1}\left[\frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx+\frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx\right] \\ &\quad +B(s_1+1,s_2+1)\left[\frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx+\frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx\right]. \end{aligned} \tag{2.4}$$

By applying (1.3) to right hand side of (2.4), and we proceed similarly as in the proof of Theorem 7, we can write

$$\begin{aligned} &\frac{1}{(d-c)(b-a)}\int_a^b\int_c^d f(x,y)g(x,y)dxdy \\ &\leq \frac{1}{(s_1+s_2+1)^2}\left[f(a,c)g(a,c)+f(b,c)g(b,c)+f(a,d)g(a,d)+f(b,d)g(b,d)\right] \\ &\quad +\frac{B(s_1+1,s_2+1)}{s_1+s_2+1}\left[f(a,c)g(b,c)+f(b,c)g(a,c)+f(a,d)g(b,d)+f(b,d)g(a,d)\right] \\ &\quad +\frac{B(s_1+1,s_2+1)}{s_1+s_2+1}\left[f(a,c)g(a,d)+f(b,c)g(b,d)+f(a,d)g(a,c)+f(b,d)g(b,c)\right] \\ &\quad +\left[B(s_1+1,s_2+1)\right]^2\left[f(a,c)g(b,d)+f(b,c)g(a,d)+f(a,d)g(b,c)+f(b,d)g(a,c)\right]. \end{aligned}$$

That is;

$$\begin{aligned} &\frac{1}{(d-c)(b-a)}\int_a^b\int_c^d f(x,y)g(x,y)dxdy \\ &\leq \frac{1}{(s_1+s_2+1)^2}L(a,b,c,d)+\frac{B(s_1+1,s_2+1)}{s_1+s_2+1}M(a,b,c,d)+\left[B(s_1+1,s_2+1)\right]^2N(a,b,c,d) \\ &= \frac{1}{(s_1+s_2+1)^2}\left[L(a,b,c,d)+\frac{s_1s_2\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+s_2+1)}M(a,b,c,d)+\left[\frac{s_1s_2\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+s_2+1)}\right]^2N(a,b,c,d)\right]. \end{aligned}$$

Which completes the proof. □

**Theorem 9.** *Let $f:\Delta\to[0,\infty)$ be convex function on the co-ordinates and $g:\Delta\subset[0,\infty)^2\to[0,\infty)$ be $s-$convex function on the co-ordinates with $a<b$, $c<d$ and $f_x(y)g_x(y)$, $f_y(x)g_y(x)\in L^1[\Delta]$ for some fixed $s\in(0,1]$. Then one has the inequality:*

$$\begin{aligned} &2^{2s}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\ &\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dxdy+\frac{2s+3}{(s+1)^2(s+2)^2}L(a,b,c,d) \\ &\quad +\frac{s^2+3s+3}{(s+1)^2(s+2)^2}M(a,b,c,d)+\frac{s^2+4s+3}{(s+1)^2(s+2)^2}N(a,b,c,d) \end{aligned} \tag{2.5}$$

*where $L(a,b,c,d)$, $M(a,b,c,d)$ and $N(a,b,c,d)$ are as in Theorem 5.*

**Proof.** Since $f$ is co-ordinated convex and $g$ is co-ordinated $s-$convex, from Lemma 1 and Lemma 2, the partial mappings

$$\begin{aligned} f_y &: [a,b] \to [0,\infty), \ f_y(x) = f(x,y) \\ f_x &: [c,d] \to [0,\infty), \ f_x(y) = f(x,y) \end{aligned} \tag{2.6}$$

are convex on $[a,b]$ and $[c,d]$, respectively, where $x \in [a,b]$, $y \in [c,d]$. Similarly;

$$\begin{aligned} g_y &: [a,b] \to [0,\infty), \ g_y(x) = g(x,y), \ y \in [c,d] \\ g_x &: [c,d] \to [0,\infty), \ g_x(y) = g(x,y), \ x \in [a,b] \end{aligned}$$

are $s-$convex on $[a,b]$ and $[c,d]$, respectively, where $x \in [a,b]$, $y \in [c,d]$.

Using (1.4) and multiplying both sides of the inequalities by $2^s$, we get

$$\begin{aligned} &2^{2s} f\left(\frac{a+b}{2},\frac{c+d}{2}\right) g\left(\frac{a+b}{2},\frac{c+d}{2}\right) - \frac{2^s}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right) g\left(x,\frac{c+d}{2}\right) dx \\ &\leq \frac{2^s}{(s+1)(s+2)}\left[f\left(a,\frac{c+d}{2}\right) g\left(a,\frac{c+d}{2}\right) + f\left(b,\frac{c+d}{2}\right) g\left(b,\frac{c+d}{2}\right)\right] \\ &\quad + \frac{2^s}{s+2}\left[f\left(a,\frac{c+d}{2}\right) g\left(b,\frac{c+d}{2}\right) + f\left(b,\frac{c+d}{2}\right) g\left(a,\frac{c+d}{2}\right)\right] \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} &2^{2s} f\left(\frac{a+b}{2},\frac{c+d}{2}\right) g\left(\frac{a+b}{2},\frac{c+d}{2}\right) - \frac{2^s}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right) g\left(\frac{a+b}{2},y\right) dy \\ &\leq \frac{2^s}{(s+1)(s+2)}\left[f\left(\frac{a+b}{2},c\right) g\left(\frac{a+b}{2},c\right) + f\left(\frac{a+b}{2},d\right) g\left(\frac{a+b}{2},d\right)\right] \\ &\quad + \frac{2^s}{s+2}\left[f\left(\frac{a+b}{2},c\right) g\left(\frac{a+b}{2},d\right) + f\left(\frac{a+b}{2},d\right) g\left(\frac{a+b}{2},c\right)\right]. \end{aligned} \tag{2.8}$$

Now, by adding (2.7) and (2.8), we obtain

$$\begin{aligned} &2^{2s+1} f\left(\frac{a+b}{2},\frac{c+d}{2}\right) g\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\ &\quad - \frac{2^s}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right) g\left(x,\frac{c+d}{2}\right) dx - \frac{2^s}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right) g\left(\frac{a+b}{2},y\right) dy \\ &\leq \frac{1}{(s+1)(s+2)}\left[2^s f\left(a,\frac{c+d}{2}\right) g\left(a,\frac{c+d}{2}\right) + 2^s f\left(b,\frac{c+d}{2}\right) g\left(b,\frac{c+d}{2}\right)\right] \\ &\quad + \frac{1}{s+2}\left[2^s f\left(a,\frac{c+d}{2}\right) g\left(b,\frac{c+d}{2}\right) + 2^s f\left(b,\frac{c+d}{2}\right) g\left(a,\frac{c+d}{2}\right)\right] \\ &\quad + \frac{1}{(s+1)(s+2)}\left[2^s f\left(\frac{a+b}{2},c\right) g\left(\frac{a+b}{2},c\right) + 2^s f\left(\frac{a+b}{2},d\right) g\left(\frac{a+b}{2},d\right)\right] \\ &\quad + \frac{1}{s+2}\left[2^s f\left(\frac{a+b}{2},c\right) g\left(\frac{a+b}{2},d\right) + 2^s f\left(\frac{a+b}{2},d\right) g\left(\frac{a+b}{2},c\right)\right]. \end{aligned} \tag{2.9}$$

Applying (1.4) to each term of right hand side of the above inequality, we have

$$\begin{aligned}
&2^s f\left(a,\frac{c+d}{2}\right) g\left(a,\frac{c+d}{2}\right)\\
&\quad\leq \frac{1}{d-c}\int_c^d f(a,y)g(a,y)dy+\frac{1}{(s+1)(s+2)}\left[f(a,c)g(a,c)+f(a,d)g(a,d)\right]\\
&\quad\quad+\frac{1}{s+2}\left[f(a,c)g(a,d)+f(a,d)g(a,c)\right],\\
&2^s f\left(b,\frac{c+d}{2}\right) g\left(b,\frac{c+d}{2}\right)\\
&\quad\leq \frac{1}{d-c}\int_c^d f(b,y)g(b,y)dy+\frac{1}{(s+1)(s+2)}\left[f(b,c)g(b,c)+f(b,d)g(b,d)\right]\\
&\quad\quad+\frac{1}{s+2}\left[f(b,c)g(b,d)+f(b,d)g(b,c)\right],\\
&2^s f\left(a,\frac{c+d}{2}\right) g\left(b,\frac{c+d}{2}\right)\\
&\quad\leq \frac{1}{d-c}\int_c^d f(a,y)g(b,y)dy+\frac{1}{(s+1)(s+2)}\left[f(a,c)g(b,c)+f(a,d)g(b,d)\right]\\
&\quad\quad+\frac{1}{s+2}\left[f(a,c)g(b,d)+f(a,d)g(b,c)\right],\\
&2^s f\left(b,\frac{c+d}{2}\right) g\left(a,\frac{c+d}{2}\right)\\
&\quad\leq \frac{1}{d-c}\int_c^d f(b,y)g(a,y)dy+\frac{1}{(s+1)(s+2)}\left[f(b,c)g(a,c)+f(b,d)g(a,d)\right]\\
&\quad\quad+\frac{1}{s+2}\left[f(b,c)g(a,d)+f(b,d)g(a,c)\right],\\
&2^s f\left(\frac{a+b}{2},c\right) g\left(\frac{a+b}{2},c\right)\\
&\quad\leq \frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx+\frac{1}{(s+1)(s+2)}\left[f(a,c)g(a,c)+f(b,c)g(b,c)\right]\\
&\quad\quad+\frac{1}{s+2}\left[f(a,c)g(b,c)+f(b,c)g(a,c)\right],\\
&2^s f\left(\frac{a+b}{2},d\right) g\left(\frac{a+b}{2},d\right)\\
&\quad\leq \frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx+\frac{1}{(s+1)(s+2)}\left[f(a,d)g(a,d)+f(b,d)g(b,d)\right]\\
&\quad\quad+\frac{1}{s+2}\left[f(a,d)g(b,d)+f(b,d)g(a,d)\right],\\
&2^s f\left(\frac{a+b}{2},c\right) g\left(\frac{a+b}{2},d\right)\\
&\quad\leq \frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx+\frac{1}{(s+1)(s+2)}\left[f(a,c)g(a,d)+f(b,c)g(b,d)\right]\\
&\quad\quad+\frac{1}{s+2}\left[f(a,c)g(b,d)+f(b,c)g(a,d)\right]
\end{aligned}$$

and

$$\begin{aligned}
&2^s f\left(\frac{a+b}{2}, d\right) g\left(\frac{a+b}{2}, c\right) \\
&\leq \frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx + \frac{1}{(s+1)(s+2)}\left[f(a,d)g(a,c)+f(b,d)g(b,c)\right] \\
&\quad + \frac{1}{s+2}\left[f(a,d)g(b,c)+f(b,d)g(a,c)\right].
\end{aligned}$$

Using these inequalities in (2.9), we get

$$\begin{aligned}
&2^{2s+1} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) g\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
&\quad -\frac{2^s}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right) g\left(x, \frac{c+d}{2}\right) dx - \frac{2^s}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right) g\left(\frac{a+b}{2}, y\right) dy \\
&\leq \frac{1}{(s+1)(s+2)(d-c)}\left[\int_c^d f(a,y)g(a,y)dy + \int_c^d f(b,y)g(b,y)dy\right] \\
&\quad + \frac{1}{(s+2)(d-c)}\left[\int_c^d f(a,y)g(b,y)dy + \int_c^d f(b,y)g(a,y)dy\right] \\
&\quad + \frac{1}{(s+1)(s+2)(b-a)}\left[\int_a^b f(x,c)g(x,c)dx + \int_a^b f(x,d)g(x,d)dx\right] \\
&\quad + \frac{1}{(s+2)(b-a)}\left[\int_a^b f(x,c)g(x,d)dx + \int_a^b f(x,d)g(x,c)dx\right] \\
&\quad + \frac{2}{(s+1)^2(s+2)^2}L(a,b,c,d) + \frac{2}{(s+1)(s+2)^2}M(a,b,c,d) + \frac{2}{(s+2)^2}N(a,b,c,d). \qquad (2.10)
\end{aligned}$$

By applying (1.4) to $2^s f(\frac{a+b}{2}, y) g(\frac{a+b}{2}, y)$, integrating over $[c,d]$ and dividing both sides by $(d-c)$, we obtain

$$\begin{aligned}
&\frac{2^s}{(d-c)}\int_c^d f\left(\frac{a+b}{2}, y\right) g\left(\frac{a+b}{2}, y\right) dy - \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dxdy \\
&\leq \frac{1}{(s+1)(s+2)}\left[\frac{1}{(d-c)}\int_c^d f(a,y)g(a,y)dy + \frac{1}{(d-c)}\int_c^d f(b,y)g(b,y)dy\right] \\
&\quad + \frac{1}{s+2}\left[\frac{1}{(d-c)}\int_c^d f(a,y)g(b,y)dy + \frac{1}{(d-c)}\int_c^d f(b,y)g(a,y)dy\right]. \qquad (2.11)
\end{aligned}$$

Similarly by applying (1.4) to $2^s f\left(x, \frac{c+d}{2}\right) g\left(x, \frac{c+d}{2}\right)$, integrating over $[a,b]$ and dividing both sides by $(b-a)$, we have

$$\begin{aligned}
&\frac{2^s}{(b-a)}\int_a^b f\left(x, \frac{c+d}{2}\right) g\left(x, \frac{c+d}{2}\right) dx - \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dxdy \\
&\leq \frac{1}{(s+1)(s+2)}\left[\frac{1}{(b-a)}\int_a^b f(x,c)g(x,c)dx + \frac{1}{(b-a)}\int_a^b f(x,d)g(x,d)dx\right] \\
&\quad + \frac{1}{s+2}\left[\frac{1}{(b-a)}\int_a^b f(x,c)g(x,d)dx + \frac{1}{(b-a)}\int_a^b f(x,d)g(x,c)dx\right]. \qquad (2.12)
\end{aligned}$$

By adding (2.11) and (2.12), we have

$$\begin{aligned}
&\frac{2^s}{(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)g\left(\frac{a+b}{2},y\right)dy+\frac{2^s}{(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)g\left(x,\frac{c+d}{2}\right)dx\\
&\quad-\frac{2}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dxdy\\
&\quad\le\frac{1}{(s+1)(s+2)}\left[\frac{1}{(d-c)}\int_c^d f(a,y)g(a,y)dy+\frac{1}{(d-c)}\int_c^d f(b,y)g(b,y)dy\right.\\
&\quad\left.+\frac{1}{(b-a)}\int_a^b f(x,c)g(x,c)dx+\frac{1}{(b-a)}\int_a^b f(x,d)g(x,d)dx\right]\\
&\quad+\frac{1}{s+2}\left[\frac{1}{(d-c)}\int_c^d f(a,y)g(b,y)dy+\frac{1}{(d-c)}\int_c^d f(b,y)g(a,y)dy\right.\\
&\quad\left.+\frac{1}{(b-a)}\int_a^b f(x,c)g(x,d)dx+\frac{1}{(b-a)}\int_a^b f(x,d)g(x,c)dx\right]. \qquad (2.13)
\end{aligned}$$

From (2.10) and (2.13) and simplifying we get

$$\begin{aligned}
&2^{2s}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\quad\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dx+\frac{2s+3}{(s+1)^2(s+2)^2}L(a,b,c,d)\\
&\quad+\frac{s^2+3s+3}{(s+1)^2(s+2)^2}M(a,b,c,d)+\frac{s^2+4s+3}{(s+1)^2(s+2)^2}N(a,b,c,d).
\end{aligned}$$

Which completes the proof. □

**Theorem 10.** *Let $f,g:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be convex functions on the co-ordinates with $a<b$, $c<d$ and $f_x(y)g_x(y)$, $f_y(x)g_y(x)\in L^1[\Delta]$. Then one has the inequality:*

$$\begin{aligned}
&\frac{1}{(b-a)^2(d-c)^2}\left[f(a,c)\int_a^b\int_c^d (x-b)\left(y-d\right)g(x,y)dydx\right.\\
&\quad+f(b,c)\int_a^b\int_c^d (a-x)\left(y-d\right)g(x,y)dydx+f(a,d)\int_a^b\int_c^d (x-b)\left(c-y\right)g(x,y)dydx\\
&\quad+f(b,d)\int_a^b\int_c^d (a-x)\left(c-y\right)g(x,y)dydx+g(a,c)\int_a^b\int_c^d (x-b)\left(y-d\right)f(x,y)dydx\\
&\quad+g(b,c)\int_a^b\int_c^d (a-x)\left(y-d\right)f(x,y)dydx+g(a,d)\int_a^b\int_c^d (x-b)\left(c-y\right)f(x,y)dydx\\
&\quad\left.+g(b,d)\int_a^b\int_c^d (a-x)\left(c-y\right)f(x,y)dydx\right]\\
&\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dx+\frac{1}{9}L(a,b,c,d)+\frac{1}{18}M(a,b,c,d)+\frac{1}{36}N(a,b,c,d)
\end{aligned}$$

*where $L(a,b,c,d)$, $M(a,b,c,d)$, $N(a,b,c,d)$ defined as in Theorem 5.*

**Proof.** Since $f$ and $g$ are co-ordinated convex functions on the co-ordinates on $\Delta$, from the definition of co-ordinated convexity, we can write

$$f(ta+(1-t)b, sc+(1-s)d) \leq tsf(a,c)+t(1-s)f(a,d)+s(1-t)f(b,c)+(1-t)(1-s)f(b,d)$$

and

$$g(ta+(1-t)b, sc+(1-s)d) \leq tsg(a,c)+t(1-s)g(a,d)+s(1-t)g(b,c)+(1-t)(1-s)g(b,d)$$

holds for all $t,s \in [0,1]$. By using the elementary inequality, if $e \leq f$ and $p \leq r$, then $er + fp \leq ep + fr$ for all $e, f, p, r \in \mathbb{R}$, we get

$$\begin{aligned}
& f(ta+(1-t)b, sc+(1-s)d) \\
& \quad \times \left[tsg(a,c)+t(1-s)g(a,d)+s(1-t)g(b,c)+(1-t)(1-s)g(b,d)\right] \\
& \quad + g(ta+(1-t)b, sc+(1-s)d) \\
& \quad \times \left[tsf(a,c)+t(1-s)f(a,d)+s(1-t)f(b,c)+(1-t)(1-s)f(b,d)\right] \\
& \leq \left[f(ta+(1-t)b, sc+(1-s)d)g(ta+(1-t)b, sc+(1-s)d)\right] \\
& \quad + \left[tsf(a,c)+t(1-s)f(a,d)+s(1-t)f(b,c)+(1-t)(1-s)f(b,d)\right] \\
& \quad \times \left[tsg(a,c)+t(1-s)g(a,d)+s(1-t)g(b,c)+(1-t)(1-s)g(b,d)\right].
\end{aligned}$$

By integrating the above integral on $[0,1]\times[0,1]$ with respect to $t$, $s$ and by taking into account the change of variables $ta+(1-t)b = x$, $(a-b)dt = dx$ and $sc+(1-s)d = y$, $(c-d)ds = dy$, we obtain the desired result. □

**3. Conclusion**

In this section, we will give some remarks of our reults for special cases of the parameters $s, s_1, s_2$ and the function $f(x)$.

**Remark 1.** In (2.1), if we choose $s = 1$, (1.6) is obtained.

**Remark 2.** In (2.1), if we choose $s = 1$ and $f(x) = 1$ which is convex, we get the second inequality in (1.5):

$$\frac{1}{(d-c)(b-a)}\int_a^b\int_c^d g(x,y)\,dx\,dy \leq \frac{g(a,c)+g(b,c)+g(a,d)+g(b,d)}{4}.$$

**Remark 3.** In (2.3) if we choose $s_1 = s_2 = 1$, (2.3) reduces to (1.6).

**Remark 4.** In (2.5), if we choose $s = 1$, we obtained (1.7).

**Remark 5.** In (2.5), if we choose $s = 1$ and $f(x) = 1$ which is convex, we have the following Hadamard-type inequality like (1.5)

$$4g\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d g(x,y)\,dx \leq \frac{3\left[g(a,c)+g(b,c)+g(a,d)+g(b,d)\right]}{4}.$$

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