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ON THE SECOND GAUSSIAN CURVATURE OF RULED SURFACES IN EUCLIDEAN 3-SPACE

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Abstract. In this paper, we mainly investigate non-developable ruled surface in a 3-dimensional Euclidean space satisfying the equation $K_{II} = KH$ along each ruling, where K is the Gaussian curvature, H is the mean curvature and K_{II} is the second Gaussian curvature.

1. Introduction

The inner geometry of the second fundamental form has been a popular research topic for ages. It is readily seen that the second fundamental form of a surface is non-degenerate if and only if a surface is non-developable.

On a non-developable surface M, we can consider the Gaussian curvature K_{II} of the second fundamental form which is regarded as a new Riemannian metric. The curvature K_{II} will be called the second Gaussian curvature of the surface M (cf. [2]).

For the study of the second Gaussian curvature, D. Koutroufiotis ([9]) has shown that a closed ovaloid is a sphere if $K_{II} = cK$ for some constant c or if $K_{II} = \sqrt{K}$, where K is the Gaussian curvature. Th. Koufogiorgos and T. Hasanis ([8]) proved that the sphere is the only closed ovaloid satisfying $K_{II} = H$, where H is the mean curvature. Also, W. Kühnel ([10]) studied surfaces of revolution satisfying $K_{II} = H$. One of the natural generalizations of surfaces of revolution is the helicoidal surfaces. In [1] C. Baikoussis and Th. Koufogiorgos proved that the helicoidal surfaces satisfying $K_{II} = H$ are locally characterized by constancy of the ratio of the principal curvatures. On the other hand, D. E. Blair and Th. Koufogiorgos ([2]) investigated a non-developable ruled surface in a 3-dimensional Euclidean space $\mathbb{E}^3$ satisfying the condition

$$aK_{II} + bH = \text{constant}, \quad 2a + b \neq 0, \tag{1.1}$$

along each ruling. Also, they proved that a ruled surface with vanishing second Gaussian curvature is a helicoid.

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Recently, the present author ([16]) studied a non-developable ruled surface in a 3-dimensional Euclidean space $\mathbb{E}^3$ satisfying the conditions

$$aH + bK = \text{constant}, \quad a \neq 0, \tag{1.2}$$
$$aK_{II} + bK = \text{constant}, \quad a \neq 0, \tag{1.3}$$

along each ruling.

In particular, if it satisfies the condition (1.2), then a surface is called a linear Weingarten surface (see [11]).

On the other hand, in [7] Y. H. Kim and the present author investigated a non-developable ruled surface in a 3-dimensional Lorentz-Minkowski space $\mathbb{L}^3$ satisfying the conditions (1.1), (1.2) and (1.3). In [13] W. Sodsiri studied a non-developable ruled surface in $\mathbb{L}^3$ with non-null rulings such that the linear combination $aK_{II} + bH + cK$ is constant along ruling.

In this article, we investigate a non-developable ruled surface in a Euclidean 3-space $\mathbb{E}^3$ satisfying the condition

$$K_{II} = KH, \tag{1.4}$$

along each ruling.

On the other hand, many geometers have been interested in studying submanifolds of Euclidean and pseudo-Euclidean space in terms of the so-called finite type immersion ([3]). Also, such a notion can be extended to smooth maps on submanifolds, namely the Gauss map ([4]). The Gauss map G on a submanifold M of a Euclidean space or pseudo-Euclidean space is said to be of pointwise 1-type if $\Delta G = fG$ for some smooth function f on M where Δ denotes the Laplace operator defined on M ([6]).

In [5] M. Choi and Y. H. Kim proved the following theorem which will be useful to prove our theorems in this paper.

Theorem 1.1.([5]) *Let M be a non-cylindrical ruled surface in a 3-dimensional Euclidean space. Then, the Gauss map is of pointwise 1-type if and only if M is an open part of a helicoid.*

2. Preliminaries

Let $\mathbb{E}^3$ be a 3-dimensional Euclidean space with the metric $<,>= dx_1^2 + dx_2^2 + dx_3^2$, where (x_1, x_2, x_3) is a standard rectangular coordinate system of $\mathbb{E}^3$.

We denote a surface M in $\mathbb{E}^3$ by

$$x(s,t) = \{x_1(s,t), x_2(s,t), x_3(s,t)\}.$$

Then the first fundamental form I of the surface M is defined by

$$I = Eds^2 + 2Fdsdt + Gdt^2,$$

$$E =< x_s, x_s >, \quad F =< x_s, x_t >, \quad G =< x_t, x_t >, \quad x_s = \frac{\partial x(s,t)}{\partial s}.$$

We define the second fundamental form II of M by

$$
\begin{aligned}
II &= eds^2 + 2fdsdt + gdt^2,\\
e &= \frac{1}{\sqrt{EG-F^2}}\det(x_s\ x_t\ x_{ss}),\\
f &= \frac{1}{\sqrt{EG-F^2}}\det(x_s\ x_t\ x_{st}),\\
g &= \frac{1}{\sqrt{EG-F^2}}\det(x_s\ x_t\ x_{tt}).
\end{aligned}
$$

Using classical notation above, the Gaussian curvature K is defined by (See, [14, p. 112])

$$K = \frac{1}{(EG-F^2)^2}\left\{\begin{vmatrix} -\frac{1}{2}E_{tt} + F_{st} - \frac{1}{2}G_{ss} & \frac{1}{2}E_s & F_s - \frac{1}{2}E_t \\ F_t - \frac{1}{2}G_s & E & F \\ \frac{1}{2}G_t & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_t & \frac{1}{2}G_s \\ \frac{1}{2}E_t & E & F \\ \frac{1}{2}G_s & F & G \end{vmatrix}\right\}, \tag{2.1}$$

or equivalently,

$$K = \frac{eg-f^2}{EG-F^2}.$$

On the other hand, the mean curvature H is given by

$$H = \frac{1}{2}\frac{Eg-2Ff+Ge}{EG-F^2}.$$

At this stage we are able to compute the second Gaussian curvature K_{II} of a non-developable surface in $\mathbb{E}^3$ by replacing E, F, G by the components of the second fundamental form e, f, g respectively in (2.1). Thus, the second Gaussian curvature K_{II} is given by

$$K_{II} = \frac{1}{(eg-f^2)^2}\left\{\begin{vmatrix} -\frac{1}{2}e_{tt} + f_{st} - \frac{1}{2}g_{ss} & \frac{1}{2}e_s & f_s - \frac{1}{2}e_t \\ f_t - \frac{1}{2}g_s & e & f \\ \frac{1}{2}g_t & f & g \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}e_t & \frac{1}{2}g_s \\ \frac{1}{2}e_t & e & f \\ \frac{1}{2}g_s & f & g \end{vmatrix}\right\}. \tag{2.2}$$

It is well known that a minimal surface has vanishing second Gaussian curvature but that a surface with vanishing second Gaussian curvature need not be minimal.

3. Main Results

In this section we classify a non-developable ruled surface in a Euclidean 3-space $\mathbb{E}^3$ satisfying the equations (1.4). It is well known that a cylindrical ruled surface is developable, i.e., the Gaussian curvature K is identically zero. Therefore, the second fundamental form II is degenerate. Thus, non-cylindrical ruled surfaces are meaningful for our study.

Theorem 3.1. *A non-developable ruled surface in a Euclidean* 3*-space* $\mathbb{E}^3$ *satisfying the condition* $K_{II} = KH$ *along each ruling is a piece of a helicoid.*

Proof. Let M be a non-developable ruled surface in $\mathbb{E}^3$. Then the parametrization for M is given by

$$x = x(s,t) = \alpha(s) + t\beta(s)$$

such that $<\beta,\beta>= 1, <\beta',\beta'>= 1$ and $<\alpha',\beta'>= 0$. In this case α is the striction curve of x, and the parameter is the arc-length on the spherical curve β. And we have the natural frame $\{x_s, x_t\}$ given by $x_s = \alpha' + t\beta'$ and $x_t = \beta$. Then, the first fundamental form of the surface is given by $E =<\alpha',\alpha'> +t^2$, $F =<\alpha',\beta>$ and $G = 1$. For later use, we define the smooth functions Q, J and D as follows :

$$Q =<\alpha',\beta\times\beta'>\neq 0, \quad J =<\beta'',\beta'\times\beta>, \quad D = \sqrt{EG - F^2}.$$

In terms of the orthonormal basis $\{\beta,\beta',\beta\times\beta'\}$ we obtain

$$\alpha' = F\beta + Q\beta\times\beta', \quad \beta'' = -\beta - J\beta\times\beta', \quad \alpha'\times\beta = Q\beta',$$

which imply $EG - F^2 = Q^2 + t^2$ and the unit normal vector N is given by

$$N = \frac{1}{D}(\alpha'\times\beta + t\beta'\times\beta) = \frac{1}{D}(Q\beta' - t\beta\times\beta').$$

Therefore, the components e, f and g of the second fundamental form are expressed as

$$e = \frac{1}{D}(Q(F + QJ) - Q't + Jt^2), \quad f = \frac{Q}{D}\neq 0, \quad g = 0.$$

Thus, using the dates described above and (2.2), we obtain

$$\begin{aligned} K_{II} &= \frac{1}{f^4}\left(ff_t(f_s - \frac{1}{2}e_t) - f^2(-\frac{1}{2}e_{tt} + f_{st})\right) \\ &= \frac{1}{2Q^2D^3}\left(Jt^4 + Q(F + 2QJ)t^2 - 2Q^2Q't + Q^3(QJ - F)\right). \end{aligned} \tag{3.1}$$

Furthermore, the mean curvature H and the Gaussian curvature K are given respectively by

$$H = \frac{1}{2D^3}\left(Jt^2 - Q't + Q(QJ - F)\right), \tag{3.2}$$

and

$$K = -\frac{Q^2}{D^4}. \tag{3.3}$$

We now differentiate K_{II}, H and K with respect to t, the results are

$$(K_{II})_t = \frac{1}{2Q^2D^5}\left(Jt^5 + Q(2QJ - F)t^3 + 4Q^2Q't^2 + Q^3(5F + QJ)t - 2Q^4Q'\right), \tag{3.4}$$

$$H_t = \frac{1}{2D^5}(-Jt^3 + 2Q't^2 + Q(-QJ + 3F)t - Q^2Q') \tag{3.5}$$

and

$$K_t = \frac{4Q^2}{D^6} t. \tag{3.6}$$

Suppose that the surface satisfies the condition (1.4). Then, by (3.1)-(3.6) we can show that the coefficients of the power of t are as follows:

t^9 : $J = 0$,
t^7 : $4Q^2J - QF = 0$,
t^6 : $4Q^2Q' = 0$,
t^5 : $6Q^4J + 3Q^3F = 0$,
t^4 : $6Q^4Q' = 0$,
t^3 : $4Q^6J + 9Q^5F - 5Q^4J = 0$,
t^2 : $6Q^4Q' = 0$,
t^1 : $5Q^7F + Q^8J + 7Q^5F - 5Q^6J = 0$,
t^0 : $Q^8Q' + Q^6Q' = 0$,

which imply $J = F = Q' = 0$. Thus, from (3.1) and (3.2) the second Gaussian curvature and the mean curvature are identically zero. Consequently, the surface is locally a helicoid. This completes the proof.

Combining the results of Theorem 1.1, our Theorem 3.1 and main Theorems in [16], we have

Theorem 3.2. *Let M be a non-developable ruled surface in a Euclidean 3-space. Then, the following are equivalent :*

1. *M is piece of a helicoid.*
2. *M has pointwise 1-type Gauss map.*
3. *M satisfies the equation $aK_{II} + bK + c = 0, a \neq 0, b, c \in \mathbb{R}$ along each ruling.*
4. *M satisfies the equation $aH + bH + c = 0, a \neq 0, b, c \in \mathbb{R}$ along each ruling.*
5. *M satisfies the equation $K_{II} = KH$ along each ruling.*

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