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FUGLEDE-PUTNAM THEOREM AND QUASI-NILPOTENT PART OF n-POWER NORMAL OPERATORS

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Abstract. In this article we show that the following properties hold for n-power normal operators T:
(i) T has the Bishop's property(β).
(ii) T is isoloid.
(iii) T is invariant under tensor product.
(iv) T satisfies the Fuglede-Putnam theorem.
(v) T is of finite ascent and descent.
(vi) The Quasi-nilpotent part of T reduces T.

1. Introduction

In this introductory section, we indicate the main trend of the ideas to be developed in this paper. Let H and K be complex Hilbert spaces and T a bounded linear operator on H, whose domain, range and null space lie in H. Let $L(H)$ denote the algebra of all bounded linear operators acting on H. An operator T is said to be n-power normal if $T^*T^n = T^nT^*$ where $n \in \mathbb{N}$. The class of n-power normal operators is denoted by $[nN]$. The class $[nN]$ was introduced by A. S. Jibril [15] and he characterized several properties of class $[nN]$. One of the properties frequently used in this paper is that $T \in [nN]$ if and only if T^n is normal. The normality of T^n enable us to study several properties of class $[nN]$. For example, in section 2 we give matrix representation for T and prove property(β).

Definition 1.1. An operator $T \in B(H)$ is said to have the property(β) at $\lambda \in \mathbb{C}$ if the following assertion holds:-

If $D \subset \mathbb{C}$ is an open neighbourhood of λ and if $f_n : D \to H(n = 1,2,\dots)$ are vector valued analytic functions such that $(T-\mu)f_n(\mu) \to 0$ uniformly on every compact subset of D, then $f_n(\mu) \to 0$, again uniformly on every compact subset of D, for all $\mu \in D$.

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Property(β) has been proved for several operators such as hyponormal operators [21], (p,k)-quasi-hyponormal operators [26], class A operators [5], class $A(k)$ operators [19], paranormal operators [27], $*$-paranormal operators [7] and k-quasi-M-hyponormal operators [24].

Definition 1.2. An operator $T \in L(H)$ is said to be isoloid if every isolated point of $\sigma(T)$ is an eigen value of T.

Throughout this paper, the range, null space and the closure of the range of a bounded linear operator T, are denoted by ran T, ker T and $[ranT]$ respectively. For convenience we write $(T-\lambda)$ in the place of $(T-\lambda I)$.

Two important subspaces in local spectral theory are $\chi_T(F)$, the glocal spectral subspace and $\chi_T(\mathbb{C}-\{\lambda\})$.

Definition 1.3. For $T \in B(H)$ and a closed subset F of $\mathbb{C}$ the glocal spectral subspace $\chi_T(F)$ is defined as the set of all $x \in H$ such that there is an analytic H-valued function $f : \mathbb{C}\backslash F \to H$ for which $(T-\lambda)f(\lambda) = x$ for all $\lambda \in \mathbb{C}\backslash F$.

The quasinilpotent part of $(T-\lambda)$ is denoted by $H_0(T-\lambda)$ and defined as follows:

Definition 1.4.

$$H_0(T-\lambda) = \left\{x \in H : \lim_{n\to\infty} \left\|(T-\lambda)^n x\right\|^{\frac{1}{n}} = 0\right\}.$$

Note that the subspace $\chi_T(\{\lambda\})$ coincides with the quasinilpotent part of $(T-\lambda)$ while $\chi_T(\mathbb{C}-\{0\})$ coincides with the analytic core $K(T)$ defined as the set $K(T-\lambda)$ of all $x \in H$ such that there exists $c > 0$ and a sequence $\{x_n\} \in H$ for which $(T-\lambda)x_1 = x, (T-\lambda)x_{n+1} = x_n$ and $\|x_n\| \le c^n \|x\|$ for all $n \in \mathbb{N}$.

Vrbova' [28] introduced the subspace $K(T)$ which is the analytic counter part of the algebraic core $C(T)$. Saphar [25] introduced the subspace $C(T)$ in purely algebraic terms.

Definition 1.5. Let T be a linear operator on H. The algebraic core $C(T)$ is defined to be the greatest subspace M of H for which $T(M) = M$.

We note that $T^n(M) = M$ for all $n \in \mathbb{N}$.

The class of all upper semi-Fredholm operators is denoted by $\Phi_+(H)$ and is defined as,

$$\Phi_+(H) = \{T \in L(H) : \alpha(T) < \infty \text{ and } T(H) \text{ is closed}\}$$

and the class of all lower semi-Fredholm operators is denoted by $\Phi_-(H)$ and is defined as,

$$\Phi_-(H) = \left\{T \in L(H) : \beta(T) < \infty\right\}$$

where $\alpha(T)$ and $\beta(T)$ denote the dimension of the kernel of T and the codimension of the range of T. The class of all semi-Fredholm operators is denoted by $\Phi_{\pm}(H)$ and is defined as $\Phi_{\pm}(H) = \Phi_{+}(H) \cup \Phi_{-}(H)$ and the class of Fredholm operators is denoted by $\Phi(H)$ and is defined as $\Phi(H) = \Phi_{+}(H) \cap \Phi_{-}(H)$.

Recall that the ascent $p(T)$ of an operator T is the smallest non-negative integer p such that $ker\ T^p = ker\ T^{p+1}$ and if such an integer does not exist then we put $p(T) = \infty$. Analogously, descent $q(T)$ of the operator T is the smallest non-negative integer q such that $ran\ T^q = ran\ T^{q+1}$ and if such an integer does not exist then we put $q(T) = \infty$. If $p(T)$ and $q(T)$ are finite then $p(T) = q(T)$ [12, Proposition 38.3].

The class of all Weyl operators denoted by $W(H)$ is defined by,

$$W(H) = \left\{T \in \Phi(H) : \text{ind } T = 0 \text{ where } \text{ind } T = \alpha(T) - \beta(T)\right\}.$$

2. Main results

We begin with the matrix representation for $T \in [nN]$.

Lemma 2.1 ([15])**.** *$T \in [nN]$ if and only if T^n is normal.*

Lemma 2.2. *Suppose $T \in [nN]$ then $[ranT^n]$ reduces T.*

Proof. Since $T \in [nN]$, $T^nT^* = T^*T^n$. $[ranT^n]$ is invariant under T is obvious. We shall show that $[ranT^n]$ is invariant under T^*. Let $x \in ranT^n$. Then $x = T^n y$ for some $y \in H$ and $T^*x = T^*T^n y = T^nT^*y \in ranT^n$.

Suppose z is a limit point of ran T^n, then there is a sequence $\{z_n\}$ in ran(T^n) such that $z_n \to z$. Since $\{z_n\}$ is a sequence in ran T^n, $z_n = T^n x_n, n = 1,2,\dots,n \in \mathbb{N}, x_n \in H$. $T^*z_n = T^*T^n x_n = T^nT^*x_n \in ranT^n$.

So $\left\{T^*z_n\right\}$ is a sequence in ranT^n. By the continuity of T^*, the sequence $\left\{T^*z_n\right\} \to T^*z \in [ranT^n]$. Thus $[ranT^n]$ is invariant under T^* and $[ranT^n]$ reduces T. □

Theorem 2.3. *If T is n-power normal then T has the following matrix representation, $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ on $H = [ranT^n] \oplus kerT^{*n}$ where $T_1 = T|_{[ranT^n]}$ is also an n-power normal operator and T_2 is a nilpotent operator with nilpotency n. Futhermore $\sigma(T) = \sigma(T_1) \cup \{0\}$.*

Proof. By Lemma 2.2, $[ranT^n]$ reduces T. Hence T has the matrix representation, $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ on $H = [ranT^n] \oplus kerT^{*n}$. Let P be the orthogonal projection onto $[ranT^n]$. Then

$\begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} = TP = PT = PTP.$

$P(T^nT^*)P = \begin{pmatrix} T_1^nT_1^* & 0 \\ 0 & 0 \end{pmatrix}.$

Also $P(T^*T^n)P = \begin{pmatrix} T_1^*T_1^n & 0 \\ 0 & 0 \end{pmatrix}.$

Since $T \in [nN], P(T^nT^*)P = P(T^*T^n)P$, implying $T_1^nT_1^* = T_1^*T_1^n$. Hence $T_1 \in [nN]$.

For any $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in H$,

$$\begin{aligned} \langle T_2^n z_2, z_2 \rangle &= \langle T^n(I-P)z, (I-P)z \rangle \\ &= \langle (I-P)z, T^{*n}(I-P)z \rangle \\ &= 0. \end{aligned}$$

Therefore $T_2^n = 0$. Since $[ranT^n]$ reduces T, $\sigma(T) = \sigma(T_1) \cup \sigma(T_2) = \sigma(T_1) \cup \{0\}$. □

Lemma 2.4. *If T is an n-power normal operator and M is a reducing subspace of T then $T|_M$ is also an n-power normal operator.*

Proof. Since M is a reducing subspace of T, it has the matrix representation, $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ on $H = M \oplus M^{\perp}$. Let P be the orthogonal projection onto M. Then $\begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} = TP = PT = PTP$.

$P(T^nT^*)P = \begin{pmatrix} T_1^nT_1^* & 0 \\ 0 & 0 \end{pmatrix}$. $P(T^*T^n)P = \begin{pmatrix} T_1^*T_1^n & 0 \\ 0 & 0 \end{pmatrix}$.

Since $T \in [nN]$, $T_1^nT_1^* = T_1^*T_1^n$. Therefore $T_1 \in [nN]$. Hence $T|_M$ is n-power normal. □

Theorem 2.5. *If $T \in [nN]$ then T has the property(β).*

Proof. Consider an open neighbourhood $D \subset \mathbb{C}$ of $\lambda \in \mathbb{C}$ and $f_m (m = 1, 2, \ldots)$, the vector valued analytic functions on D such that $(T-\mu)f_m(\mu) \to 0$ uniformly on every compact subset of D.

Decompose H as $H = [ranT^n] \oplus kerT^{*n}$, by Theorem 2.3, $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ where $T_1 \in [nN]$ and T_2 is a nilpotent operator with nilpotency n.
$(T-\mu)f_m(\mu) \to 0$ implies,

$$\begin{pmatrix} T_1-\mu & 0 \\ 0 & T_2-\mu \end{pmatrix}\begin{pmatrix} f_{m_1}(\mu) \\ f_{m_2}(\mu) \end{pmatrix} = \begin{pmatrix} (T_1-\mu)f_{m_1}(\mu) \\ (T_2-\mu)f_{m_2}(\mu) \end{pmatrix} \to 0.$$

Since T_2 is nilpotent, it has property(β) and therefore $f_{m_2}(\mu) \to 0$.

Also since T_1^n is normal, it has property(β) and therefore by Theorem 3.39 [17], T has property(β). ☐

Corollary 2.6. *If $T \in [nN]$ then T has the single-valued extension property.*

The following two Examples show that for a n- power normal operator T, the corresponding eigenspaces need not be reducing subspaces of T.

Example 2.7. $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Clearly T is a 2-power normal operator and the eigenspace of T is $\begin{pmatrix} x \\ 0 \end{pmatrix}$ but it is not a reducing subspace of T.

Example 2.8. $T = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$. Here T is a 2-power normal operator and the corresponding eigenspaces of T are $\begin{pmatrix} x \\ 0 \end{pmatrix}$ and $\begin{pmatrix} x \\ -x \end{pmatrix}$ but these are not reducing subspaces of T.

Also the n-power normal operators are not semiregular. For example consider the multiplication operator T defined by $(Tf)(t) = tf(t)$ for $f \in L^2[0,1]$ and $t \in [0,1]$. Then T is normal, injective and has dense range. Since the range of T is not closed, T is not semiregular.

Lemma 2.9. *Let $T \in [nN]$ and $\lambda \in \sigma(T)$ be an isolated point.Then λ^n is an isolated point of $\sigma(T^n)$.*

Proof. Since $\lambda \in \sigma(T)$ is an isolated point there is a neighbourhood V of λ with radius δ which contains no point of $\sigma(T)$ other than λ. By Spectral mapping Theorem, $\lambda^n \in \sigma(T^n)$. Suppose $\lambda^n \in \sigma(T^n)$ is not an isolated point of $\sigma(T^n)$, then every neighbourhood of λ^n contains atleast one point of $\sigma(T^n)$ other than λ^n. Consequently, let μ^n in $\sigma(T^n)$ be a point in a neighbourhood V_n of λ^n with radius $\frac{\delta}{2}\rho$ where $\rho = \left|\sum_{k=0}^{n-1} \lambda^{n-k-1}\mu^k\right|$. It follows from Spectral mapping Theorem that $\mu \in \sigma(T)$. Then

$$|\lambda^n - \mu^n| \le \frac{\delta}{2}\rho \tag{2.1}$$

$$\begin{aligned} |\lambda^n - \mu^n| &= |\lambda - \mu| \left|\sum_{k=0}^{n-1} \lambda^{n-k-1}\mu^k\right| \\ &= |\lambda - \mu|\rho \end{aligned}$$

Consequently, $|\lambda - \mu| \le \frac{\delta}{2}$ by (2.1).

This shows that $\mu \neq \lambda$ is a point in V, contradicting the hypothesis that λ is an isolated point of $\sigma(T)$. Thus, λ^n is an isolated point of $\sigma(T^n)$. ☐

Lemma 2.10. *Let $T \in [nN]$ then T is isoloid.*

Proof. Let $\lambda \in \sigma(T)$ be an isolated point. By Lemma 2.9, λ^n is an isolated point of $\sigma(T^n)$. Since T^n is normal, it is isoloid. Therefore λ^n is in the point spectrum of T^n. This implies that $(\lambda^n - T^n)^{-1}$ does not exist.

$$(\lambda^n - T^n)^{-1} = \frac{1}{\lambda^n}\left(I - \frac{T^n}{\lambda^n}\right)^{-1} = \mu^n (I - \mu^n T^n)^{-1} \quad \text{where} \quad \mu = \frac{1}{\lambda}.$$

Recall the identity,

$$\begin{aligned} (I - \mu^n T^n)^{-1} &= \frac{1}{n}[(I - \mu T)^{-1} + (I - \mu w T)^{-1} + (I - \mu w^2 T)^{-1} + \cdots + (I - \mu w^{n-1} T)^{-1}] \\ (I - \mu^n T^n)^{-1} &= \frac{1}{n}[(I - \mu T)^{-1} + \sum_{k=1}^{n-1} (I - \mu w^k T)^{-1}] \end{aligned} \tag{2.2}$$

where w is the primitive root of unity. Since $(I - \mu^n T^n)^{-1}$ does not exist, atleast one term of the expression on the righthand side of (2.2) does not exist. Hence there exist two cases.

Case(i):
Suppose $(I - \mu T)^{-1}$ does not exist. Since $\mu = \frac{1}{\lambda}$, $(\lambda I - T)^{-1}$ does not exist, which implies $\lambda \in P_\sigma(T)$.

Case(ii):
Suppose $(I - \mu w^k T)^{-1}$ does not exist for some $k = 1, 2, 3, \ldots, n-1$. From (2.2),

$$\begin{aligned} n(I - \mu^n T^n)^{-1} - (I - \mu w^k T)^{-1} &= (I - \mu T)^{-1} + \cdots + (I - \mu w^{k-1} T)^{-1} \\ &\quad + (I - \mu w^{k+1} T)^{-1} + \cdots + (I - \mu w^{n-1} T)^{-1}. \end{aligned}$$

Since $n(I - \mu^n T^n)^{-1} - (I - \mu w^k T)^{-1}$ does not exist the expression on the otherside also does not exist. In that expression atleast one term does not exist. On repeating a similar argument as above, we arrive at a stage where $(I - \mu T)^{-1}$ does not exist. That is $\frac{1}{\lambda}(\lambda I - T)^{-1}$ does not exist, hence $\lambda \in P_\sigma(T)$. It follows that T is isoloid. □

Tensor product of class[nN] operators

For $A, B \in L(H)$, a number of authors have considered variously ,the tensor product $A \otimes B$, on the product space $H \otimes H$. The operation of taking tensor products $A \otimes B$ preserves many a property of $A, B \in L(H)$, but by no means all of them. For instance the normaloid property is invariant under tensor products, whereas the spectroloid property is not [23, pp.623 and 631]. H. Jinchuan [16] proved that $A \otimes B$ is normal if and only if A and B are so, where A and B are non-zero operators. Similar results were proved for subnormal operators [18], hyponormal operators [13], $p-$ hyponormal operators [6], class A operators [14] and $p-$ quasihyponormal operators [9]. But there exists paranormal operators A and B such that $A \otimes B$ is not paranormal [3]. We show that if A and B are of class n-power normal then $A \otimes B$ is also of the class n-power normal.

Lemma 2.11 ([13]). *If $A \in L(H)$ and $B \in L(K)$ are non-zero operators, then $A \otimes B$ is normal if and only if so are A and B.*

Theorem 2.12. *$T_1 \otimes T_2$ is an n-power normal operator if and only if T_1 and T_2 are so.*

Proof. First we begin with the observations that $(T_1 \otimes T_2)^*(T_1 \otimes T_2) = T_1^* T_1 \otimes T_2^* T_2$ and $(T_1 \otimes T_2)^n = T_1^n \otimes T_2^n$. Suppose T_1 and T_2 are n-power normal operators, then

$$\begin{aligned}(T_1 \otimes T_2)^n (T_1 \otimes T_2)^* &= T_1^n T_1^* \otimes T_2^n T_2^* \\ &= T_1^* T_1^n \otimes T_2^* T_2^n \\ &= (T_1 \otimes T_2)^* (T_1 \otimes T_2)^n.\end{aligned}$$

Therefore $T_1 \otimes T_2$ is an n-power normal operator. Conversely, suppose $T_1 \otimes T_2$ is an n-power normal operator, then $(T_1 \otimes T_2)^n$ is normal. By Lemma 2.11, we have T_1^n and T_2^n are normal. Then by Lemma 2.1, T_1 and T_2 are n-power normal operators. □

Fuglede-Putnam Theorem for n-power normal operators

Fuglede-Putnam Theorem is well known in operator theory. It affirms that if A and B are normal operators and $AX = XB$ for some operator X then $A^* X = XB^*$. First, Fuglede [10] proved it in the case when $A = B$ and then Putnam [22] proved it in a general case. There exists many generalizations of this Theorem of which most of them go into unwinding the normality of A and B (see [11, 20] and some of the references cited in these papers).

Berbarian [4] unwinds the hypothesis on A and B by assuming A and B^* are hyponormal operators and X to be a Hilbert-Schmidt class. The operators in H which are of Hilbert-Schmidt class form an ideal $\mathbb{H}$ in the algebra $L(H)$ of all operators in H. $\mathbb{H}$ itself is a Hilbert space for the inner product

$$\langle X, Y \rangle = \sum \langle Xe_i, Ye_i \rangle = Tr(Y^* X) = Tr(XY^*),$$

where $\{e_i\}$ is any orthonormal basis of H. For each pair of operators $A, B \in L(H)$, there is an operator Γ defined on $L(\mathbb{H})$ via the formula $\Gamma(X) = AXB$ as in [4]. Obviously, $\|\Gamma\| \le \|A\|\,\|B\|$. The adjoint of Γ is given by the formula $\Gamma^*(X) = A^* X B^*$. Also if $A \ge 0, B \ge 0$ then $\Gamma \ge 0$ [4].

Lemma 2.13. *If A and B^* are of class$[nN]$ then the operator Γ is of class$[nN]$.*

Proof. By hypothesis, $A^* A^n = A^n A^*, BB^{*^n} = B^{*^n} B$.
Since, $\Gamma(X) = AXB$ and $\Gamma^*(X) = A^* X B^*$ for any pair $A, B \in L(H)$,

$$\begin{aligned}(\Gamma^* \Gamma^n - \Gamma^n \Gamma^*)X &= \Gamma^* \Gamma^n X - \Gamma^n \Gamma^* X \\ &= \Gamma^*(A^n X B^{*^n}) - \Gamma^n (A^* X B)\end{aligned}$$

$$\begin{aligned} &= A^*A^nXB^{*^n}B - A^nA^*XBB^{*^n} \\ &= A^*A^nXB^{*^n}B - A^*A^nXB^{*^n}B \\ &= 0. \end{aligned}$$

The above equality shows that $\Gamma \in [nN]$. □

Lemma 2.14. *If $A \in [nN]$ and A is invertible, then $A^{-1} \in [nN]$.*

Proof. By hypothesis $A^*A^n = A^nA^*$, we need to prove that $A^{*^{-1}}A^{n^{-1}} = A^{n^{-1}}A^{*^{-1}}$.

$$A^{*^{-1}}A^{n^{-1}} = (A^nA^*)^{-1} = (A^*A^n)^{-1} = A^{n^{-1}}A^{*^{-1}}.$$

Hence $A^{-1} \in \text{class}[nN]$. □

Lemma 2.15 ([15])**.** *Let $T \in L(H)$ such that $T \in [2N] \cap [3N]$, then $T \in [nN]$ for all positive integers $n \geq 4$.*

Theorem 2.16. *Let A and B^* be in class$[2N \cap 3N]$ such that B^* is invertible and X be a Hilbert-Schmidt operator. Suppose that $AX = XB$ then $A^*X = XB^*$.*

Proof. Let Γ be the Hilbert-Schmidt operator defined by, $\Gamma Y = AYB^{-1}$, where $Y \in L(H)$. By hypothesis A and B^* are of class$[nN]$, by Lemma 2.14 $(B^*)^{-1}$ is of class$[nN]$. Since $(B^*)^{-1} = (B^{-1})^*$, it follows by Lemma 2.13 that Γ is of class$[nN]$. The hypothesis $AX = XB$ implies that $\Gamma X = X$ and also by Lemma 2.15, $T \in [nN]$ for all $n \geq 2$, it follows that,

$$\begin{aligned} \left\|\Gamma^*X\right\|^2 &= \left\langle \Gamma^*X, \Gamma^*X \right\rangle \\ &= \left\langle \Gamma^*\Gamma^nX, \Gamma^*\Gamma^nX \right\rangle \\ &= \left\langle \Gamma\Gamma^{*n}\Gamma^*\Gamma^nX, X \right\rangle \\ &= \left\langle \Gamma\Gamma^{*n+1}\Gamma^nX, X \right\rangle \\ &= \left\langle \Gamma^{*n+1}\Gamma\Gamma^nX, X \right\rangle \\ &= \left\langle \Gamma^{n+1}X, \Gamma^{n+1}X \right\rangle \\ &= \|X\|^2. \end{aligned}$$

The above equality gives,

$$\begin{aligned} \left\|\Gamma^*X - X\right\|^2 &= \left\langle \Gamma^*X - X, \Gamma^*X - X \right\rangle \\ &= \left\langle \Gamma^*X, \Gamma^*X \right\rangle - \left\langle \Gamma^*X, X \right\rangle - \left\langle X, \Gamma^*X \right\rangle + \langle X, X \rangle \\ &= \left\|\Gamma^*X\right\|^2 - \langle X, \Gamma X \rangle - \langle \Gamma X, X \rangle + \|X\|^2 \\ &= \|X\|^2 - \langle X, X \rangle - \langle X, X \rangle + \|X\|^2 \end{aligned}$$

$$= 0.$$

Therefore $\Gamma^* X = X$ and hence $A^* X = XB^*$. □

Ascent and Descent

The non-negative integers $p(T)$ and $q(T)$ known as the ascent and descent of T respectively play a vital role to generate several classes of Browder operators and related spectrum. So we may anticipate if an n-power normal operator T have finite ascent(descent) or not. Infact, T^n has finite ascent since it is normal. Indeed, the following Lemma shows that the ascent and descent of $T \in [nN]$ are finite.

Lemma 2.17. *For any operator $T \in L(H)$ with T^n normal, the following assertions hold:*

(i) $p(T) = q(T) \leq n$.

(ii) $N^\infty(T) = kerT^n$ *and* $T^\infty(H) = ranT^n$*, where* $N^\infty(T) = \bigcup_{k\in\mathbb{N}} kerT^k$ *and* $T^\infty(H) = \bigcap_{k\in\mathbb{N}} T^k(H)$ *are the hyper kernel and hyper range respectively.*

Proof. (i) It is well known that, for any normal operator A, $kerA^2 = kerA$ and $[ranA^2] = [ranA]$.

Since T^n is normal, $kerT^{2n} = kerT^n$ and $[ranT^{2n}] = [ranT^n]$. Consequently, from the chain relations $kerT \subseteq kerT^2 \subseteq \cdots \subseteq kerT^n \subseteq kerT^{n+1} \subseteq \cdots \subseteq kerT^{2n} = kerT^n \subseteq kerT^{2n+1} \cdots$ and $\cdots [ranT^n] = [ranT^{2n}] \subseteq [ranT^{2n-1}] \subseteq \cdots \subseteq [ranT^{n+1}] \subseteq [ranT^n] \subseteq [ranT^{n-1}] \subseteq \cdots \subseteq [ranT]$, we obtain, $kerT^n = kerT^{n+1}$ and $ranT^n = ranT^{n+1}$. By the definition of $p(T)$ and $q(T)$, we have $p(T) \leq n$ and $q(T) \leq n$. Since both are finite $p(T) = q(T)$ [12].

(ii) Also

$$N^\infty(T) = \bigcup_{k\in\mathbb{N}} kerT^k = kerT^n, T^\infty(H) = \bigcap_{k\in\mathbb{N}} T^k(H) = ranT^n.$$ □

Nullity and Deficiency

The role of nullity $\alpha(T)$ and deficiency $\beta(T)$ of an operator T are crucial in the class of Fredholm operators and Weyl operators. The following Theorem concerning $\alpha(T)$ and $\beta(T)$ is useful to explore if $T \in [nN]$ fit into the class of Weyl operators or not. Infact, Aiena [1] proved a Theorem connecting ascent and descent with nullity and deficiency, which is stated below.

Theorem 2.18 ([1],Theorem 3.4)**.** *If T is a linear operator on a vector space X and if $p(T) = q(T) < \infty$ then $\alpha(T) = \beta(T)$ (possibly infinity).*

Theorem 2.19. *Suppose $T \in [nN]$ such that $\alpha(T)$ or $\beta(T)$ is finite and $T(H)$ is closed then T is a Weyl operator.*

Proof. We have $p(T) = q(T) \leq n$ by Lemma 2.17. It immediately follows from Theorem 2.18 that $\alpha(T) = \beta(T) < \infty$.

Consequently T is a Fredholm operator with ind T=0 and hence Weyl. □

Theorem 2.20. *Suppose that $C(T)$ is the algebraic core of $T \in [nN]$ then the following assertions hold:*

(i) *$C(T)$ is invariant under T^{*n}.*

(ii) $T^*(C(T)) \subseteq C(T^n)$.

Proof. (i) Since $T \in [nN]$, $T^*T^n = T^nT^*$ or $T^{*n}T = TT^{*n}$. Also by the definition of algebraic core of T, $T(C(T)) = C(T)$ or $T^n(C(T)) = C(T)$ for all $n \in \mathbb{N}$. $T^{*n}T = TT^{*n}$ implies $T^{*n}T(C(T)) = TT^{*n}(C(T))$ or $T^{*n}(C(T)) = TT^{*n}(C(T))$.

$C(T)$ being the greatest subspace satisfying $T(C(T)) = C(T)$, we have $T^{*n}(C(T)) \subseteq C(T)$. Thus $C(T)$ is invariant under T^{*n}.

(ii) $T \in [nN]$ implies $T^*T^nC(T) = T^nT^*C(T)$ or $T^*(C(T)) = T^nT^*(C(T))$. It follows that $T^*(C(T)) \subseteq C(T^n)$, the algebraic core of T^n. □

A. S. Jibril [15] proved that if $T \in [2N] \cap [3N]$, then $T \in [nN]$ for all positive integers $n \geq 4$.

Theorem 2.21. *If $T \in [2N] \cap [3N]$, then*

(i) *$H_0(T)$ is a reducing subspace of T.*

(ii) *$x \in H_0(T)$ if and only if $T^*x \in H_0(T)$ where*

$$H_0(T) = \left\{x \in H : \lim_{n\to\infty} \left\|T^n x\right\|^{\frac{1}{n}} = 0\right\}.$$

(iii) *$ker(T-\lambda) \cap H_0(T) = \{0\}$ for every $\lambda \neq 0$.*

Proof. (i) Let $F \subset \mathbb{C}$ be a closed set. The glocal spectral subspace $\chi_T(F)$ is defined as, $\chi_T(F) = \left\{x \in H : \exists\, analytic\ f(z) : (T-z)f(z) = x \text{ on } \mathbb{C}\backslash F\right\}$. By Theorem 2.20 [1], we have $H_0(T-\lambda) = \chi_T(\{\lambda\})$. By Theorem 2.5, T has property(β). Also by Proposition 1.2.19 [17], $\chi_T(F)$ is closed and $\sigma(T|_{\chi_T(F)}) \subset F$. Hence $H_0(T-\lambda)$ is closed for $\lambda \in \mathbb{C}$, which implies $H_0(T)$ is closed. If $x \in H_0(T)$ then from the inequality $\|T^nTx\| \leq \|T\|\,\|T^nx\|$, it is easily seen that $Tx \in H_0(T)$ and $H_0(T)$ is invariant under T.

$$\begin{aligned}
\left\|T^nT^*x\right\|^2 &= \left\langle T^nT^*x, T^nT^*x\right\rangle \\
&= \left\langle TT^{*n}T^nT^*x, x\right\rangle \\
&= \left\langle T^{*n+1}x, T^{*n+1}x\right\rangle \\
&= \left\|T^{n+1}x\right\|^2 since\ T^{n+1}\ is\ normal \\
\left\|T^nT^*x\right\| &= \left\|T^{n+1}x\right\| \qquad (2.3)
\end{aligned}$$

If $x \in H_0(T)$ then, $\|T^n T^* x\|^{\frac{1}{n}} = \left(\left\|T^{n+1}x\right\|^{\frac{1}{n+1}}\right)^{\frac{n+1}{n}}$ by (2.3). It follows that $T^*x \in H_0(T)$ and $H_0(T)$ is invariant under T^*.

(ii) $x \in H_0(T)$ implies $T^*x \in H_0(T)$ follows by (i). Conversely let $T^*x \in H_0(T)$. Since by (2.3) $\left\|T^{n+1}x\right\| = \|T^n T^* x\|$,

$$\lim_{n\to\infty}\left\|T^{n+1}x\right\|^{\frac{1}{n+1}} = \lim_{n\to\infty}\left(\left\|T^n T^* x\right\|^{\frac{1}{n}}\right)^{\frac{n}{n+1}} = 0.$$

Thus $x \in H_0(T)$.

(iii) Suppose $x \neq 0 \in ker(T-\lambda) \cap H_0(T)$. Then $x \in ker(T-\lambda)$ implies,

$$(T-\lambda)x = 0 \Rightarrow Tx = \lambda x \Rightarrow T^n x = \lambda^n x.$$

By (ii) $x \in H_0(T)$ if and only if $T^*(x) \in H_0(T)$ and hence,

$$\begin{aligned} 0 &= \lim_{n\to\infty}\left\|T^n T^* x\right\|^{\frac{1}{n}} \\ &= \lim_{n\to\infty}\left\|T^* T^n x\right\|^{\frac{1}{n}} \\ &= \lim_{n\to\infty}\left\|T^* \lambda^n x\right\|^{\frac{1}{n}} \\ &= \lim_{n\to\infty}|\lambda|\left\|T^* x\right\|^{\frac{1}{n}} \\ &= |\lambda| \lim_{n\to\infty}\left\|T^* x\right\|^{\frac{1}{n}} \\ &= |\lambda|. \end{aligned}$$

Which is a contradiction and therefore $T^*x \notin H_0(T) \Rightarrow x \notin H_0(T)$. Hence $ker(T-\lambda) \cap H_0(T) = \{0\}$ for every $\lambda \neq 0$. □

Remark 2.22. For $T \in [nN]$ the restriction $T^n|_M$ of T^n to a closed invariant subspace M is a hyponormal operator, since $T^n|_M$ is subnormal.

Theorem 2.23. *Suppose $T \in [2N] \cap [3N]$, then for every $m \geq 2, m \in \mathbb{N}$ the following properties hold:*

(i) *$H_0(T^m - \lambda)$ is a reducing subspace of T.*
(ii) *$H_0(T^m - \lambda) = ker(T^m - \lambda) = ker(T^{*m} - \lambda)$. In particular $H_0(T^m) = ker\,T^m = ker\,T^{*m}$.*
(iii) *If M is an invariant subspace of T and $T_1 = T|_M$ on $H = M \oplus M^\perp$ then $H_0(T_1^m - \lambda) = ker(T_1^m - \lambda) \subseteq ker(T_1^m - \lambda)^*$.*
(iv) *$H_0(T^m - \lambda^m) \supset H_0(T-\lambda)$ and $H_0(T^m - \lambda^m) = H_0(T-\lambda)$ if $S = T^{m-1} + \lambda T^{m-2} + \cdots + \lambda^{m-2}T + \lambda^{m-1}$ is invertible.*
(v) *$H_0(T-\lambda) \subset ker(T^m - \lambda^m)$ and $H_0(T-\lambda) = ker(T-\lambda)$ if S is invertible.*

Proof. (i)

$$H_0(T^m-\lambda)=\left\{x\in H:\lim_{n\to\infty}\left\|(T^m-\lambda)^n x\right\|^{\frac{1}{n}}=0\right\}.$$

Since $T\in[2N]\cap[3N]$, T is n-power normal for all $n\geq 2$. Therefore $(T^m)^nT^*=T^*(T^m)^n$ for all $m\geq 2, n\geq 1$. Consequently, $(T^m-\lambda)^nT^*=T^*(T^m-\lambda)^n$ and hence for $x\in H_0(T^m-\lambda)$, we have

$$\begin{aligned}\lim_{n\to\infty}\left\|(T^m-\lambda)^nT^*x\right\|^{\frac{1}{n}} &= \lim_{n\to\infty}\left\|T^*(T^m-\lambda)^n x\right\|^{\frac{1}{n}}\\ &\leq \lim_{n\to\infty}\left\|T^*\right\|^{\frac{1}{n}}\lim_{n\to\infty}\left\|(T^m-\lambda)^n x\right\|^{\frac{1}{n}}\\ &= 0.\end{aligned}$$

Thus $T^*x\in H_0(T^m-\lambda)$. That $Tx\in H_0(T^m-\lambda)$ is obvious.

(ii) It is well known that for a totally paranormal operator T, $H_0(T-\lambda)=ker(T-\lambda)$ for all $\lambda\in\mathbb{C}$ [2]. The class of totally paranormal operators includes the class of hyponormal operators and hence normal operators. Since T^m is normal for all $m\geq 2$, we have

$$H_0(T^m-\lambda)=ker(T^m-\lambda)=ker(T^m-\lambda)^*.$$

For $\lambda=0$, $H_0(T^m)=ker(T^m)=ker(T^{*m})$.

(iii) By Remark 2.22, $T_1^m=T^m|_M$ is hyponormal and hence $H_0(T_1^m-\lambda)=ker(T_1^m-\lambda)\subseteq ker(T_1^m-\lambda)^*$.

(iv) Let $x\in H_0(T-\lambda)$ then

$$\lim_{n\to\infty}\left\|(T-\lambda)^n x\right\|^{\frac{1}{n}}=0.$$

Since $T^m-\lambda^m=(T-\lambda)(T^{m-1}+\lambda T^{m-2}+\cdots+\lambda^{m-2}T+\lambda^{m-1})=(T-\lambda)S$, where $S=(T^{m-1}+\lambda T^{m-2}+\cdots+\lambda^{m-2}T+\lambda^{m-1})$, we have,

$$\begin{aligned}\lim_{n\to\infty}\left\|(T^m-\lambda^m)^n x\right\|^{\frac{1}{n}} &= \lim_{n\to\infty}\left\|(T-\lambda)^nS^n x\right\|^{\frac{1}{n}}\\ &\leq \lim_{n\to\infty}\left\|S^n\right\|^{\frac{1}{n}}\lim_{n\to\infty}\left\|(T-\lambda)^n x\right\|^{\frac{1}{n}}\\ &\leq \|S\|\lim_{n\to\infty}\left\|(T-\lambda)^n x\right\|^{\frac{1}{n}}\\ &= 0.\end{aligned}$$

Therefore $x\in H_0(T^m-\lambda^m)$ and $H_0(T-\lambda)\subset H_0(T^m-\lambda^m)$.

On the otherhand if S is invertible then $(T-\lambda) = S^{-1}(T^m-\lambda^m)$. For $x \in H_0(T^m-\lambda^m)$, we have,

$$\begin{aligned}\lim_{n\to\infty}\left\|(T-\lambda)^n x\right\|^{\frac{1}{n}} &= \lim_{n\to\infty}\left\|S^{-n}(T^m-\lambda^m)^n x\right\|^{\frac{1}{n}} \\ &\leq \left\|S^{-1}\right\| \lim_{n\to\infty}\left\|(T^m-\lambda^m)^n x\right\|^{\frac{1}{n}} \\ &= 0.\end{aligned}$$

Consequently, $H_0(T^m-\lambda^m) = H_0(T-\lambda)$ for all $m \geq 2$.
(v) $H_0(T-\lambda) \subset ker(T^m-\lambda^m)$ follows from (ii) and (iv).

That S is invertible yields $ker(T^m-\lambda^m) = ker(T-\lambda)$. Again by (ii) and (iv) $H_0(T-\lambda) = ker(T-\lambda)$. □

In general $T \in [nN]$ is not translation invariant.

Example 2.24. It is easily seen that, for $T = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \in [3N]$,

$$(T-i)^3(T-i)^* = \begin{pmatrix} -6i-8 & -4i+7 \\ 10i-1 & -4i-7 \end{pmatrix}.$$

$$(T-i)^*(T-i)^3 = \begin{pmatrix} -6i-8 & -10i+1 \\ 4i-7 & -4i-7 \end{pmatrix}.$$

Therefore $(T-i) \notin [3N]$. Thus $T \in [nN]$ is not translation invariant.

Naturally in view of the above statement, the following question arises: What could be the nature of class $[nN]$ operators satisfying the translation invariant property?

In [8] Eungil Ko proved that if the k^{th} root of a hyponormal operator is translation invariant then it is hyponormal. We use the same technique to prove the following Theorem.

Theorem 2.25. *Suppose $T \in [nN]$ is translation invariant then T is normal.*

Proof.

$$\begin{aligned}(T-\lambda)^n(T-\lambda)^* &= (T-\lambda)^*(T-\lambda)^n \\ 0 &= (T-\lambda)^*(T-\lambda)^n - (T-\lambda)^n(T-\lambda)^* \\ 0 &= (T-\lambda)^*\left[\sum_{k=0}^{n}\binom{n}{k}T^{n-k}(-\lambda)^k\right] - \left[\sum_{k=0}^{n}\binom{n}{k}T^{n-k}(-\lambda)^k\right](T-\lambda)^* \end{aligned} \quad (2.4)$$

Put $\lambda = \rho e^{i\theta}, \rho > 0, 0 \leq \theta < 2\pi$, in (2.4) and dividing the simplified equation by ρ^{n-1} gives,

$$0 = n(T^*T - TT^*)e^{(n-1)i\theta} + \frac{1}{\rho}(\text{the other terms}).$$

Taking limit as $\rho \to \infty$ gives, $TT^* = T^*T$. □

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