ON THE SECOND GAUSSIAN CURVATURE OF RULED SURFACES
IN EUCLIDEAN 3-SPACE

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Abstract. In this paper, we mainly investigate non-developable ruled surface in a 3-dimensional
Euclidean space satisfying the equation $K_{II} = KH$ along each ruling, where $K$ is the Gaussian
curvature, $H$ is the mean curvature and $K_{II}$ is the second Gaussian curvature.

1. Introduction

The inner geometry of the second fundamental form has been a popular research
topic for ages. It is readily seen that the second fundamental form of a surface is non-
degenerate if and only if a surface is non-developable.

On a non-developable surface $M$, we can consider the Gaussian curvature $K_{II}$ of the
second fundamental form which is regarded as a new Riemannian metric. The curvature $K_{II}$ will be called the second Gaussian curvature of the surface $M$ (cf. [2]).

For the study of the second Gaussian curvature, D. Koutroufiotis ([9]) has shown that
a closed ovaloid is a sphere if $K_{II} = cK$ for some constant $c$ or if $K_{II} = \sqrt{K}$, where $K$ is
the Gaussian curvature. Th. Koufogiorgos and T. Hasanis ([8]) proved that the sphere
is the only closed ovaloid satisfying $K_{II} = H$, where $H$ is the mean curvature. Also,
W. Kühnel ([10]) studied surfaces of revolution satisfying $K_{II} = H$. One of the natural
generalizations of surfaces of revolution is the helicoidal surfaces. In [1] C. Baikoussis
and Th. Koufogiorgos proved that the helicoidal surfaces satisfying $K_{II} = H$ are locally
characterized by constancy of the ratio of the principal curvatures. On the other hand,
D. E. Blair and Th. Koufogiorgos ([2]) investigated a non-developable ruled surface in a
3-dimensional Euclidean space $\mathbb{E}^3$ satisfying the condition

$$aK_{II} + bH = \text{constant}, \quad 2a + b \neq 0,$$  \hspace{1cm} (1.1)

along each ruling. Also, they proved that a ruled surface with vanishing second Gaussian
curvature is a helicoid.

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Recently, the present author ([16]) studied a non-developable ruled surface in a 3-dimensional Euclidean space $\mathbb{E}^3$ satisfying the conditions

\begin{align}
aH + bK &= \text{constant}, \quad a \neq 0, \\
aK_{II} + bK &= \text{constant}, \quad a \neq 0,
\end{align}

(1.2) (1.3)

along each ruling.

In particular, if it satisfies the condition (1.2), then a surface is called a linear Wein- garten surface (see [11]).

On the other hand, in [7] Y. H. Kim and the present author investigated a non-developable ruled surface in a 3-dimensional Lorentz-Minkowski space $\mathbb{L}^3$ satisfying the conditions (1.1), (1.2) and (1.3). In [13] W. Sodsiri studied a non-developable ruled surface in $\mathbb{L}^3$ with non-null rulings such that the linear combination $aK_{II} + bH + cK$ is constant along ruling.

In this article, we investigate a non-developable ruled surface in a Euclidean 3-space $\mathbb{E}^3$ satisfying the condition

\begin{equation}
K_{II} = KH,
\end{equation}

(1.4)

along each ruling.

On the other hand, many geometers have been interested in studying submanifolds of Euclidean and pseudo-Euclidean space in terms of the so-called finite type immersion ([3]). Also, such a notion can be extended to smooth maps on submanifolds, namely the Gauss map ([4]). The Gauss map $G$ on a submanifold $M$ of a Euclidean space or pseudo-Euclidean space is said to be of pointwise 1-type if $\Delta G = fG$ for some smooth function $f$ on $M$ where $\Delta$ denotes the Laplace operator defined on $M$ ([6]).

In [5] M. Choi and Y. H. Kim proved the following theorem which will be useful to prove our theorems in this paper.

**Theorem 1.1.** ([5]) Let $M$ be a non-cylindrical ruled surface in a 3-dimensional Euclidean space. Then, the Gauss map is of pointwise 1-type if and only if $M$ is an open part of a helicoid.

**2. Preliminaries**

Let $\mathbb{E}^3$ be a 3-dimensional Euclidean space with the metric $<,> = dx_1^2 + dx_2^2 + dx_3^2$, where $(x_1, x_2, x_3)$ is a standard rectangular coordinate system of $\mathbb{E}^3$.

We denote a surface $M$ in $\mathbb{E}^3$ by

$$x(s, t) = \{x_1(s, t), x_2(s, t), x_3(s, t)\}.$$

Then the first fundamental form $I$ of the surface $M$ is defined by

\begin{equation}
I = Eds^2 + 2Fdsdt + Gdt^2,
\end{equation}

$$E = <x_s, x_s>, \quad F = <x_s, x_t>, \quad G = <x_t, x_t>, \quad x_s = \frac{\partial x(s, t)}{\partial s}.$$
We define the second fundamental form $II$ of $M$ by

$$II = eds^2 + 2f ds dt + g dt^2,$$

$$e = \frac{1}{\sqrt{EG-F^2}} \det(x_s x_t x_{ss}),$$

$$f = \frac{1}{\sqrt{EG-F^2}} \det(x_s x_t x_{st}),$$

$$g = \frac{1}{\sqrt{EG-F^2}} \det(x_s x_t x_{tt}).$$

Using classical notation above, the Gaussian curvature $K$ is defined by (See, [14, p. 112])

$$K = \frac{1}{(EG-F^2)^2} \left\{ \begin{array}{ccc} -\frac{1}{2}E_{tt} + F_{st} - \frac{1}{2}G_{ss} & \frac{1}{2}E_s F_s - \frac{1}{2}E_t F_t & 0 \frac{1}{2}E_t + \frac{1}{2}G_s \\ \frac{1}{2}G_t & F & E \\ F & G & \frac{1}{2}G_s & E F \end{array} \right\}, \quad (2.1)$$

or equivalently,

$$K = \frac{eg - f^2}{EG - F^2}.$$

On the other hand, the mean curvature $H$ is given by

$$H = \frac{1}{2} \frac{Eg - 2Ff + Gc}{EG - F^2}.$$

At this stage we are able to compute the second Gaussian curvature $K_{II}$ of a non-developable surface in $\mathbb{E}^3$ by replacing $E, F, G$ by the components of the second fundamental form $e, f, g$ respectively in (2.1). Thus, the second Gaussian curvature $K_{II}$ is given by

$$K_{II} = \frac{1}{(eg-f^2)^2} \left\{ \begin{array}{ccc} -\frac{1}{2}E_{tt} + f_{st} - \frac{1}{2}g_{ss} & \frac{1}{2}E_s f_s - \frac{1}{2}E_t f_t & 0 \frac{1}{2}E_t + \frac{1}{2}g_s \\ f_t - \frac{1}{2}g_s & e & f \\ \frac{1}{2}g_t & f & g \end{array} \right\}, \quad (2.2)$$

It is well known that a minimal surface has vanishing second Gaussian curvature but that a surface with vanishing second Gaussian curvature need not be minimal.

3. Main Results

In this section we classify a non-developable ruled surface in a Euclidean 3-space $\mathbb{E}^3$ satisfying the equations (1.4). It is well known that a cylindrical ruled surface is developable, i.e., the Gaussian curvature $K$ is identically zero. Therefore, the second fundamental form $II$ is degenerate. Thus, non-cylindrical ruled surfaces are meaningful for our study.
Theorem 3.1. A non-developable ruled surface in a Euclidean 3-space \( \mathbb{E}^3 \) satisfying the condition \( K_{II} = KH \) along each ruling is a piece of a helicoid.

Proof. Let \( M \) be a non-developable ruled surface in \( \mathbb{E}^3 \). Then the parametrization for \( M \) is given by

\[
x = x(s,t) = \alpha(s) + t\beta(s)
\]

such that \( < \beta, \beta > = 1, < \beta', \beta' > = 1 \) and \( < \alpha', \beta' > = 0 \). In this case \( \alpha \) is the striction curve of \( x \), and the parameter is the arc-length on the spherical curve \( \beta \). And we have the natural frame \( \{ x_s, x_t \} \) given by

\[
x_s = \alpha' + t\beta' \quad \text{and} \quad x_t = \beta.
\]

Then, the first fundamental form of the surface is given by

\[
E = < \alpha', \alpha' > + t^2, \quad F = < \alpha', \beta >, \quad G = 1.
\]

For later use, we define the smooth functions \( Q, J \) and \( D \) as follows:

\[
Q = < \alpha', \beta \times \beta' >, \quad J = < \beta'', \beta' \times \beta >, \quad D = \sqrt{EG - F^2}.
\]

In terms of the orthonormal basis \( \{ \beta, \beta', \beta \times \beta' \} \) we obtain

\[
\alpha' = F\beta + Q\beta \times \beta', \quad \beta'' = -\beta - J\beta \times \beta', \quad \alpha \times \beta = Q\beta',
\]

which imply \( EG - F^2 = Q^2 + t^2 \) and the unit normal vector \( N \) is given by

\[
N = \frac{1}{D}(\alpha' \times \beta + t\beta' \times \beta) = \frac{1}{D}(Q\beta' - t\beta \times \beta').
\]

Therefore, the components \( e, f \) and \( g \) of the second fundamental form are expressed as

\[
e = \frac{1}{D}(Q(F + QJ) - Q't + Jt^2), \quad f = \frac{Q}{D} \ne 0, \quad g = 0.
\]

Thus, using the dates described above and (2.2), we obtain

\[
K_{II} = \frac{1}{f^4} \left( fJ(f_s - \frac{1}{2}e_t) - f^2(-\frac{1}{2}e_{tt} + f_{st}) \right) = \frac{1}{2Q^2D^2} \left( Jt^4 + Q(F + 2QJ)t^2 - 2Q^2Q't + Q^3(QJ - F) \right). \tag{3.1}
\]

Furthermore, the mean curvature \( H \) and the Gaussian curvature \( K \) are given respectively by

\[
H = \frac{1}{2D^3} \left( Jt^2 - Q't + Q(QJ - F) \right), \tag{3.2}
\]

and

\[
K = \frac{Q^2}{D^4}. \tag{3.3}
\]

We now differentiate \( K_{II}, H \) and \( K \) with respect to \( t \), the results are

\[
(K_{II})_t = \frac{1}{2Q^2D^5} \left( Jt^5 + Q(2QJ - F)t^3 + 4Q^2Q't^2 + Q^3(5F + QJ)t - 2Q^4Q' \right), \tag{3.4}
\]

\[
H_t = \frac{1}{2D^5} \left( -Jt^3 + 2Q^2t^2 + Q(-QJ + 3F)t - Q^2Q' \right) \tag{3.5}
\]

and
\[ K_t = \frac{4Q^2}{D^6} t. \]  

(3.6)

Suppose that the surface satisfies the condition (1.4). Then, by (3.1)-(3.6) we can show that the coefficients of the power of \( t \) are as follows:

- \( t^9 \): \( J = 0 \),
- \( t^7 \): \( 4Q^2J - QF = 0 \),
- \( t^6 \): \( 4Q^2Q' = 0 \),
- \( t^5 \): \( 6Q^4J + 3Q^3F = 0 \),
- \( t^4 \): \( 6Q^4Q' = 0 \),
- \( t^3 \): \( 4Q^6J + 9Q^5F - 5Q^4J = 0 \),
- \( t^2 \): \( 6Q^4Q' = 0 \),
- \( t^1 \): \( 5Q^7F + Q^8J + 7Q^5F - 5Q^6J = 0 \),
- \( t^0 \): \( Q^8Q' + Q^6Q' = 0 \),

which imply \( J = F = Q' = 0 \). Thus, from (3.1) and (3.2) the second Gaussian curvature and the mean curvature are identically zero. Consequently, the surface is locally a helicoid. This completes the proof.

Combining the results of Theorem 1.1, our Theorem 3.1 and main Theorems in [16], we have

**Theorem 3.2.** Let \( M \) be a non-developable ruled surface in a Euclidean 3-space. Then, the following are equivalent:

1. \( M \) is piece of a helicoid.
2. \( M \) has pointwise 1-type Gauss map.
3. \( M \) satisfies the equation \( aK_{II} + bK + c = 0 \), \( a \neq 0 \), \( b, c \in \mathbb{R} \) along each ruling.
4. \( M \) satisfies the equation \( aH + bH + c = 0 \), \( a \neq 0 \), \( b, c \in \mathbb{R} \) along each ruling.
5. \( M \) satisfies the equation \( K_{II} = KH \) along each ruling.

**References**


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