TAMKANG JOURNAL OF MATHEMATICS
Volume 37, Number 3, 207-209, Autumn 2006

ANOTHER PROOF OF MONOTONICITY FOR THE EXTENDED MEAN VALUES

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Abstract. We provide another proof of monotonicity for the extended mean values.

Stolarsky defined in [5] the extended mean values $E(r,s;x,y)$ by

$$
\begin{aligned}
E(r,s;x,y) &= \Big(\frac{r}{s}\cdot\frac{y^s-x^s}{y^r-x^r}\Big)^{1/(s-r)}, && rs(r-s)(x-y)\neq 0;\\
E(r,0;x,y) &= \Big(\frac{1}{r}\cdot\frac{y^r-x^r}{\ln y-\ln x}\Big)^{1/r}, && r(x-y)\neq 0;\\
E(r,r;x,y) &= \frac{1}{e^{1/r}}\Big(\frac{x^{x^r}}{y^{y^r}}\Big)^{1/(x^r-y^r)}, && r(x-y)\neq 0;\\
E(0,0;x,y) &= \sqrt{xy}, && x\neq y;\\
E(r,s;x,x) &= x, && x=y
\end{aligned}
$$

and proved that it is continuous on the domain $\{(r,s;x,y): r,s\in\mathbb{R},\ x,y>0\}$.

Leach and Sholander showed in [1, 2] that $E(r,s;x,y)$ is increasing with both r and s, and with both x and y. The monotonicities of E has also been researched in [3, 4] using different ideas and simpler methods.

The aim of this article is to give another proof of monotonicity for the extended mean values $E(r,s;x,y)$.

The variables x and y are, in this article, positive.

Theorem 1. *$E(r,s;x,y)$ is strictly increasing with both r and s.*

Proof. Since $E(r,s;x,y)$ is symmetric on r and s, it suffices to prove its monotonicity of $E(r,s;x,y)$ with respect to r. Since $E(r,s;x,y)$ is symmetric between x and y, without loss of generality, assume $x<y$.

Received December 30, 2004.
2000 *Mathematics Subject Classification*. 26A48.
Key words and phrases. Monotonicity, extended mean values.
The authors were supported in part by the SF of Henan Innovation Talents at Universities, China

Define for $r \in (-\infty, +\infty)$,

$$\varphi(r) = \begin{cases} \ln \dfrac{y^r - x^r}{r}, & r \neq 0; \\ \ln \ln(y/x), & r = 0. \end{cases}$$

Then,

$$\ln E(r, s; x, y) = \begin{cases} \dfrac{\varphi(r) - \varphi(s)}{r - s}, & r \neq s; \\ \varphi'(s), & r = s. \end{cases}$$

To prove that $\ln E(r, s; x, y)$ is strictly increasing with respect to r it suffices to show that φ is strictly convex on $(-\infty, +\infty)$. Easy computation reveals that

$$\varphi(-r) = \varphi(r) - r \ln(xy).$$

which implies that $\varphi''(-r) = \varphi''(r)$, and then φ has the same convexity on both $(-\infty, 0)$ and $(0, +\infty)$. Hence, it is sufficient to prove that φ is strictly convex on $(0, +\infty)$.

A simple computation yields

$$r^2 \varphi''(r) = 1 - \frac{(x/y)^r [\ln(x/y)^r]^2}{[1 - (x/y)^r]^2}.$$

Define for $0 < t < 1$,

$$\omega(t) = \frac{t(\ln t)^2}{(1 - t)^2}.$$

Differentiation yields

$$(1 - t)t \ln t \frac{\omega'(t)}{\omega(t)} = (1 + t) \ln t + 2(1 - t) = -\sum_{n=2}^{\infty} \frac{n - 1}{n(n + 1)} (1 - t)^{n+1} < 0,$$

which means that $\omega'(t) > 0$ for $0 < t < 1$, and then, $\omega(t) < \lim_{t \to 1} \omega(t) = 1$ for $0 < t < 1$. Clearly, $0 < (x/y)^r < 1$ for $y > x > 0$ and $r > 0$, and thus, $\varphi''(r) > 0$ for $r > 0$. The proof is complete.

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