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PRINCIPAL PIVOT TRANSFORMS OF RANGE SYMMETRIC MATRICES IN MINKOWSKI SPACE

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Abstract. It is shown that the property of a matrix being range symmetric in Minkowski space $\mathfrak{m}$ is preserved under the principal pivot transformation.

1. Introduction

Throughout we shall deal with $C^{n\times n}$, the space of $n\times n$ complex matrices. Let C^n be the space of complex n-tuples, we shall index the components of a complex vector in C^n from 0 to $n-1$, that is $u=(u_0,u_1,u_2,\dots,u_{n-1})$. Let G be the Minkowski metric tensor defined by $Gu=(u_0,-u_1,-u_2,\dots,-u_{n-1})$. Clearly, the Minkowski metric matrix

$$G=\begin{bmatrix}1 & 0\\ 0 & -I_{n-1}\end{bmatrix},\quad G=G^* \text{ and } G^2=I_n. \tag{1.1}$$

In [8], Minkowski inner product on C^n is defined by $(u,v)=[u,Gv]$, where $[.,.]$ denotes the conventional Hilbert (unitary) space inner product. A space with Minkowski inner product is called a Minkowski space and denoted as $\mathfrak{m}$.

For $A\in C^{n\times n}$, $x,y\in C^n$ and by using (1.1), we get

$$\begin{aligned}(Ax,y)&=[Ax,Gy]\\ &=[x,A^*Gy]\\ &=[x,G(GA^*G)y]\\ &=[x,GA^{\sim}y]\\ &=(x,A^{\sim}y)\qquad \text{where } A^{\sim}=GA*G.\end{aligned} \tag{1.2}$$

$A^{\sim}$ is called the Minkowski adjoint of A in $\mathfrak{m}$ (A^* is usual Hermitian adjoint of A). Naturally, we call a matrix $A\in C^{n\times n}$ $\mathfrak{m}$-symmetric in $\mathfrak{m}$ if $A=A^{\sim}$. For $A\in C^{n\times n}$, $\text{rk}(A)$, $N(A)$ and $R(A)$ are respectively the rank of A, the null space of A and the range space of A. By a generalized inverse of A we mean a solution of the equation $A\times A=A$ and is denoted as $A^{(1)}$. $A\{1\}$ is the set of all generalized inverses of A. Throughout I refers to identity matrix of appropriate order unless otherwise specified.

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Definition 1.1.(p.7 [2]) For $A \in C^{m\times n}$, A^+ is the Moore-Penrose inverse of A if $AA^+A = A$, $A^+AA^+ = A^+$, AA^+ and A^+A are hermitian.

Theorem 1.2.([4]) *For $A, B, C \in C^{n\times n}$, the following are equivalent*
(i) $CA^{(1)}B$ *is invariant for every* $A^{(1)} \in A\{1\}$.
(ii) $N(A) \subseteq N(C)$ *and* $N(A^*) \subseteq N(B^*)$.
(iii) $C = CA^{(1)}A$ *and* $B = AA^{(1)}B$ *for every* $A^{(1)} \in A\{1\}$.

Theorem 1.3.(Lemma 3.3 [7]) *Let A and B be matrices in $\mathfrak{m}$. Then* $N(A^*) \subseteq N(B^*) \Leftrightarrow N(A^\sim) \subseteq N(B^\sim)$.

Theorem 1.4.(Lemma 2.3 [7]) *For* $A_1, A_2 \in C^{n\times n}$ $(A_1A_2)^\sim = A_2^\sim A_1^\sim$ *and* $(A_1^\sim)^\sim = A_1$.

A matrix $A \in C^{n\times n}$ is said to be range symmetric is unitary space (or) equivalently A is said to be EP if $N(A) = N(A^*)$ [or $AA^+ = A^+A$] [p.163(2)]. For further properties of EP matrices one may refer [1, 2, 4 and 9].

In [6], the concept of range symmetric matrix in $\mathfrak{m}$ is introduced and developed analogous to that of EP matrices in unitary space. A matrix $A \in C^{n\times n}$ is said to be range symmetric in $m \Leftrightarrow N(A) = N(A^\sim)$. In the sequel, we shall make use of the following results.

Theorem 1.5.(Theorem 2.2 [6]) *For $A \in C^{n\times n}$, the following are equivalent:*
i) *A is range symmetric in $\mathfrak{m}$*
ii) *GA is EP*
iii) *AG is EP*
iv) $N(A^*) = N(AG)$
v) $R(A) = R(A^\sim)$
vi) $A^\sim = HA = AK$ *for some non-singular matrices H and K.*
vii) $R(A^*) = R(GA)$

Definition 1.6.(p.291 [3]) Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be an $n \times n$ matrix. The schur complement of A in M, denoted by S is defined as $D - CA^{(1)}B$, where $A^{(1)}$ is a generalized inverse of A.

Theorem 1.7.(Theorem 1 [4]) *Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be an $n \times n$ matrix with $N(A) \subseteq N(C)$ and $N(S) \subseteq N(B)$. Then M is EP if and only if A and S the schur complement of A in M are EP, $N(A^*) \subseteq N(B^*)$ and $N(S^*) \subseteq N(C^*)$.*

2. Principal Pivot on a Matrix

Let us consider a system of linear equations $Mz = t$, where $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ satisfying

$N(A) \subseteq N(C)$ and $N(A^*) \subseteq N(B^*)$. If z and t are partitioned conformally as $z = \begin{bmatrix} x \\ y \end{bmatrix}$ and $t = \begin{bmatrix} u \\ v \end{bmatrix}$ then the system becomes $Ax + By = u$; $Cx + Dy = v$. Since M satisfy $N(A) \subseteq N(C)$ and $N(A^*) \subseteq N(B^*)$ by using Theorem 1.2 and Theorem 1.3, we can express x and v in terms of u and y as $x = A^+u - A^+By$; $v = CA^+u - (D - CA^+B)y$. Thus $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ which satisfies $N(A) \subseteq N(C)$, $N(A^*) \subseteq N(B^*)$ can be transformed into the matrix

$$\hat{M} = \begin{bmatrix} A^+ & -A^+B \\ CA^+ & S \end{bmatrix} \tag{2.1}$$

where $S = D - CA^+B$ is the schur complement of A in M. $\hat{M}$ is called a principal pivot transform of M. The operation that transforms $M \to \hat{M}$ is called a principal pivot. If A is non-singular it reduces to the principal pivot by pivoting the block A [10]. Properties and applications of the principal pivot transforms are well recognized in mathematical programming [10 and 11].

Theorem 2.1. *Let* $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be a matrix with $N(A) \subseteq N(C)$ and $N(A^*) \subseteq N(B^*)$, S the schur complement of A in M, then the following are equivalent:

i) *M is range symmetric in $\mathfrak{m}$.*

ii) *A is range symmetric in $\mathfrak{m}$ and S is EP, $N(A^\sim) \subseteq N(B^\sim)$ and $N(S^\sim) \subseteq N(C^\sim)$.*

Proof. (i) $\Rightarrow$ (ii) Let us consider the matrices $P = \begin{bmatrix} I & O \\ CA^{(1)} & I \end{bmatrix}$; $Q = \begin{bmatrix} I & BS^{(1)} \\ O & I \end{bmatrix}$ for $A^{(1)} \in A\{1\}$ and $S^{(1)} \in S\{1\}$, and $L = \begin{bmatrix} A & O \\ O & S \end{bmatrix}$. $\begin{bmatrix} A^{(1)} & O \\ O & S^{(1)} \end{bmatrix}$ is one choice of $L^{(1)}$. P, Q are non-singular. Since $N(A) \subseteq N(C)$, $N(S) \subseteq N(B)$, by Theorem 1.2, we have $C = CA^{(1)}A$ and $B = BS^{(1)}S$. Thus M can be factorized as

$$\begin{aligned} PQL &= \begin{bmatrix} I & O \\ CA^{(1)} & I \end{bmatrix} \begin{bmatrix} I & BS^{(1)} \\ O & I \end{bmatrix} \begin{bmatrix} A & O \\ O & S \end{bmatrix} \\ &= \begin{bmatrix} A & BS^{(1)}S \\ CA^{(1)}A & CA^{(1)}BS^{(1)}S + S \end{bmatrix} = \begin{bmatrix} A & BS^{(1)}S \\ CA^{(1)}A & CA^{(1)}B + S \end{bmatrix} = M \end{aligned}$$

Since $M = PQL$, P and Q are non-singular and $N(L) \subseteq N(M)$. Also $\text{rk}(M) = \text{rk}(PQL) = \text{rk}(L)$ therefore $N(L) = N(M)$. Also M is range symmetric in $\mathfrak{m}$, we have

$N(M^{\sim}) = N(M) = N(L)$ hence by using Theorem 1.2, we get

$$\begin{aligned}
M^{\sim} &= M^{\sim}L^{(1)}L \\
GM^*G &= GM^*GL^{(1)}L && \text{[By using (1.2)]} \\
M^* &= M^*GL^{(1)}LG && \text{[By using (1.1)]} \\
\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} &= \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} \begin{bmatrix} G_1 & O \\ O & -I \end{bmatrix} \begin{bmatrix} A^{(1)} & O \\ O & S \end{bmatrix} \begin{bmatrix} A & O \\ O & S \end{bmatrix} \begin{bmatrix} G_1 & O \\ O & -I \end{bmatrix} \\
\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} &= \begin{bmatrix} A^*G_1A^{(1)}AG_1 & C^*S^{(1)}S \\ B^*G_1A^{(1)}AG_1 & D^*S^{(1)}S \end{bmatrix}.
\end{aligned}$$

Equating the corresponding blocks, we get

$$\begin{aligned}
A^* &= A^*G_1A^{(1)}AG_1 \\
G_1A^*G_1 &= G_1A^*G_1A^{(1)}A \\
A^{\sim} &= A^{\sim}A^{(1)}A && \text{[By using (1.2)]}
\end{aligned}$$

By using Theorem 1.2 and Theorem 1.3, for $A^{\sim} = A^{\sim}A^{(1)}A$, we get $N(A) \subseteq N(A^{\sim})$ also $\text{rk}(A) = \text{rk}(A^{\sim})$ implies $N(A) = N(A^{\sim})$. Thus A is range symmetric in $\mathfrak{m}$.

$$\begin{aligned}
B^* &= B^*G_1A^{(1)}AG_1 \\
G_1B^*G_1 &= G_1B^*G_1A^{(1)}A \\
B^{\sim} &= B^{\sim}A^{(1)}A.
\end{aligned}$$

Again by Theorem 1.2, $N(A) \subseteq N(B^{\sim})$, since A is range symmetric in $\mathfrak{m}$, we have $N(A^{\sim}) = N(A) \subseteq N(B^{\sim})$. Using $C^* = C^*S^{(1)}S$ and $D^* = D^*S^{(1)}S$ in $S = D - CA^{(1)}B$, we get

$$\begin{aligned}
S^*S^{(1)}S &= (D - CA^{(1)}B)^*S^{(1)}S \\
&= D^*S^{(1)}S - (CA^{(1)}B)^*S^{(1)}S \\
&= D^*S^{(1)}S - B^*(A^{(1)})^*C^*S^{(1)}S \\
&= D^* - B^*(A^{(1)})^*C^* \\
&= (D - CA^{(1)}B)^* \\
S^*S^{(1)}S &= S^*.
\end{aligned}$$

By applying Theorem 1.2, we get $N(S) \subseteq N(S^*)$. Since $\text{rk}(S) = \text{rk}(S^*)$, $N(S) = N(S^*)$ from this it follows that S is EP. Again applying Theorem 1.2, for $C^* = C^*S^{(1)}S$ implies $N(S) \subseteq N(C^*)$ also S is EP. We have $N(S^*) = N(S) \subseteq N(C^*)$. By using Theorem 1.3, we have $N(S^{\sim}) \subseteq N(C^{\sim})$. Hence (ii) is proved.

(ii) $\Rightarrow$ (i) By hypothesis A is range symmetric, S is EP, $N(A) \subseteq N(C)$, $N(A^{\sim}) \subseteq N(B^{\sim})$, $N(S) \subseteq N(B)$ and $N(S^{\sim}) \subseteq N(C^{\sim})$. Since A is range symmetric in $\mathfrak{m}$ by

Theorem 1.5 (ii), G_1A is EP. Therefore

$$\begin{aligned} G_1A(G_1A)^+ &= (G_1A)^+G_1A \\ G_1AA^+G_1 &= A^+G_1G_1A \\ G_1AA^+G_1 &= A^+A \qquad \text{[By using (1.1)]} \qquad (2.2) \end{aligned}$$

By using Theorem 1.2 and Theorem 1.3 we have $N(A) \subseteq N(C)$, $N(A^*) \subseteq N(B^*)$, $N(S) \subseteq N(B)$ and $N(S^*) \subseteq N(C^*)$ hold. According to the assumption of Theorem 1 (v) [3], we have

$$M^+ = \begin{bmatrix} A^+ + A^+BS^+CA^+ & -A^+BS^+ \\ -S^+CA^+ & S^+ \end{bmatrix} \tag{2.3}$$

Under the condition $N(A^\sim) \subseteq N(B^\sim)$ and $N(S^\sim) \subseteq N(C^\sim)$, we have $AA^+B = B$, $C = SS^+C$

$$\begin{aligned} \text{Now } MM^+ &= \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} A^+ + A^+BS^+CA^+ & -A^+BS^+ \\ -S^+CA^+ & S^+ \end{bmatrix} \\ &= \begin{bmatrix} AA^+ + AA^+BS^+CA^+ - BS^+CA^+ & -AA^+BS^+ + BS^+ \\ CA^+ + CA^+BS^+CA^+ - DS^+CA^+ & -CA^+BS^+ + DS^+ \end{bmatrix} \\ &= \begin{bmatrix} AA^+ + AA^+BS^+CA^+ - BS^+CA^+ & -AA^+BS^+ + BS^+ \\ CA^+ - (D - CA^+B)S^+CA^+ & (D - CA^+B)S^+ \end{bmatrix} \\ MM^+ &= \begin{bmatrix} AA^+ & O \\ O & SS^+ \end{bmatrix} \qquad \text{[By using (2.4)]} \qquad (2.5) \end{aligned}$$

Similarly by using Theorem 1.2, $N(A) \subseteq N(C)$ and $N(S) \subseteq N(B)$, we have $C = CA^+A$; $B = BS^+S$,

$$M^+M = \begin{bmatrix} A^+A & O \\ O & S^+S \end{bmatrix}. \tag{2.6}$$

We claim GM is EP, for

$$\begin{aligned} GM(GM)^+ &= GMM^+G \\ &= G\begin{bmatrix} AA^+ & O \\ O & SS^+ \end{bmatrix} G = \begin{bmatrix} G_1 & O \\ O & -I \end{bmatrix} \begin{bmatrix} AA^+ & O \\ O & SS^+ \end{bmatrix} \begin{bmatrix} G_1 & O \\ O & -I \end{bmatrix} \\ &= \begin{bmatrix} G_1AA^+G_1 & O \\ O & SS^+ \end{bmatrix} \\ &= \begin{bmatrix} A^+A & O \\ O & S^+S \end{bmatrix} \qquad \text{[By using (2.2) and } S \text{ is } EP] \\ &= M^+M = M^+GGM \qquad \text{[By using (1.1)]} \qquad (2.7) \end{aligned}$$

Hence GM is EP again by Theorem 1.5 (ii), M is range symmetric in $\mathfrak{m}$.

Theorem 2.2. *Let* $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *be range symmetric in* $\mathfrak{m}$ *with* $N(A) \subseteq N(C)$, $N(A^\sim) \subseteq N(B^\sim)$, *then* $\hat{M}$ *the principal pivot transform of* M *is range symmetric in* $\mathfrak{m}$.

Proof. Since M is range symmetric in $\mathfrak{m}$ with $N(A) \subseteq N(C)$, $N(A^{\sim}) \subseteq N(B^{\sim})$ by applying Theorem 2.1, A is range symmetric in $\mathfrak{m}$. Again by Theorem 1.5 (ii) both GM and G_1A are EP, where G_1 is Minkowski tensor of order as that of A. Thus

$$GM = \begin{bmatrix} G_1 & O \\ O & -I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} G_1A & G_1B \\ -C & -D \end{bmatrix}.$$

Since $N(A) \subseteq N(C)$, $N(G_1A) = N(A) \subseteq N(C)$ and $N(A^{\sim}) \subseteq N(B^{\sim})$ implies $N(A^{\sim}G_1) \subseteq N(B^{\sim}G_1) \Rightarrow N(G_1A)^{\sim} \subseteq N(G_1B)^{\sim}$ [By Theorem 1.4]. Thus $N(G_1A) \subseteq N(C)$ and $N(G_1A)^{\sim} \subseteq N(G_1B)^{\sim}$ hold for GM. Hence by using (2.1) GM can be transformed into its principal pivot

$$G\hat{M} = \begin{bmatrix} A^+G_1 & -A^+B \\ -CA^+G_1 & -S \end{bmatrix}$$

$$\text{Now } \hat{M}G = \begin{bmatrix} A^+ & -A^+B \\ CA^+ & S \end{bmatrix} \begin{bmatrix} G_1 & O \\ O & -I \end{bmatrix} = \begin{bmatrix} A^+G_1 & A^+B \\ CA^+G_1 & -S \end{bmatrix}$$

$$\text{Consider } P = \begin{bmatrix} I & O \\ O & -I \end{bmatrix};\ P^{\sim} = GP^*G = \begin{bmatrix} I & O \\ O & -I \end{bmatrix} = P^*$$

$$\text{Now } P\hat{M}GP^* = \begin{bmatrix} I & O \\ O & -I \end{bmatrix} \begin{bmatrix} A^+G_1 & A^+B \\ CA^+G_1 & -S \end{bmatrix} \begin{bmatrix} I & O \\ O & -I \end{bmatrix}$$

$$= \begin{bmatrix} A^+G_1 & -A^+B \\ -CA^+G_1 & -S \end{bmatrix} = G\hat{M}$$

Since GM is EP, by Theorem 1 [5], $G\hat{M}$ is EP. Hence $\hat{M}G = P^*G\hat{M}P$ is EP, again by Theorem 1.5 (iii) $\hat{M}$ is range symmetric in $\mathfrak{m}$.

Lemma 2.3. *Let* $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, $G = \begin{bmatrix} G_1 & O \\ O & -I \end{bmatrix}$ *be partitioned in confirmity with that of* M*.* S *and* S_1 *be the schur complements of* A *and* D *is* M *respectively. Let* $N(A) \subseteq N(C)$ *and* $N(D) \subseteq N(B)$*, then the following are equivalent.*

i) M *is range symmetric in* $\mathfrak{m}$ *with* $N(S) \subseteq N(B)$, $N(S_1) \subseteq N(C)$.

ii) A, G_1D *are range symmetric in* $\mathfrak{m}$, S, G_1S_1 *are* EP, $N(A) = N(S_1) \subseteq N(B^{\sim})$ *and* $N(D) = N(S) \subseteq N(C^{\sim})$.

Proof. (i)⇒(ii) Since M is range symmetric in $\mathfrak{m}$ with $N(A) \subseteq N(C)$ and $N(S) \subseteq N(B)$ by Theorem 2.1, A is range symmetric in $\mathfrak{m}$ and S is EP, $N(A^{\sim}) \subseteq N(B^{\sim})$ and $N(S^{\sim}) \subseteq N(C^{\sim})$. Since A is range symmetric in $\mathfrak{m}$, $N(A) = N(A^{\sim}) \subseteq N(B^{\sim})$ and S is EP implies $N(S) = N(S^*)$ by using Theorem 1.7. $N(S^*) \subseteq N(C^*)$ again by Theorem 1.3, $N(S^{\sim}) \subseteq N(C^{\sim})$ and hence $N(S) \subseteq N(C^{\sim})$. Since M is range symmetric in $\mathfrak{m}$, by Theorem 1.5 (ii), GM is EP. Then the principal rearrangement $P^*GMP = \begin{bmatrix} -D & -C \\ G_1B & G_1A \end{bmatrix}$ is EP with $N(D) \subseteq N(B) = N(G_1B)$ and $N(G_1S_1) = N(S_1) \subseteq N(C)$. Now the schur complement of D in P^*GMP is $G_1A - G_1BD^+C = G_1(A - BD^+C) =$

G_1S_1 and therefore by Theorem 1.7, D and S_1 are EP. $N(D^*) \subseteq N(C^*)$, $N(G_1S_1)^* \subseteq N(G_1B)^*$. Since D, G_1S_1 are EP by using Theorem 1.5 (ii), G_1D and S_1 using Theorem 1.3, it follows that $N(G_1D)^\sim = N(G_1D) = N(D) \subseteq N(C^\sim)$ and $N(S_1^\sim) = N(S_1) \subseteq N(B^\sim)$. Since A is range symmetric in $\mathfrak{m}$, by Theorem 1.5 (ii), G_1A is EP and hence $G_1AA^+G_1 = A^+A$. Also $N(A) \subseteq N(C)$, $N(S) \subseteq N(B)$, $N(A^\sim) \subseteq N(B^\sim)$ and $N(S^\sim) \subseteq N(C^\sim)$ hold, by applying Theorem 1 (v) [3], we have M^+ of the form (2.3). By using Theorem 1.2 and Theorem 1.3, for the conditions $N(A^\sim) \subseteq N(B^\sim)$ and $N(S^\sim) \subseteq N(C^\sim)$. We get $MM^+ = \begin{bmatrix} AA^+ & O \\ O & SS^+ \end{bmatrix}$ [By using (2.5)]. Since $N(A) \subseteq N(C)$, $N(S) \subseteq N(B)$, $N(A^\sim) \subseteq N(B^\sim)$ and $N(S^\sim) \subseteq N(C^\sim)$ hold for A as well as D according to the assumptions of Theorem 1(v) [3], M^+ is also given by

$$M^+ = \begin{bmatrix} S_1^+ & -A^+BS^+ \\ -D^+CS_1^+ & S^+ \end{bmatrix}. \tag{2.8}$$

Again by using Theorem 1.2 and Theorem 1.3, for $N(A^\sim) \subseteq N(B^\sim)$, $N(D^\sim) \subseteq N(C^\sim)$ we have $B = AA^+B$ and $C = DD^+C$, hence

$$MM^+ = \begin{bmatrix} AA^+ & O \\ O & SS \end{bmatrix} = \begin{bmatrix} S_1S_1^+ & O \\ O & SS^+ \end{bmatrix}.$$

Since M is range symmetric in $\mathfrak{m}$ by Theorem 1.5 (ii), GM is EP, by using (2.7) $MM^+ = GM^+MG$ implies $G_1A^+AG_1 = G_1S_1^+S_1G_1$ and hence $N(A) = N(S_1)$. Similarly, using the formulae (2.4) and (2.8), we obtain two more expressions for M^+M comparing the coresponding block yields $D^+D = S^+S$ which implies $N(D) = N(S)$. Thus (ii) holds.

(ii)⇒(i) $N(S) \subseteq N(B)$ follows directly from $N(S) = N(D) \subseteq N(B)$. Similarly, $N(S_1) \subseteq N(C)$ follows from $N(S_1) = N(A) \subseteq N(C)$. Since A, G_1D are range symmetric in $\mathfrak{m}$ and S, G_1S_1 are EP satisfying $N(A) \subseteq N(C)$, $N(S) \subseteq N(B)$, $N(D) \subseteq N(B)$ and $N(S_1) \subseteq N(C)$. Hence by Theorem 2.1, M is range symmetric in $\mathfrak{m}$. Thus (i) holds.

Theorem 2.4. *Let* $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, *$S$ and S_1 be the schur complements of A and D in M respectively, If M is range symmetric in $\mathfrak{m}$ with $N(A) \subseteq N(C)$, $N(D) \subseteq N(B)$, $N(S) \subseteq N(B)$ and $N(S_1) \subseteq N(C)$. The following hold:*

i) *Principal submatrices A is range symmetric in $\mathfrak{m}$ and D is EP.*
ii) *The schur complements S and G_1S_1 are EP.*
iii) *The principal pivot transform $\hat{M}$ of M by pivoting the block A is range symmetric in $\mathfrak{m}$ and $\mathrm{rk}(\hat{M}) = r$.*

Proof. (i) and (ii) are consequence of Lemma 2.3.

(iii) By Lemma 2.3, M satisfies $N(A) \subseteq N(C)$ and $N(S) \subseteq N(B)$ hence by pivoting the block A, the principal pivot transform $\hat{M}$ of M is $\hat{M} = \begin{bmatrix} A^+ & -A^+B \\ CA^+ & S \end{bmatrix}$. In $\hat{M}$,

$N(A^+) \subseteq N(CA^+)$ and $N(A^+)^* \subseteq N(A^+B)^*$. Futher the schur complement of A^+ in $\hat{M}$ is $\hat{S} = S + CA^+(A^+)^+A^+B = S + CA^+B = D$. By assumption $N(\hat{S}) = N(D) \subseteq N(B)$. By using Lemma 2.3, A and G_1D are range symmetric in $\mathfrak{m}$. Again by Theorem 1.5 (ii), G_1A and D are EP. $N(D^*) = N(\hat{S})^* \subseteq N(C^*)$, by using Theorem 1.3, we get $N(\hat{S}^\sim) \subseteq N(C^\sim)$. Now applying Theorem 2.2, we have $\hat{M}$ is range symmetric in $\mathfrak{m}$. Finally, we prove $\mathrm{rk}(\hat{M}) = \mathrm{rk}(M) = r$. The proof runs as follows:

$$\begin{aligned}\mathrm{rk}(\hat{M}) &= \mathrm{rk}(A^+) + \mathrm{rk}(\hat{S})\\ &= \mathrm{rk}(A) + \mathrm{rk}(D)\\ &= \mathrm{rk}(A) + \mathrm{rk}(S) && [\text{By using } N(D) = N(S)]\\ &= \mathrm{rk}(M) = r.\end{aligned}$$

Remark 2.5. In the special case, when M is non-singular with A and D non-singular then the conditions $N(A) \subseteq N(C)$ and $N(D) \subseteq N(B)$ automatically hold and by Theorem 1 in [3], S and S_1 are non-singular further $\mathrm{rk}(\hat{M}) = \mathrm{rk}(A) + \mathrm{rk}(D)$. Hence it follows that the principal pivot transform $\hat{M}$ of M if non-singular. However, we note that the non-singularity of $\hat{M}$ need not imply that M is non-singular. This is illustrated in the following example.

Example 2.6. Let $M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ be range symmetric in $\mathfrak{m}$.

$$\text{For } M^\sim = GM^*G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = M.$$

Thus $N(M) = N(M^\sim)$ implies M is range symmetric in $\mathfrak{m}$.

Let $B = C^* = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$; $D = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Here A and D are non-singular

and $S = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$$S = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

By Theorem 2.4, S is EP and $\text{rk}(S) = 1$ and hence S is EP_1. Therefore $\text{rk}(M) = \text{rk}(A) + \text{rk}(S) = 2 + 1 = 3$. By using (2.1), $\hat{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ is non-singular and by using Theorem 2.4, $\text{rk}(\hat{M}) = \text{rk}(A) + \text{rk}(D) = 2 + 2 = 4$.

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